Iterates of the core model

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Abstract

Let N be a transitive model of ZFC such that ${}^{\omega}N \subset N$ and $\mathcal{P}(\mathbb{R}) \subset N$. Assume that both V and N satisfy "the core model K exists." Then K^N is an iterate of K, i.e., there exists an iteration tree $\mathcal{T}$ on K such that $\mathcal{T}$ has successor length and $\mathcal{M}^{\mathcal{T}}_{\infty} = K^N$. Moreover, if there exists an elementary embedding $\pi: V \to N$ then the iteration map associated to the main branch of $\mathcal{T}$ equals $\pi \restriction K$. (This answers a question of W.H. Woodin, M. Gitik, and others.) The hypothesis that $\mathcal{P}(\mathbb{R}) \subset N$ is not needed if there does not exist a transitive model of ZFC with infinitely many Woodin cardinals.

MSC: 03E15; 03E45; 03E55.

Keywords: Set theory; Core models; Large cardinals

0 Introduction.

The present paper isolates a sufficient criterion for when a given premouse is an iterate of the core model.

The condensation lemma for L says that Σ_1 hulls taken inside initial segments of L condense to initial segments of L; i.e., if $\pi: M \to J_\alpha$ is Σ_1 elementary and M is transitive then $M = J_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$. Such a strong form of condensation need not be true for the core model. Let K_{DJ} denote the core model of Dodd and Jensen (i.e., the core model below one measurable cardinal; cf. [1]). If κ is Ramsey in K_{DJ}, say, then it is easy to construct a fully elementary $\pi: M \to K_{\mathrm{DJ}}||\kappa$ such that M is transitive but not an initial segment of K_{DJ} (cf. [1, Sec. 17]; $K_{\mathrm{DJ}}||\kappa$ denotes the initial segment of K_{DJ} of height κ). Nevertheless, there are weaker variants of the condensation lemma for L which still hold true for K_{DJ}.

If $\pi: \mathcal{M} \to K_{\mathrm{DJ}}||\alpha$ is Σ_1 elementary, where $\mathcal{M}$ is transitive (and hence an iterable premouse), then $\mathcal{M}$ is in fact an *iterate* of K_{DJ}; i.e., there is an iteration $\mathcal{T}$ of K_{DJ} with a last model $\mathcal{M}^{\mathcal{T}}_{\infty}$ such that $\mathcal{M}$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_{\infty}$. However, even this is no longer true if we allow an inner model with a measurable cardinal and replace K_{DJ} by the core model below $0^\dagger$ (cf. [2]). The appropriate substitute for many purposes and at higher levels says that for every α there are stationarily many submodels of $K||\alpha$ of any given uncountable size smaller than the cardinality of α which condense to iterates of K. Here the letter K is supposed to denote *the* core model. Let us make more precisely what we have in mind.

We build upon the core model theory of Steel's papers [10] and [11]. Let us assume that there is no inner model with a Woodin cardinal, and that Ω is a measurable cardinal. Then the core model K of height Ω exists (cf. [10, §5]). Suppose now we are given a premouse $\mathcal{M}$. We may ask whether there is an iteration tree $\mathcal{T}$ on K with a last model $\mathcal{M}^{\mathcal{T}}_\infty$ such that $\mathcal{M}$ is an initial segment of $\mathcal{M}^{\mathcal{T}}_\infty$. If this is the case then we say that $\mathcal{M}$ is an *iterate* of K.

We shall prove that a premouse $\mathcal{M}$ is an iterate of K provided that $\mathcal{M}$ is the core model of a transitive structure $\mathcal{H}$ which to a certain extent resembles V. (Basically, we shall require that $\mathcal{H}$ be a coarse premouse in the sense of [4].) We shall in fact prove this result without having to assume that there be no inner model with a Woodin cardinal; cf. our Theorems 2.1 and 2.2.

Let us warn the unexperienced reader, though, that if an iteration tree $\mathcal{T}$ on K witnesses that $\mathcal{M}$ is an iterate of K then there does not have to be an iteration map from K to $\mathcal{M}$ being associated to $\mathcal{T}$; the reason for this is that $\mathcal{T}$ might involve "drops." If, however, $\mathcal{H}$ is a coarse premouse and there is an elementary embedding $\pi: V \to \mathcal{H}$ then, as we shall prove, $\pi \restriction K$ arises from an iteration tree on K (with no drops on the main branch); cf. our Corollary 3.1. As a second warning, this result is much less obvious than it might appear at first glance; if π is given by an iteration tree U on V then the iteration tree $\mathcal{T}$ on K from which $\pi \restriction K$ arises can have a tree structure which is significantly more complicated than the one of U.

Let us add some historical background information. Jensen has shown (cf. [3, §5.3 Lemma 6]) that every universal weasel is an iterate of K, provided that $0^\P$ does not exist. His proof in fact shows that if there are no ordinals $\kappa < \mu$ such that

$$(1) \qquad K||\mu \models \text{“}\kappa \text{ is strong,”} \text{ and } K \models \text{“}\mu \text{ is measurable,”}$$

then every universal weasel is an iterate of K. (In fact, the iteration trees on K witnessing this will all be "almost linear" in the sense of [10, p. 83].) On the other hand, Steel has shown (cf. [8, Lemma 8.21]) that if there are $\kappa < \mu$ such that (1) holds then inside $K^{Col(\omega,<\mu)}$ there is a universal weasel which is not an iterate of K. Steel has also shown in general that K embeds into all universal weasels (cf. [10, Theorem 8.10] for a precise statement).

It was open for many years whether the core model of an ultrapower $Ult(V;U)$ of V by a measure U witnessing that some cardinal is measurable has to be an iterate of K. W.H. Woodin, M. Gitik, and others had asked this question. Our Corollary 3.1 gives an affirmative answer to this question.

The main bulk of work, however, which is necessary for proving this theorem is provided by the paper [5].

