Proper forcing and remarkable cardinals

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The present paper investigates the power of proper forcings to change the shape of the universe, in a certain well-defined respect. It turns out that the ranking among large cardinals can be used as a measure for that power. However, in order to establish the final result I had to isolate a new large cardinal concept, which I dubbed "remarkability." Let us approach the exact formulation of the problem – and of its solution – at a slow pace.

Breathtaking developments in the mid 1980's found one of its culminations in the theorem, due to Martin, Steel, and Woodin, that the existence of infinitely many Woodin cardinals with a measurable cardinal above them all implies that AD, the axiom of determinacy, holds in the least inner model containing all the reals, $L(\mathbb{R})$ (cf. [6] p. 91). One of the nice things about AD is that the theory $ZF + AD + V = L(\mathbb{R})$ appears as a choiceless "completion" of ZF in that any interesting question (in particular, about sets of reals) seems to find an at least attractive answer in that theory (cf. for example [5] Chap. 6). (Compare with $ZF + V = L$!) Beyond that, AD is very canonical as may be illustrated as follows.

Let us say that $L(\mathbb{R})$ is absolute for set-sized forcings if for all posets $P \in V$, for all formulae ϕ, and for all $\vec{x} \in \mathbb{R}$ do we have that

$$L(\mathbb{R}) \models \phi(\vec{x}) \;\Leftrightarrow\; P \Vdash \text{“}L(\dot{\mathbb{R}}) \models \phi(\check{\vec{x}}),\text{”} \tag{1}$$

where $\dot{\mathbb{R}}$ is a name for the set of reals in the extension. The existence of a proper class of Woodin cardinals, say, implies that $L(\mathbb{R})$ *is* absolute for set-sized forcings (by Woodin, cf. [12] Cor. 4.6). Now AD is canonical in the sense that $L(\mathbb{R}) \models AD$ is actually provable from the assumption that $L(\mathbb{R})$ is absolute for set-sized forcings (together with an extra technical hypothesis; this is due to Steel and Woodin independently, cf. [13]).

By looking at things a bit closerly one arrives at a circle of implications, *cum grano salis*. Let A_∞ denote the assumption that there is an inner model M together with a countable ordinal δ such that

$$M = L(V_\delta^M) \models \text{“}\delta \text{ is the sup of } \omega \text{ many Woodin cardinals,”}$$

and M has a unique iteration strategy for iteration trees of length $< \infty$ (I have borrowed the notation from [7]). Then A_∞ implies that $L(\mathbb{R})$ is absolute for set-sized forcings; if there exists a measurable cardinal and $L(\mathbb{R})$ is absolute for set-sized forcings then $L(\mathbb{R}) \models AD$; and if $L(\mathbb{R}) \models AD$ then we don't quite get A_∞, but a theorem of Woodin gives us an inner model with infinitely many Woodin cardinals (cf. [14] part III). Hence A_∞ as well as $L(\mathbb{R})$ absoluteness for set-sized forcings are both very tightly connected to the consistency strength of AD.

A harder question, which goes beyond $L(\mathbb{R})$ absoluteness for set-sized forcings, asks whether $L(\mathbb{R})$ can provide a counterexample to the continuum hypothesis, i.e., whether $\theta^{L(\mathbb{R})}$ can be larger than ω_2, in the presence of large cardinals. (Recall that $\theta^{L(\mathbb{R})} = sup\{\alpha \mid \exists f \in L(\mathbb{R})\ f\colon \mathbb{R} \to \alpha \text{ onto }\}$.) In 1991, Woodin settled this in the negative by showing that, for example, Martin's Maximum yields that $\boldsymbol{\delta}^1_2 = \omega_2$ (cf. [15] Thm. 1.2).