The papers [5] and [9] provide examples of situations in which it appears necessary to know that certain premice are iterates of K. In particular, the "covering argument" (by which we mean the argument of [5] leading towards a proof of weak covering for the core model K) is to a large extent a verification of the fact that certain submodels of initial segments of K condense to iterates thereof. Specifically, the paper [5] for example shows that if $\beta^{\aleph_0} < \alpha \leq \Omega$ then there are stationarily many submodels of $K||\alpha$ of size $\beta^{\aleph_0}$ which condense to an iterate of K. Our Theorem 2.2, when used as a black box, will also simplify the proof of [10, Theorem 3.1].

1 Preliminaries.

We need a word for transitive structures in which the core model can be built. We picked "coarse premouse," as ours will resemble the coarse premice of [4] to a large extent (and we want to avoid an inflation of terminology). In this paper, by "coarse premouse" we shall always mean a structure in the following sense.

Definition 1.1 *A transitive structure $\mathcal{H} = (H; \in, \Omega^{\mathcal{H}}, U^{\mathcal{H}})$ is called a* coarse premouse *provided the following hold.*
(1) $\mathcal{H} \models ZFC^-$ (in the language $\{\dot{\in}\}$), where $ZFC^- = ZFC-$ the power set axiom,
(2) $\mathcal{H} \models$ "$U^{\mathcal{H}}$ witnesses that $\Omega^{\mathcal{H}}$ is a measurable cardinal,"
(3) $\mathcal{H} \models$ "every premouse is tame, and $\emptyset$ is excellent," and
(4) ${}^{\omega}H \subset H$.

A comment on "$\emptyset$ is excellent" should be in order. Suppose that $\mathcal{V} = (V; \in, \Omega, U)$ is a coarse premouse. We can then define the K^c of [11] of height Ω. Let Σ be the following strategy for iterating K^c. (Cf. Definition 1.2 below.) If $\mathcal{T}$ is an iteration tree on K^c of limit length λ then we let $\Sigma(\mathcal{T}) = c$, where c is the unique cofinal branch b through $\mathcal{T}$ such that there is a Q-structure $\mathcal{P} \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ for $\mathcal{T}$, if such b exists (otherwise we let $\Sigma(\mathcal{T})$ being undefined). $\mathcal{P}$ is a Q-structure for $\mathcal{T}$ if $\mathcal{P}$ is a premouse such that $\delta(\mathcal{T})$ is a strong cutpoint of $\mathcal{P}$, there is some $k < \omega$ such that $\mathcal{P}$ kills the Woodin property (of $\delta(\mathcal{T})$) at k, and countable submodels of $\mathcal{P}$ are $\omega_1 + 1$ iterable above $\delta(\mathcal{T})$ (cf. [11, Definition 2.1]). The assertion that $\emptyset$ is excellent means that Σ is an iteration strategy for (stacks of) normal trees on K^c. In particular, the excellence of $\emptyset$ (together with the measurability of Ω) implies that the true core model K of height Ω can be isolated. (This is shown in [11]; cf. [11, Theorem 2.20].)

As this paper will be based on the theory of [11], we invite the reader to consult the first two sections of [11] for details. In particular, [11, Definition 2.14] provides the definition of the concept of excellence. Any reader who does not know [11] (but who knows [10]) may in what follows replace in his mind the assumption " $\emptyset$ is excellent" by "there is no inner model with a Woodin cardinal;" all Q-structures will then be of the form $J_\alpha(\mathcal{M}_b^\mathcal{T}||\delta(\mathcal{T}))$.

Let $\mathcal{H} = (H; \in, \Omega^\mathcal{H}, U^\mathcal{H})$ be a coarse premouse. We shall denote by $K^\mathcal{H}$ the true core model of height $\Omega^\mathcal{H}$ constructed inside $\mathcal{H}$. We now aim to prove that any $K^\mathcal{H}$ is sufficiently iterable.

Definition 1.2 *We define a partial function*

$$\Sigma : \mathbb{T} \to V,$$

where $\mathbb{T}$ is the class of all iteration trees of limit length on a premouse or phalanx. Let $\mathcal{T} \in \mathbb{T}$. Then we let $\Sigma(\mathcal{T})$ be the unique cofinal branch b through $\mathcal{T}$ such that $\mathcal{Q} = \mathcal{Q}(\mathcal{M}_\infty(\mathcal{T}))$ exists (cf. [11, Definitions 2.1 and 2.2]) and $\mathcal{Q} \trianglelefteq \mathcal{M}_b^\mathcal{T}$. If no such branch b exists then we let $\Sigma(\mathcal{T})$ undefined.

We have that Σ (restricted to iteration trees on $\mathcal{M}$) is a putative iteration strategy for any premouse or phalanx $\mathcal{M}$. [11, Section 2] provides important information on this strategy.

Convention. Throughout this paper we shall assume that Ω and U are such that $\mathcal{V} = (V; \in, \Omega, U)$ is a coarse premouse. We shall write K for $K^\mathcal{V}$.

The fact that $\emptyset$ is excellent implies that Σ is an Ω iteration strategy for K and therefore witnesses that K is $\Omega + 1$ iterable. By abuse of language, we sometimes refer to Σ as an iteration strategy, although it might not witness the iterability of a given premouse $\mathcal{M}$.

Definition 1.3 *Let $\mathcal{M}$ and $\mathcal{N}$ be premice. We say that $\mathcal{M}$ and $\mathcal{N}$ are coiterable via the iteration strategy Σ if there are (padded) normal iteration trees $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{U}$ on $\mathcal{N}$ of successor length $\theta + 1$ such that*
(1) the pair $E_\alpha^\mathcal{T}$, $E_\alpha^\mathcal{U}$ constitutes the least disagreement between $\mathcal{M}_\alpha^\mathcal{T}$ and $\mathcal{M}_\alpha^\mathcal{U}$ for all $\alpha < \theta$,
(2) $[0,\lambda)_\mathcal{T} = \Sigma(\mathcal{T} \restriction \lambda)$ and $[0,\lambda)_\mathcal{U} = \Sigma(\mathcal{U} \restriction \lambda)$ for all limit ordinals $\lambda \leq \theta$, and
(3) $\mathcal{M}_\theta^\mathcal{T}$ and $\mathcal{M}_\theta^\mathcal{U}$ are lined up.