However, Foreman and Magidor had shown prior to Woodin's result that (granted the existence of certain large cardinals) proper forcing can*not* change the value of $\theta^{L(\mathbb{R})}$ (cf. [3]). Shelah had introduced proper forcing (cf. [11]) as a fruitful unifying concept for which he could prove a very useful iteration lemma. Recall that a poset $P \in V$ is called proper if for any $\alpha \geq \omega_1$ and for every stationary $S \subset [\alpha]^\omega$ do we have that

$$P \Vdash \text{“}\check{S} \text{ is stationary.”}$$

Having heard about [3], Neeman and Zapletal found a strong generalization of the Foreman-Magidor result. They can show that A_∞ implies that $L(\mathbb{R})$ is absolute for proper forcings in the stronger sense that for all proper posets $P \in V$, for all formulae ϕ, for all $\vec{x} \in \mathbb{R}$, and for all $\vec{\alpha} \in OR$ do we have that

$$L(\mathbb{R}) \models \phi(\vec{x}, \vec{\alpha}) \ \Leftrightarrow \ P \Vdash \text{“}L(\dot{\mathbb{R}}) \models \phi(\check{\vec{x}}, \check{\vec{\alpha}}) \tag{2}$$

(cf. [7] and [8]; Woodin pointed out later that this stronger form of $L(\mathbb{R})$ absoluteness for proper forcings can smoothly be derived from [3] when combined with a theorem of his according to which $L(\mathbb{R})$ is a symmetric extension of its HOD.) Woodin's above-mentioned result of 1991 in fact shows that “proper” can't be replaced by “semi-proper” here.

Is the conclusion of the Neeman-Zapletal theorem also tightly connected to the strength of AD? (After all, their use of A_∞ as a hypothesis seems very natural!) This question (and certain variations of it) has turned out to be an interesting one, as it leads towards the theory of coding into L (and, more generally, into core models). My main theorem, 1.3 below, will provide a straight answer to this question.

Let $\mathcal{P} \subset V$ be a class of posets. Let us say that $L(\mathbb{R})$ is absolute for $\mathcal{P}$-forcings if for all posets $P \in \mathcal{P}$, for all formulae ϕ, and for all $\vec{x} \in \mathbb{R}$ do we have that (1) above holds. We say that $L(\mathbb{R})$ is "boldface" absolute for $\mathcal{P}$-forcings if for all posets $P \in \mathcal{P}$, for all formulae ϕ, for all $\vec{x} \in \mathbb{R}$, and for all $\vec{\alpha} \in OR$ do we have that (2) above holds. (Notice that "boldface" $L(\mathbb{R})$ absoluteness for $\mathcal{P}$-forcings can only hold if no $P \in \mathcal{P}$ collapses ω_1.)

The first main theorem of [7] even says that A_∞ implies $L(\mathbb{R})$ is "boldface" absolute for reasonable forcings. The class of reasonable forcings, introduced by Foreman and Magidor in [3], extends the class of proper ones. For $P \in V$ to be reasonable it is only required that for any $\alpha \geq \omega_1$ and for $S = [\alpha]^\omega$ do we have that

$$P \Vdash \text{"}\check{S} \text{ is stationary."}$$

On the other hand, in [9] I showed the following. Recall that a cardinal κ is called strong if for all regular cardinals $\theta \in OR$ there is some elementary embedding $\pi: V \rightarrow M$ with M transitive and critical point κ such that $H_\theta \subset M$. (Here, H_θ is the set of all sets which are hereditarily smaller than θ.)

Theorem 1.1 *If $L(\mathbb{R})$ is absolute for reasonable forcings then for every real x there is an inner model with a strong cardinal containing x.*

As of today, the conclusion of 1.1 gives the best known lower bound for the strength of its assumption. Notice however that this conclusion implies that $\mathbb{R}$ is closed under $\sharp$'s which by a theorem of Martin gives $\mathbf{\Pi}^1_1$ determinacy; i.e., we immediately get:

Corollary 1.2 *If $L(\mathbb{R})$ is absolute for reasonable forcings then $\mathbf{\Pi}^1_1$ determinacy holds.*

It is conjectured in [9] that $L(\mathbb{R})$ absoluteness for reasonable forcings implies (at least) projective determinacy. On the other side, Kunen (unpublished) showed that $L(\mathbb{R})$ absoluteness for c.c.c. forcings is equiconsistent with a weakly compact cardinal. In particular, $L(\mathbb{R})$ absoluteness for c.c.c. forcings does *not* yield $\mathbf{\Pi}^1_1$ determinacy.

It is open whether we may replace "reasonable" by "semi-proper" in the statements of 1.1 and 1.2. However, as pointed out in [9], "stationary preserving" *can* replace "reasonable" in the statements of 1.1 and 1.2. Recall that a poset $P \in V$ is called stationary preserving if for all stationary $S \subset \omega_1$ do we have that

$$P \Vdash \text{"}\check{S} \text{ is stationary."}$$

It might thus seem mildly plausible that working a bit harder should give that "reasonable" can also be replaced by "proper" in the statement of 1.2. Specifically, does ("boldface") $L(\mathbb{R})$ absoluteness for proper forcings imply $\mathbf{\Pi}^1_1$ determinacy? This is the problem referred to in the first paragraph of this paper.

The theorem which I shall now announce will provide a completely satisfactory answer to this question (cf. 1.5 below).

Theorem 1.3 *The following theories are equiconsistent.*

(a) ZFC+ $L(\mathbb{R})$ absoluteness holds for proper forcings,

(b) ZFC+ "boldface" $L(\mathbb{R})$ absoluteness holds for proper forcings,

(c) $ZFC + V \neq L(\mathbb{R})+$ for all sets $B \subset OR$ and for all proper posets $P \in V$ do we have that

$$B \in L(\mathbb{R}) \;\Leftrightarrow\; P \Vdash \text{“}\check{B} \in L(\dot{\mathbb{R}}),\text{”}$$

and (d) ZFC+ there is a remarkable cardinal.

Here, item (c) states an anti-coding property, and it is the conclusion of the second main theorem of [7]; (c) is of independent interest, but as a matter of fact we shall have to verify Con(c) $\Rightarrow$ Con(d) when proving Con(a) $\Rightarrow$ Con(d). Item (d) mentions the new large cardinal concept which I am advertizing here. We now finally want to see the official definition.

Definition 1.4 *A cardinal κ is called* remarkable *iff for all regular cardinals $\theta > \kappa$ there are π, M, $\bar{\kappa}$, σ, N, and $\bar{\theta}$ such that the following hold:*

- *$\pi\colon M \to H_\theta$ is an elementary embedding,*
- *M is countable and transitive,*
- *$\pi(\bar{\kappa}) = \kappa$,*
- *$\sigma\colon M \to N$ is an elementary embedding with critical point $\bar{\kappa}$,*
- *N is countable and transitive,*
- *$\bar{\theta} = M \cap OR$ is a regular cardinal in N, $\sigma(\bar{\kappa}) > \bar{\theta}$, and*
- *$M = H^N_{\bar{\theta}}$, i.e., $M \in N$ and $N \models$ "M is the set of all sets which are hereditarily smaller than $\bar{\theta}$."*

We may view that cardinal whose existence is equiconsistent with $L(\mathbb{R})$ absoluteness for $\mathcal{P}$-forcings as a measure for the power of $\mathcal{P}$-forcings to change the $L(\mathbb{R})$-part of the universe. With 1.3, we have arrived at the following picture.

$\mathcal{P}$	$\mathcal{P}$'s power
c.c.c.	weakly compact cardinal
proper	remarkable cardinal
semi-proper	?
reasonable, stationary preserving	$\geq$ strong cardinal
all forcings	$\approx A_\infty$

It is easy to verify that every remarkable cardinal has to be totally indescribable. It is also not hard to see that every remarkable cardinal κ is n-ineffable for every $n < \omega$. (Recall that κ is called n-ineffable if every $f\colon [\kappa]^{n+1} \to 2$ admits an f-homogeneous set $X \subset \kappa$ which is stationary in κ.)