Lemma 1.4 *Let $\mathcal{H} = (H; \in, \Omega^\mathcal{H}, U^\mathcal{H})$ be a coarse premouse with $\Omega^\mathcal{H} \leq \Omega$. Suppose that $\mathcal{P}(\mathbb{R}) \subset H$. Then $K^\mathcal{H}$ is coiterable with K via the iteration strategy Σ.*

PROOF. Let $\mathcal{T}$ and $\mathcal{U}$ be the (padded) putative iteration trees on K and $K^{\mathcal{H}}$, respectively, arising from the comparison of K with $K^{\mathcal{H}}$ as being induced by the iteration strategy Σ. We let θ be the least ordinal such that either θ is a limit ordinal and $\Sigma(\mathcal{U} \restriction \theta)$ is not defined, or else θ is a successor ordinal and $\mathcal{M}^{\mathcal{U}}_{\theta-1}$ is ill-founded or $\mathcal{M}^{\mathcal{T}}_{\theta-1}$ and $\mathcal{M}^{\mathcal{U}}_{\theta-1}$ are lined up. Standard arguments give that $\theta = lh(\mathcal{T}) = lh(\mathcal{U}) \leq \Omega + 1$. We have to show that θ is a successor ordinal and $\mathcal{M}^{\mathcal{T}}_{\theta-1}$ and $\mathcal{M}^{\mathcal{U}}_{\theta-1}$ are lined up.

Suppose not. Let us first assume that θ is a limit ordinal. Let $b = \Sigma(\mathcal{T})$. We may let $\mathcal{Q}$ be the shortest initial segment of $\mathcal{M}^{\mathcal{T}}_b$ such that a counterexample to $\delta(\mathcal{T})$ being Woodin is definable over $\mathcal{Q}$, i.e., $\mathcal{Q} = \mathcal{Q}(b, \mathcal{T})$. Pick $\sigma \colon \bar{V} \rightarrow V_{\Omega+\omega}$ such that $\bar{V}$ is countable and transitive and such that $\{K, K^{\mathcal{H}}, \mathcal{T}, \mathcal{U}, \mathcal{Q}\} \subset ran(\sigma)$. Notice that $\{\bar{V}, \sigma \restriction \sigma^{-1}(K^{\mathcal{H}})\} \subset \mathcal{H}$.

For every limit ordinal $\lambda < \sigma^{-1}(\theta)$ we have that $[0, \lambda)_{\sigma^{-1}(\mathcal{U})}$ is the unique cofinal branch c through $\sigma^{-1}(\mathcal{U}) \restriction \lambda$ such that $\mathcal{Q}(c, \sigma^{-1}(\mathcal{U}) \restriction \lambda)$ is $\omega_1 + 1$ iterable. This fact relativizes down to $\mathcal{H}$, as $\mathcal{P}(\mathbb{R}) \subset \mathcal{H}$. As $\Sigma^{\mathcal{H}}$ is an iteration strategy for $K^{\mathcal{H}}$ inside $\mathcal{H}$, we shall get that in $\mathcal{H}$ there is a cofinal branch b through $\sigma^{-1}(\mathcal{U})$ such that $\mathcal{Q}(b, \sigma^{-1}(\mathcal{U}))$ is $\omega_1 + 1$ iterable. [Working inside $\mathcal{H}$, we may copy $\sigma^{-1}(\mathcal{U})$ onto $K^{\mathcal{H}}$. It is straightforward to see that the copying construction goes through at limit stages (!). In the end, this will also give us the desired object b.] The fact that $\mathcal{Q}(b, \sigma^{-1}(\mathcal{U}))$ is $\omega_1 + 1$ iterable will hold in V as well, as $\mathcal{P}(\mathbb{R}) \subset \mathcal{H}$.

A standard comparison argument now shows that $\mathcal{Q}(b, \sigma^{-1}(\mathcal{U})) = \sigma^{-1}(\mathcal{Q})$. We may therefore define b inside $\bar{V}^{Col(\omega, \sigma^{-1}(\delta(\mathcal{T})))}$ as the unique cofinal branch c through $\sigma^{-1}(\mathcal{U})$ such that $\mathcal{Q}(c, \sigma^{-1}(\mathcal{U})) = \sigma^{-1}(\mathcal{Q})$. Thus $b \in \bar{V}$. Furthermore, in $\bar{V}$ we have that b is the unique cofinal branch c through $\sigma^{-1}(\mathcal{Q})$ such that $\mathcal{Q}(c, \sigma^{-1}(\mathcal{U})) = \sigma^{-1}(\mathcal{Q})$.

But now we may just apply $\sigma \colon \bar{V} \rightarrow V_{\Omega+\omega}$ to deduce that there is a unique cofinal branch d through $\mathcal{U}$ such that $\mathcal{Q}(d, \mathcal{U}) = \mathcal{Q}$. Hence $\Sigma(\mathcal{U})$ is well-defined after all. Contradiction!

In the case that θ is a successor cardinal an argument which is a simple variant of the argument just given will yield that $\mathcal{M}^{\mathcal{U}}_{\theta-1}$ is well-founded. Therefore $\mathcal{M}^{\mathcal{T}}_{\theta-1}$ and $\mathcal{M}^{\mathcal{U}}_{\theta-1}$ must be lined up.

$\square$ (1.4)

The reader will have noticed that the only use of $\mathcal{P}(\mathbb{R}) \subset \mathcal{H}$ in the proof of Lemma 1.4 was that it implies that $\omega_1 + 1$ iterability of the relevant countable Q-structures is absolute between V and $\mathcal{H}$. The requirement that $\mathcal{P}(\mathbb{R}) \subset \mathcal{H}$ might often be an overkill. If for example every premouse is ω-small then $\mathbb{R} \subset \mathcal{H}$ will be enough to get the desired absoluteness. Of course, $\mathbb{R} \subset \mathcal{H}$ follows from (4) of Definition 1.1.