On the other hand, let κ be strong. If $\theta > \kappa$ is a regular cardinal then we may pick $\tilde{\pi}\colon V \to M$ with M transitive and critical point κ such that $H_\theta \subset M$ (i.e., $H_\theta^M = H_\theta$). Let $\pi\colon \bar{H} \to H_\rho$, where $\rho >> \tilde{\pi}(\theta)$ is regular, be such that $\bar{H}$ is countable and transitive, and $\{\kappa,\, H_\theta,\, H^M_{\tilde{\pi}(\theta)},\, \tilde{\pi} \restriction H_\theta\} \subset ran(\pi)$. Then $\pi \restriction \pi^{-1}(H_\theta)$, $\pi^{-1}(H_\theta)$, $\pi^{-1}(\kappa)$, $\pi^{-1}(\tilde{\pi} \restriction H_\theta)$, $\pi^{-1}(H^M_{\tilde{\pi}(\theta)})$ and $\pi^{-1}(\theta)$ witness that κ is remarkable w.r.t. θ. Thus strong cardinals are remarkable; in fact, remarkable cardinals are, loosely speaking, a poor man's version of strong cardinals.

However, consistency-wise are remarkable cardinals much weaker than strong cardinals. Every Silver indiscernible can be verified to be remarkable in L. Actually, $ZFC+$ "there is an ω-Erdös cardinal" proves the consistency of $ZFC+$ "there is a remarkable cardinal". Moreover, if κ is remarkable then $L \models$ "κ is remarkable". 1.3 hence immediately gives:

Corollary 1.5 *$L(\mathbb{R})$ absoluteness for proper forcings does not imply $\mathbf{\Pi}^1_1$ determinacy.*

The following picture illustrates where the assertion that there is a remarkable cardinal sits in the consistency strength hierarchy on various weak large cardinal hypotheses (">" means "is consistency-wise stronger"). Notice that the existence of any of the cardinals listed is compatible with $V = L$.

$$... > \kappa \to (\omega)^{<\omega} \;>\; \text{remarkable} \;>\; n\text{−ineffable} \;>$$

$$\text{subtle} \;>\; \text{unfoldable} \;>\; \text{totally indescribable} \;>\; ...$$

The following lemma will give a very useful characterization of remarkability.

Definition 1.6 *Let κ be a cardinal. Let G be $Col(\omega, < \kappa)$-generic over V, let $\theta > \kappa$ be a regular cardinal, and let $X \in [H_\theta^{V[G]}]^\omega$. We say that X* condenses remarkably *if $X = ran(\pi)$ for some elementary*

$$\pi\colon (H_\beta^{V[G\cap H_\alpha^V]}; \in, H_\beta^V, G \cap H_\alpha^V) \to (H_\theta^{V[G]}; \in, H_\theta^V, G)$$

where $\alpha = crit(\pi) < \beta < \kappa$ and β is a regular cardinal (in V).

Lemma 1.7 *A cardinal κ is remarkable if and only if for all regular cardinals $\theta > \kappa$ do we have that*

$$\Vdash^V_{Col(\omega,<\kappa)} \text{“}\{X \in [H_{\check\theta}]^{\check\omega} \colon X \text{ condenses remarkably}\} \text{ is stationary,”}$$

where $H_{\check\theta}$ is meant as describing the set of all sets which are hereditarily smaller than θ in the extension.

We should at least indicate how to prove the “only if” direction of 1.7. Let π, M, $\bar\kappa$, σ, N, and $\bar\theta$ be as in the statement of 1.4. In V we may pick g and g' such that $g' \supset g$, g is $Col(\omega, < \bar\kappa)$-generic over M, g' is $Col(\omega, < \sigma(\bar\kappa))$-generic over N, and σ extends to $\tilde\sigma \colon M[g] \to N[g']$. If $\mathfrak{M} = (H^{M[g]}_\vartheta; \in, H^M_\vartheta, g, ...) \in M[g]$ is a model of finite type with $\bar\kappa < \vartheta < \bar\theta$ and ϑ regular in M, then $\tilde\sigma \restriction H^{M[g]}_\vartheta \in V$ witnesses that $ran(\tilde\sigma \restriction H^{M[g]}_\vartheta)$ condenses remarkably and is the universe of a submodel of $\tilde\sigma(\mathfrak{M})$. By absoluteness (notice $M[g]$ is countable in $N[g']$!), there is some $\sigma' \in N[g']$ such that $N[g']$ knows that $ran(\sigma')$ condenses remarkably and is the universe of a submodel of $\tilde\sigma(\mathfrak{M})$. By pulling this statement back via $\tilde\sigma$ we conclude that in $M[g]$,

$$\{X \in [H^{M[g]}_\vartheta]^\omega \,:\, X \text{ condenses remarkably } \} \text{ is stationary.}$$

Lifting up via π yields the displayed assertion in 1.7 (with θ replaced by ϑ).