Lemma 1.5 *Let $\mathcal{H} = (H; \in, \Omega^{\mathcal{H}})$ be a coarse premouse with $\Omega^{\mathcal{H}} \leq \Omega$. Suppose that every premouse is ω-small. Then $K^{\mathcal{H}}$ is coiterable with K via the iteration strategy Σ.*

Below we shall frequently use the following notation. Let $\mathcal{M}$ be a premouse, and let $\alpha \leq \mathcal{M} \cap OR$. Then $\mathcal{M}||\alpha$ denotes $\mathcal{M}$ cut off at α with top extender $E_\alpha^{\mathcal{M}}$ (if $E_\alpha^{\mathcal{M}} \neq \emptyset$; so $\mathcal{M}||\alpha$ is active if and only if $E_\alpha^{\mathcal{M}} \neq \emptyset$). We shall also confuse $\mathcal{M}||\alpha$ with its underlying universe.

The next section contains the proof of the main result of this paper. This proof builds directly on the covering argument of [5]. The reader should have some acquaintance with that paper.

2 The main result.

Theorem 2.1 *Let $\mathcal{H} = (H; \in, \Omega^{\mathcal{H}}, U^{\mathcal{H}})$ be a coarse premouse with $\Omega^{\mathcal{H}} \leq \Omega$. Suppose that $\mathcal{P}(\mathbb{R}) \subset H$. Then $K^{\mathcal{H}}$ is a normal Σ-iterate of K.*

Proof. Let $\mathcal{T}$ and $\mathcal{U}$ denote the iteration trees on K and $K^{\mathcal{H}}$, respectively, arising from the comparison of K with $K^{\mathcal{H}}$ as being induced by the iteration strategy Σ. Lemma 1.4 tells us that we'll get last models $\mathcal{M}_\infty^{\mathcal{T}}$ and $\mathcal{M}_\infty^{\mathcal{U}}$ with $\mathcal{M}_\infty^{\mathcal{T}} \trianglerighteq \mathcal{M}_\infty^{\mathcal{U}}$. We aim to show that $\mathcal{M}_\infty^{\mathcal{U}} = K^{\mathcal{H}}$.

For any ordinal ν, we say that $\mathcal{U}$ is *beyond* ν if and only if for all $\alpha + 1 < lh(\mathcal{U})$ we have that

$$E_\alpha^{\mathcal{U}} \neq \emptyset \Rightarrow lh(E_\alpha^{\mathcal{U}}) > \nu.$$

We aim to prove that $\mathcal{U}$ is beyond ν for all $\nu < \Omega^{\mathcal{H}}$. Our proof will exploit many of the key ideas of [5]; in fact, our proof will be parallel to the one of [5] to such an extent that it will be possible to directly cite some of the lemmas appearing in [5]. It is worth emphasizing that [5] nowhere really uses the assumption that there is no inner model with a Woodin cardinal; rather, it is enough for the arguments of [5] to go through that $(V; \in, \Omega, U)$ be a coarse premouse (in our sense).

Let us suppose that there is some $\nu < \Omega^{\mathcal{H}}$ such that $\mathcal{U}$ is not beyond ν. Let ζ denote the least $\nu < \Omega^{\mathcal{H}}$ such that $\mathcal{U}$ is not beyond ν. We shall eventually derive a contradiction. Let W be the canonical very soundness witness for $K||\zeta^{+V}$, and let $W^{\mathcal{H}}$ be the canonical very soundness witness for $K^{\mathcal{H}}||\zeta^{+H}$ from the point of view of $\mathcal{H}$ (cf. [10, Theorem 8.3]). Notice that we may construe iterations of W, $W^{\mathcal{H}}$ as iterations of K, $K^{\mathcal{H}}$, respectively; in particular, W and $W^{\mathcal{H}}$ are coiterable via the iteration strategy Σ by the proof of Lemma 1.4. Let $\alpha_0 + 1 < lh(\mathcal{U})$ be such that $E_{\alpha_0}^{\mathcal{U}} \neq \emptyset$ and $lh(E_{\alpha_0}^{\mathcal{U}}) = \zeta$; we may and shall from now on construe $\mathcal{T} \restriction (\alpha_0 + 1)$ and $\mathcal{U} \restriction (\alpha_0 + 1)$ as iteration trees on W and $W^{\mathcal{H}}$, respectively.

Let $(\kappa_i \colon i \leq \gamma)$ be the order preserving enumeration of the set of cardinals of $K^{\mathcal{H}}||\zeta = W^{\mathcal{H}}||\zeta$, and let

$$\lambda_i = \kappa_i^{+K^{\mathcal{H}}||\zeta}$$

for each $i \leq \gamma$. (We understand that $\lambda_\gamma = \zeta$.) Let, for $i < \gamma$, $\beta(i)$ be the least $\beta \leq \alpha_0+1$ such that $\kappa_i < \nu(E^{\mathcal{U}}_\beta)$ if there is some such β; if not, we let $\beta(i) = \alpha_0+1$. We let $\mathcal{P}_i$ be the longest initial segment $\mathcal{P}'$ of $\mathcal{M}^{\mathcal{T}}_{\beta(i)}$ such that $\mathcal{P}(\kappa_i) \cap \mathcal{P}' = \mathcal{P}(\kappa_i) \cap K^{\mathcal{H}}||\zeta$. (Then $\mathcal{P}_i$ either is a weasel with class projectum $\leq \kappa_i$, or else $\mathcal{P}_i$ is a κ_i-sound premouse with $\rho_\omega(\mathcal{P}_i) \leq \kappa_i$.) For any $i \leq \gamma$ we let $\vec{\mathcal{P}}(\lambda_i)$ denote the phalanx

$$((\mathcal{P}_j \colon j < i)^\frown W^{\mathcal{H}}, (\lambda_j \colon j < i)).$$

In the language of [5, Definition 2.4.5], $\vec{\mathcal{P}}(\lambda_i)$ is a special phalanx for every $i \leq \gamma$.

We also want to emphasize that for any $i \leq \gamma$, if $\mathcal{V}$ is a putative iteration tree on $\vec{\mathcal{P}}(\lambda_i)$ arising from the comparison with W then $\mathcal{V}$ will be a special tree in the sense of [5, Definition 2.4.6]. The only thing to notice here is that if $E^{\mathcal{V}}_\xi$ is least with $E^{\mathcal{V}}_\xi \neq \emptyset$ then $E^{\mathcal{V}}_\xi = E^{K^{\mathcal{H}}}_\zeta$.

As a matter of fact, [5, Lemma 3.15] now shows that if $\vec{\mathcal{P}}(\zeta)$ is iterable then $\mathcal{U}$ is beyond ζ after all, giving a contradiction. Actually, in order to get the desired contradiction it obviously suffices to have that $\vec{\mathcal{P}}(\zeta)$ be coiterable with W.

Before proving that $\vec{\mathcal{P}}(\zeta)$ is coiterable with W we note the following.

Claim 1. Let $i < \gamma$. If $\vec{\mathcal{P}}(\lambda_i)$ is coiterable with W via the iteration strategy Σ then $((W^{\mathcal{H}}, \mathcal{P}_i), \lambda_i)$ is iterable via Σ.

Proof. Fix $i < \gamma$. Let $\mathcal{V}$ and $\mathcal{W}$ denote the (padded) iteration trees on $\vec{\mathcal{P}}(\lambda_i)$ and W, respectively, arising from the comparison of $\vec{\mathcal{P}}(\lambda_i)$ with W as being induced by the iteration strategy Σ. By standard arguments the main branch through $\mathcal{V}$ gives us an elementary embedding $\rho_i \colon W^{\mathcal{H}} \to \tilde{W}_i$ such that $\rho_i \restriction \kappa_i = id$, $\tilde{W}_i||\lambda_i = W^{\mathcal{H}}||\lambda_i$, and $\tilde{W}_i \trianglelefteq \mathcal{M}^{\mathcal{W}}_\infty$.

We'll of course have that $\mathcal{W} \restriction \alpha_0 + 1 = \mathcal{T} \restriction \alpha_0 + 1$. It is thus clear that $\mathcal{M}^{\mathcal{T}}_{\beta(i)} = \mathcal{M}^{\mathcal{W}}_{\beta(i)}$, so that $\mathcal{P}_i \trianglelefteq \mathcal{M}^{\mathcal{W}}_{\beta(i)}$. Because the phalanx $((\mathcal{M}^{\mathcal{W}}_\infty, \mathcal{M}^{\mathcal{W}}_{\beta(i)}), \lambda_i)$ is iterable (cf. [5, Fact 3.19.1]), the phalanx $((\tilde{W}_i, \mathcal{P}_i), \lambda_i)$ is now iterable as well. We may then use the maps ρ_i and $id \restriction \mathcal{P}_i$ and argue exactly as for [5, Lemma 3.16] to show that $((W^{\mathcal{H}}, \mathcal{P}_i), \lambda_i)$ is iterable.

[We remind the informed reader that the proof of [5, Lemma 3.16] goes beyond that of [10, Theorem 8.6] in that the slightly more sophisticated copying construction from the proof of [10, Theorem 6.11] has to be built in.]

$\square$ (Claim 1)

We have argued that in order to finish the proof of 2.1 it suffices to show the following.

Claim 2. For all $i \leq \gamma$ we have that $\vec{\mathcal{P}}(\lambda_i)$ is a phalanx which is coiterable with W via the iteration strategy Σ.

PROOF. The proof is by induction on $i \leq \gamma$. Fix $i \leq \gamma$, and suppose that for all $j < i$ we have that $\vec{\mathcal{P}}(\lambda_j)$ is a phalanx which is coiterable with W via the iteration strategy Σ. Claim 1 therefore immediately gives the following.

Subclaim 1. For all $j < i$ we have that $((W^{\mathcal{H}}, \mathcal{P}_j), \lambda_j)$ is iterable.

We now write $\nu = \lambda_i$, and $\vec{\mathcal{P}} = \vec{\mathcal{P}}(\nu)$. We let $\mathcal{V}$ and $\mathcal{W}$ be the (padded) putative iteration trees on $\vec{\mathcal{P}}$ and W, respectively, arising from the comparison of $\vec{\mathcal{P}}$ with W as being induced by the iteration strategy Σ. We let θ be the least ordinal such that either θ is a limit ordinal and $\Sigma(\mathcal{V} \restriction \theta)$ is not defined, or else θ is a successor ordinal and $\mathcal{M}^{\mathcal{V}}_{\theta-1}$ is ill-founded or $\mathcal{M}^{\mathcal{V}}_{\theta-1}$ and $\mathcal{M}^{\mathcal{W}}_{\theta-1}$ are lined up. Standard arguments give that $\theta = lh(\mathcal{V}) = lh(\mathcal{W}) \leq \Omega + 1$. We have to show that θ is a successor ordinal and that $\mathcal{M}^{\mathcal{V}}_{\theta-1}$ and $\mathcal{M}^{\mathcal{W}}_{\theta-1}$ are lined up.

Let $\sigma: \bar{V} \to V_{\Omega+\omega}$ be elementary, where $\bar{V}$ is countable and transitive, and $\{\vec{\mathcal{P}}, W, \mathcal{V}, \mathcal{W}\} \subset ran(\sigma)$. If θ is a limit ordinal then we may and shall also require that $\mathcal{Q} = \mathcal{Q}(\Sigma(\mathcal{W}), \mathcal{W}) \in ran(\sigma)$. It will be crucial that

$$\{\bar{V}, \sigma \restriction \sigma^{-1}(W^{\mathcal{H}})\} \subset \mathcal{H}, \tag{2}$$

which holds true by ${}^{\omega}\mathcal{H} \subset \mathcal{H}$.