Now that we are familiar with remarkable cardinals, let us turn towards sketching a proof of 1.3. We shall restrict ourselves to indicating the implications $\textsc{Con}$(d) $\Rightarrow$ $\textsc{Con}$(b), $\textsc{Con}$(c) $\Rightarrow$ $\textsc{Con}$(d), and $\textsc{Con}$(b) $\Rightarrow$ $\textsc{Con}$(d). A full proof of 1.3 can be found in [10].

We commence with $\textsc{Con}$(d) $\Rightarrow$ $\textsc{Con}$(b). Let κ be a remarkable cardinal in V, and let G be $Col(\omega, < \kappa)$-generic over V. We claim that $V[G]$ is a model of “boldface” $L(\mathbb{R})$ absoluteness for proper forcings.

The key observation here is the following. Suppose that Q is a proper poset in $V[G]$, H is Q-generic over $V[G]$, and $x \in \mathbb{R} \cap V[G][H]$. Then there are a poset $Q_x \in V_\kappa$ (where $V_\kappa = V^V_\kappa$!) and some $H_x \in V[G][H]$ being Q_x-generic over V such that $x \in V[H_x]$. For consider the structure $\mathfrak{M} = (H^{V[G]}_\theta; \in, H^V_\theta, Col(\omega, < \kappa), G, Q, \dot x, H)$, where $\theta >> \kappa$ is regular with $\mathcal{P}(Q) \cap V[G] \subset H^{V[G]}_\theta$, and $\dot x^H = x$. Using 1.7 as well as the fact that Q is proper we may pick some elementary embedding

$$\pi \colon (H^{V[G \cap H^V_\alpha]}_\beta; \in, H^V_\beta, Col(\omega, < \alpha), G \cap H^V_\alpha, \bar Q, \bar{\dot x}, \bar H) \to \mathfrak{M}$$

where $\alpha = crit(\pi) < \beta < \kappa$, and β is a regular cardinal in V. It is easy to see that $x = \bar{\dot x}^{\bar H}$, where $\bar H$ is $\bar Q$-generic over $H^{V[G \cap H^V_\alpha]}_\beta$ (and $G \cap H^V_\alpha$ is $Col(\omega, < \alpha)$-generic over H^V_β). However, we have that $\mathcal{P}(\bar{Q}) \cap V[G \cap H^V_\alpha] \subset H^{V[G \cap H^V_\alpha]}_\beta$, so that in fact $\bar{H}$ is $\bar{Q}$-generic over all of $V[G \cap H^V_\alpha]$ (and $G \cap H^V_\alpha$ is $Col(\omega, < \alpha)$-generic over V). We may hence set $Q_x = Col(\omega, < \alpha) \star \bar{Q}$, and $H_x = (G \cap H^V_\alpha, \bar{H})$.

We may express this fact by saying that all the reals of further proper set-generic extensions are "small generic" over V. This enables us to get a "normal form," i.e., we can now make the set of such reals the reals of some Levy collapse of κ (cf. [1] p. 1385). Namely, let E be $Col(\omega, (2^{\aleph_0})^{V[G][H]})$-generic over $V[G][H]$, and let $(e_i \colon i < \omega)$ be an enumeration of $\mathbb{R} \cap V[G][H]$ inside $V[G][H][E]$. We can easily define a sequence $(\alpha_i, G_i \colon i < \omega)$ such that for all $i < \omega$ do we have that

- $\alpha_i < \alpha_{i+1} < \kappa$,
- $G_i \subset G_{i+1}$,
- $G_i \in V[G][H]$ is $Col(\omega, < \alpha_i)$-generic over V, and
- $e_i \in V[G_i]$.