Let $j \in i \cap ran(\sigma)$. Set $\bar{\mathcal{P}}_j = \sigma^{-1}(\mathcal{P}_j)$, and let

$$\mathcal{Q}_j = Ult_n(\bar{\mathcal{P}}_j, \sigma \restriction \sigma^{-1}(K^{\mathcal{H}}||\lambda_j))$$

be the (long) ultrapower of $\bar{\mathcal{P}}_j$ by the appropriate restriction of σ, where $n \leq \omega$ is largest such that $\rho_n(\mathcal{P}_j) > \kappa_j$. Let $\bar{\sigma}_j: \bar{\mathcal{P}}_j \to \mathcal{Q}_j$ be the ultrapower map. (See [5, §2.5].) There is a canonical embedding $\sigma_j: \mathcal{Q}_j \to \mathcal{P}_j$ such that $\sigma_j \circ \bar{\sigma}_j = \sigma \restriction \bar{\mathcal{P}}_j$. Set

$$\tilde{\lambda}_j = sup\ ran(\sigma \restriction \sigma^{-1}(\lambda_j)) = sup(ran(\sigma) \cap \lambda_j) = \kappa_j^{+\mathcal{Q}_j}.$$

Unfortunately, $\mathcal{Q}_j$ will be a protomouse (rather than a premouse) if and only if $\rho_1(\mathcal{P}_j) \leq \kappa_j$ and $\mathcal{P}_j$ has a top extender with critical point $\mu < \kappa_j$ such that $cf^V(\mu^{+\mathcal{P}_j}) > \omega$ (in which case σ, and hence $\bar{\sigma}_j$, will be discontinuous at $\sigma^{-1}(\mu^{+\mathcal{P}_j})$). (See [5, Lemma 2.5.2].) In this case we shall replace $\mathcal{Q}_j$ by a premouse. (It is not hard to verify that our replacement function $\mathcal{P}_j \mapsto \mathcal{Q}_j$ is the same as the one which is defined in [5], although [5] gives its definition in a different way.)

Let still $j \in i \cap ran(\sigma)$. We aim to define $\mathcal{R}_j$.

Case 1. $\mathcal{Q}_j$ is a premouse.

In this case we just set $\mathcal{R}_j = \mathcal{Q}_j$. Notice that we may use the maps $id \restriction W^{\mathcal{H}}$ and σ_j to see that $((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ is iterable by Subclaim 1.

Case 2. $\mathcal{Q}_j$ is not a premouse.

Let μ be the critical point of the top extender of $\mathcal{P}_j$, let $\eta \geq sup(ran(\sigma) \cap \mu^{+\mathcal{P}_j})$ be least such that $\rho_\omega(\mathcal{Q}_j||\eta) \leq \mu$, let $n < \omega$ be such that $\rho_{n+1}(\mathcal{Q}_j||\eta) \leq \mu < \rho_n(\mathcal{Q}_j||\eta)$, and let F be the top extender fragment of $\mathcal{Q}_j$. Let us set

$$\bar{\mathcal{R}}_j = Ult_n(\mathcal{Q}_j||\eta, F).$$

Let G be the top extender of $\mathcal{P}_j$. We may define an $r\Sigma_0$ elementary embedding

$$\varphi_j \colon \bar{\mathcal{R}}_j \to Ult_0(\mathcal{P}_j; G)||i_G(\eta)$$

by setting

$$[a, f]_F^{\mathcal{Q}_j||\eta} \mapsto [\sigma_j(a), f]_G^{\mathcal{P}_j}.$$

Of course, $Ult_0(\mathcal{P}_j; G)||i_G(\eta) = Ult_0(W^{\mathcal{H}}; G)||i_G(\eta)$. Also, $\varphi_j \restriction \tilde{\lambda}_j = id$, because $\sigma_j \restriction \tilde{\lambda}_j = id$.

There is now a further case split. Let us denote by $(*)$ the assertion that

$$E_\eta^{\mathcal{Q}_j} = E_\eta^{K^{\mathcal{H}}} \neq \emptyset,\ \tau(E_\eta^{\mathcal{Q}_j}) = \mu, \text{ and } s(E_\eta^{\mathcal{Q}_j}) = \emptyset.$$

The object $\bar{\mathcal{R}}_j$ will be a premouse if and only if $(*)$ does not hold.

Case 2A. $(*)$ does not hold.

In this case we simply set

$$\mathcal{R}_j = \bar{\mathcal{R}}_j.$$

Using the maps $id \restriction W^{\mathcal{H}}$ and φ_j we may copy any iteration tree on the phalanx $((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ onto $((W^{\mathcal{H}}, Ult_0(W^{\mathcal{H}}; G)||i_G(\eta)), \lambda_j)$. But any iteration tree on this latter phalanx may be construed as an iteration tree on $((W^{\mathcal{H}}, \mathcal{P}_j), \lambda_j)$. Subclaim 1 therefore implies that $((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ is iterable.

Case 2B. $(*)$ does hold.

Let us write H for the top extender of $\bar{\mathcal{R}}_j$, which is the "stretch" of $E_\eta^{\mathcal{Q}_j} = E_\eta^{K^{\mathcal{H}}}$ by F. In this case H does not satisfy the initial segment condition, and $\bar{\mathcal{R}}_j$ is thus not a premouse. We hence have to define $\mathcal{R}_j$ differently.

We set

$$\mathcal{R}_j = Ult_0(K^{\mathcal{H}}; H).$$

Let $\tilde{H}$ denote the top extender of $Ult_0(\mathcal{P}_j; G)||i_G(\eta)$. As φ_j is $r\Sigma_0$ elementary, we may define a fully elementary map

$$\tilde{\varphi}_j \colon \mathcal{R}_j \to Ult_0(K^{\mathcal{H}}; \tilde{H})$$

by setting

$$[a, f]_H^{K^{\mathcal{H}}} \mapsto [\varphi_j(a), f]_{\tilde{H}}^{K^{\mathcal{H}}}.$$

We'll have $\tilde{\varphi}_j \restriction \tilde{\lambda}_j = \varphi_j \restriction \tilde{\lambda}_j = id$. Using the maps $id \restriction W^{\mathcal{H}}$ and $\tilde{\varphi}_j$ we may copy any iteration tree on the phalanx $((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ onto $((W^{\mathcal{H}}, Ult_0(K^{\mathcal{H}}; \tilde{H})), \lambda_j)$. But any iteration tree on this latter phalanx may be construed as an iteration tree on $((W^{\mathcal{H}}, \mathcal{P}_j), \lambda_j)$. Subclaim 1 therefore implies that $((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ is iterable.

This finishes the definition of $\mathcal{R}_j$. Notice that we have proved the following.

Subclaim 2. $((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ is iterable whenever $j \in i \cap ran(\sigma)$.

Now because of (2) we have that $\{(\bar{\mathcal{P}}_j \colon j \in i \cap ran(\sigma)), \sigma \restriction \sigma^{-1}(W^{\mathcal{H}})\} \subset \mathcal{H}$, and we easily get that $(\mathcal{Q}_j \colon j \in i \cap ran(\sigma)) \in \mathcal{H}$, and thus that $(\mathcal{R}_j \colon j \in i \cap ran(\sigma)) \in \mathcal{H}$. (Of course, $\mathcal{H}$ does not see all of σ, but the set $i \cap ran(\sigma)$ does exist in $\mathcal{H}$.) An argument exactly as for Lemma 1.4 hence shows that Subclaim 2 implies the following.

Subclaim 3. $\mathcal{H} \models$ "$((W^{\mathcal{H}}, \mathcal{R}_j), \tilde{\lambda}_j)$ is coiterable with $W^{\mathcal{H}}$ via the iteration strategy Σ," whenever $j \in i \cap ran(\sigma)$.

Let $\vec{\mathcal{R}}$ denote the phalanx

$$((\mathcal{R}_j \colon j \in i \cap ran(\sigma))^\frown W^{\mathcal{H}}, (\tilde{\lambda}_j \colon j \in i \cap ran(\sigma))).$$

Subclaim 4. $\mathcal{H} \models$ "$\vec{\mathcal{R}}$ is iterable with respect to special iteration trees."

PROOF. Let us work inside $\mathcal{H}$. Let $j \in i \cap ran(\sigma)$. By Subclaim 3, standard arguments yield an iterate $\tilde{W}_j$ of $W^{\mathcal{H}}$ together with an elementary embedding $\tau_j \colon \mathcal{R}_j \to \tilde{W}_j$ such that $\tau_j \restriction \tilde{\lambda}_j = id$.

Let $\vec{W}$ denote the phalanx

$$((\tilde{W}_j : j \in i \cap ran(\sigma))^\frown W^{\mathcal{H}}, (\tilde{\lambda}_j : j \in i \cap ran(\sigma))).$$

We know that $\vec{W}$ is iterable with respect to special iteration trees by [5, Fact 3.19.1] (applied inside $\mathcal{H}$). But this readily implies Subclaim 4, due to the existence of the sequence of maps $(\tau_j : j \in i \cap ran(\sigma))^\frown id \upharpoonright W^{\mathcal{H}}$.

$\square$ (Subclaim 4)

Let $\vec{\mathcal{Q}}$ denote the phalanx

$$((\mathcal{Q}_j : j \in i \cap ran(\sigma))^\frown W^{\mathcal{H}}, (\tilde{\lambda}_j : j \in i \cap ran(\sigma))).$$

By the proof of [5, Lemma 3.18] (run inside $\mathcal{H}$) we see that Subclaim 4 implies the following.

Subclaim 5. $\mathcal{H} \models$ "$\vec{\mathcal{Q}}$ is iterable with respect to special iteration trees."

We may now finish the proof of Claim 2 by an argument as for Lemma 1.4. Let us work inside $\mathcal{H}$ until further notice. Recall that for each $j \in i \cap ran(\sigma)$ we have an embedding

$$\bar{\sigma}_j : \bar{\mathcal{P}}_j \to \mathcal{Q}_j,$$

and we have

$$\sigma \upharpoonright \sigma^{-1}(W^{\mathcal{H}}) : \sigma^{-1}(W^{\mathcal{H}}) \to W^{\mathcal{H}}.$$

We have that $\bar{\sigma}_j \upharpoonright \sigma^{-1}(\lambda_k) = \bar{\sigma}_k \upharpoonright \sigma^{-1}(\lambda_k) = \sigma \upharpoonright \sigma^{-1}(\lambda_k)$ for $k \leq j \in i \cap ran(\sigma)$, so that we may now copy $\sigma^{-1}(\mathcal{V})$ (being a tree on $\sigma^{-1}(\vec{\mathcal{P}})$) onto $\vec{\mathcal{Q}}$ getting a special iteration tree on $\vec{\mathcal{Q}}$.

Let us now first assume that θ is a limit ordinal. An argument exactly as in the proof of Lemma 1.4 will then give us a unique cofinal branch $b \in \bar{V}$ such that from the point of view of $\bar{V}$, $\mathcal{Q}(b, \sigma^{-1}(\mathcal{V})) = \sigma^{-1}(\mathcal{Q})$. (Recall that $\mathcal{Q} = \mathcal{Q}(\Sigma(\mathcal{W}), \mathcal{W})$.) By finally stepping outside of $\mathcal{H}$ and applying $\sigma : \bar{V} \to V_{\Omega+\omega}$ we may hence conclude that (in V) there is a unique cofinal branch c through $\mathcal{V}$ such that $\mathcal{Q}(c, \mathcal{V}) = \mathcal{Q}$. Hence $\Sigma(\mathcal{V})$ is well-defined after all.