As $Col(\omega, < \kappa)$ has the κ-c.c., it follows that $G' = \bigcup_{i<\omega} G_i$ is $Col(\omega, < \kappa)$-generic over V, and $\mathbb{R} \cap V[G'] \subset \bigcup_{i<\omega} V[G_i] \subset V[G][H]$, and therefore $\mathbb{R} \cap V[G'] = \mathbb{R} \cap V[G][H]$.

We may then use the homogeneity of the Levy collapse to get that, for any formula ϕ, for all $\vec{x} \in \mathbb{R} \cap V[G]$, and for all $\vec{\alpha} \in OR$ do we have that

$$L(\mathbb{R}^{V[G]}) \models \phi(\vec{x}, \vec{\alpha}) \Leftrightarrow$$

$$\Vdash^{V[\vec{x}]}_{Col(\omega, <\kappa)} \text{“}L(\dot{\mathbb{R}}) \models \phi(\check{\vec{x}}, \check{\vec{\alpha}})\text{”} \Leftrightarrow$$

$$L(\mathbb{R}^{V[G']}) \models \phi(\vec{x}, \vec{\alpha}).$$

This proves that $V[G]$ is a model of "boldface" $L(\mathbb{R})$ absoluteness for proper forcings.

The method used here can be exploited for getting the following.

Theorem 1.8 (Derived model theorem) *Assume that every real is set-generic over L, and that $L(\mathbb{R})$ is absolute for proper forcings. Then (in some set-generic extension of V) there is G being $Col(\omega, < \omega_1^V)$-generic over L such that $L(\mathbb{R}^V) = L(\mathbb{R}^{L[G]})$.*

Let us now turn towards Con(c) $\Rightarrow$ Con(d). In order to establish this implication we shall prove that unless ω_1^V is remarkable in L we can "code" any $A \subset OR$ "into" $L(\mathbb{R})$ in the sense that A ends up as an element of the $L(\mathbb{R})$ of some proper set-generic extension. We aim to discuss this implication from a somewhat abstract point of view. Let us call a pair $(\vec{a}, B)$ *robust* if

- $\vec{a} = (a_i \colon i < \omega_1)$ is an ω_1 sequence of (pairwise) almost disjoint subsets of ω,
- $B \subset \omega_1$, and
- there is some $F \colon \bigcup_{\alpha<\omega_1} \mathcal{P}(\alpha) \to \mathbb{R}$ with $P \Vdash$ "$\check{F} \in L(\dot{\mathbb{R}})$" for all posets P being c.c.c., and such that $a_i = F(B \cap i)$ for all $i < \omega_1$.

Let $(\vec{a}, B)$ be robust. By almost disjoint forcing (cf. [4] p. 85 ff.) we may add $a \subset \omega$ to V such that for all $i < \omega_1$ do we have that $i \in B \Leftrightarrow a \cap a_i$ is finite. Said forcing has the c.c.c., so that an easy recursion can be run inside $L(\mathbb{R}^{V[a]})$ to prove that $(\vec{a}, B) \in L(\mathbb{R}^{V[a]})$.

Suppose now we want to "code" some given $A \subset OR$ "into" $L(\mathbb{R})$ in the above sense. We may assume w.l.o.g. that $A \subset \omega_1$ by first forcing with some $Col(\omega_1, \lambda)$, $\lambda > \omega_1$, if necessary. As a matter of fact, we should search for a robust $(\vec{a}, B)$ in some proper set-generic extension where, say, $A = B_{even} = \{i \colon 2i \in B\}$. The function F witnessing $(\vec{a}, B)$ is robust will be obtained from inner model theory; specifically, we plan on setting $F(B \cap i) =$ the $<_{L[B \cap i]}$-least $b \subset \omega$ which is almost disjoint from any element of $\{a_j \colon j < i\}$, if it exists, and $F(B \cap i) = \emptyset$ else. Here, $<_{L[B \cap i]}$ denotes the canonical well-ordering of $L[B \cap i]$.