Hence θ cannot be a limit ordinal. A variant of the argument just given then shows that $\mathcal{M}^{\mathcal{V}}_{\theta-1}$ must be well-founded. Therefore, $\mathcal{M}^{\mathcal{V}}_{\theta-1}$ and $\mathcal{M}^{\mathcal{W}}_{\theta-1}$ are lined up. This is what we were trying to show.

$\square$ (Claim 2)
$\square$ (2.1)

The same proof (with the use of Lemma 1.4 replaced by the use of Lemma 1.5) yields the following result.

Theorem 2.2 *Let $\mathcal{H} = (H; \in, \Omega^{\mathcal{H}})$ be a coarse premouse with $\Omega^{\mathcal{H}} \leq \Omega$. Suppose that every premouse is ω-small. Then $K^{\mathcal{H}}$ is a normal Σ-iterate of K.*

3 An application.

Corollary 3.1 *Suppose that the measure U witnesses that Ω is a measurable cardinal, every premouse is tame, and $\emptyset$ is excellent. Let $\pi : V \to M$ be an elementary embedding coming from a finite coarse iteration tree on V living on V_Ω (i.e., all extenders used are elements of V_Ω and images thereof) such that M is transitive and ${}^\omega M \subset M$. Let $K^{\mathcal{H}} = \pi(K)$ be the core model of $\mathcal{H}$, where $K = K^{(V; \in, \Omega, U)}$ and $\mathcal{H} = (M; \in, \Omega, \pi(U))$. Then $K^{\mathcal{H}}$ is a normal iterate of K, i.e., there is a normal iteration tree $\mathcal{T}$ on K of successor length $\leq \Omega + 1$ such that $\mathcal{M}^{\mathcal{T}}_\infty = K^{\mathcal{H}}$. Moreover, we'll have that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.*

Proof. This readily follows from Theorem 2.1 by the argument in the proof of [10, Lemma 7.13]. The existence of $\mathcal{T}$ is given by Theorem 2.1. It remains to be shown that $\pi^{\mathcal{T}}_{0\infty} = \pi \restriction K$.

Let W be a (linear) iterate of K such that Ω is thick in W and there is some $\sigma : K \to W$ with $\mathrm{Def}(W) = \sigma$"K (cf. [10, Lemmas 8.2 and 8.3]). For each $x \in K$, $\pi(\sigma(x)) = \pi(\sigma)(\pi(x))$, and hence $\pi \circ \sigma = \pi(\sigma) \circ \pi$. Let $i : W \to Q$ and $j : \pi(W) \to Q$ arise from the comparison of W with $\pi(W)$. Then for each term τ and for each vector $\vec{s}$ of fixed points under both i and j, $i(\tau^W[\vec{s}]) = \tau^Q[\vec{s}] = j \circ \pi(\tau^W[\vec{s}])$. Putting these things together buys us that

$$i \circ \sigma = j \circ \pi(\sigma) \circ \pi. \tag{3}$$

Now suppose that $\pi^{\mathcal{T}}_{0\infty}(\xi) \neq \pi(\xi)$ for some $\xi < \Omega$. (3) would then imply that

$$i \circ \sigma(\xi) \neq j \circ \pi(\sigma) \circ \pi^{\mathcal{T}}_{0\infty}(\xi). \tag{4}$$

However, both $i \circ \sigma$ and $j \circ \pi(\sigma) \circ \pi^{\mathcal{T}}_{0\infty}$ arise from iterations of K. (4) therefore contradicts the Dodd-Jensen Lemma (cf. [6, Lemma 5.3]).

$\square$ (3.1)

If $\pi : V \to M$ is an elementary embedding coming from an *infinite* coarse iteration tree on V then M cannot be closed under countable sequences. We do not know if Corollary 3.1 still holds true if in its statement the word "finite" is crossed out and the hypothesis that ${}^\omega M \subset M$ is dropped. E. Schimmerling pointed out that Corollary 3.1 still holds true if in its statement the word "finite" is replaced by "countable," the hypothesis that ${}^\omega M \subset M$ is dropped, but it is further assumed that the tree on V be ρ-maximal in the sense of Neeman's paper [7].

Corollary 3.2 (Schimmerling) *Suppose that the measure U witnesses that Ω is a measurable cardinal, every premouse is tame, and $\emptyset$ is excellent. Let $\pi : V \to M$ be an elementary embedding coming from a countable coarse iteration tree U on V living on V_Ω (i.e., all extenders used are elements of V_Ω and images thereof) such that M is transitive. Suppose that U is ρ-maximal. Let $K^{\mathcal{H}} = \pi(K)$ be the core model of $\mathcal{H}$, where $K = K^{(V;\in,\Omega,U)}$ and $\mathcal{H} = (M; \in, \Omega, \pi(U))$. Then $K^{\mathcal{H}}$ is a normal iterate of K, i.e., there is a normal iteration tree $\mathcal{T}$ on K of successor length $\leq \Omega+1$ such that $\mathcal{M}_\infty^{\mathcal{T}} = K^{\mathcal{H}}$. Moreover, we'll have that $\pi_{0\infty}^{\mathcal{T}} = \pi \restriction K$.*

PROOF SKETCH. We have to run the proof of Theorem 2.1. The assumption that ${}^\omega H \subset H$ was used there just in order to get (2), i.e., that $\{\bar{V}, \sigma \restriction \sigma^{-1}(W^{\mathcal{H}})\} \subset H$ (we borrow the notation from the proof of Theorem 2.1). In the situation of the statement of Corollary 3.2, it might be false that ${}^\omega M \subset M$. However, it'll certainly still be true that $\bar{V} \in M$ (as $\mathbb{R} \subset M$). The point is now that due to the fact that U is countable and ρ-maximal it'll also be true that the map $\sigma \restriction \sigma^{-1}(W^{\mathcal{H}})$ is *generic* over M (this is shown in [7]). This suffices for finishing off the argument which shows that $\mathcal{U}$ is trivial (again, we borrow the notation from the proof of Theorem 2.1). The "Moreover" part is shown as in the proof of Corollary 3.1.

$\square$ (3.2)

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