In order for this to work we just need that $L[B \cap i] \models$ "i is countable," for every $i < \omega_1$. Such a $B \subset \omega_1$ is called *reshaped*. It is consistent even with $\neg 0^\sharp$ that V doesn't contain a reshaped subset of ω_1. (Example: $V = L^{Col(\omega, <\kappa)}$ where κ is weakly compact in L.) However, the point is that we can force a reshaped $B \subset \omega_1$ by a proper forcing – provided that ω_1^V is not remarkable in L.

Let us call A^+ *good* if

- $A^+ \subset \omega_1$, and
- $H_{\omega_2} = L_{\omega_2}[A^+]$.

Let us now sketch the actual proof of Con(c) $\Rightarrow$ Con(d). Fix $A \subset OR$ such that $A \in V \setminus L(\mathbb{R})$. As mentioned above, w.l.o.g., $A \subset \omega_1$. We aim to prove that if ω_1^V is not remarkable in L then $A \in L(\mathbb{R}^W)$ where W is some proper set-generic extension of V. As mentioned above, every Silver indiscernible is remarkable in L, so that we may assume that $0^\sharp$ does not exist. But then there is a good A^+ in a set-generic extension obtained by an ω-closed forcing. (Hint: force with $Col(\delta^+, 2^\delta) \star Col(\omega_1, \delta) \star Q$ where δ is a singular cardinal of uncountable cofinality with $\delta^{\aleph_0} = \delta$ and Q is an almost disjoint forcing which codes some $D \subset \delta^+$ of the intermediate model, where $L_{\delta^+}[D] = H_{\delta^+}$, by $A^+ \subset \omega_1$ using δ^+ many pairwise a.d. subsets of δ provided by Jensen covering for L.) We may thereafter force with P, where $p \in P$ if and only if

- $p \colon i \to 2$ for some $i < \omega_1$, and
- $L[A^+ \cap j, p \upharpoonright j] \models$ "j is countable" for all $j \leq i$.

This is Jensen's "reshaping" (cf. [4] p. 90 f., and [2]). The upshot now is that if ω_1 is not remarkable in L then P is actually proper. (This fact is in want of a non-trivial proof which uses the key characterization 1.7 above. We emphasize that, on the other hand, reshaping is not proper if $V = L^{Col(\omega, <\kappa)}$ where κ is remarkable in L.) After having forced with P, let us finally force Martin's Axiom MA, by some c.c.c. forcing. Let W denote the final generic extension.

Let $R \subset \omega_1$ be a reshaped element of W. With $A \subset \omega_1$ being our initially given set, we have that $B = A \oplus R$ is reshaped, too. If we let $F \restriction \{B \cap i : i < \omega_1\}$ be the function defined as above then F will witness $(\vec{a}, B)$ is robust for $\vec{a} = (F(B \cap i) : i < \omega_1)$. As almost disjoint coding has the c.c.c., and $W \models MA$, we thus get that $(\vec{a}, B) \in L(\mathbb{R}^W)$; in particular, $A \in L(\mathbb{R}^W)$.

Thus if ω_1 is not remarkable in L then any $A \subset OR$ can be "coded into" $L(\mathbb{R})$ by some proper set forcing.

Let us now finally take a look at $\textsc{Con}$(a) $\Rightarrow$ $\textsc{Con}$(d). The above argument can be easily varied to give a proof of this. Instead of forcing MA in the end, we may as well produce – under the assumption that ω_1 is not remarkable in L – a proper set generic extension W' in which there is a reshaped good $B \subset \omega_1$, and the continuum hypothesis holds. We can then use almost disjoint forcing to obtain a further extension in which $\mathbb{R} \subset L[a]$ for some real a, i.e., in which there is a $\boldsymbol{\Delta}^1_2$ well-ordering of the reals.

On the other hand, by adding ω_1 many Cohen reals we can produce a model in which there is no well-ordering whatsoever of its reals in $L(\mathbb{R})$. Hence $L(\mathbb{R})$ absoluteness for proper forcings fails if ω_1 is not remarkable in L.

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