Handbook of Set Theory

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Contents

I. Fine structure

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Fine structure theory is an in-depth study of definability over levels of constructible hierarchies. It was invented by Ronald B. Jensen (cf. [3]), and later pursued by Jensen, Mitchell, Steel, and others (cf. for instance [8] and [4]). Our aim here is to give a self-contained introduction to this theory.

Fine structure theory is a necessary tool for a detailed analysis of Gödel's L and of more complicated constructible models; in fact, it is unavoidable even for the construction of an important class of such models, the so-called core models. The present chapter is thus intended as an introduction to chapters [5], [9], and [12], where core model theory is developed and applied. It may also be read as an introduction to [8], [4], or [15].

An important result of [3] is the Σ_n uniformization theorem (cf. [3, Theorem 3.1]), which implies that for any ordinal α and for any positive integer n there is a Σ_n Skolem function for J_α, i.e., a Skolem function for Σ_n relations over J_α which is itself Σ_n definable over J_α. The naïve approach for obtaining such a Skolem function only works for $n = 1$; for $n > 1$, fine structure theory is called for.

Classical applications of the fine structure theory are to establish Jensen's results that $\Box_\kappa$ holds in L for every infinite cardinal κ (cf. [3, Theorem 5.2]) and his Covering Lemma: If $0^\#$ does not exist, then every uncountable set of ordinals can be covered by a set in L of the same size (cf. [2]). We shall prove $L \models \Box_\kappa$ as well as a slight weakening of Jensen's Covering Lemma in the final section of this chapter (cf. [5] on a complete proof of the Covering Lemma); they have been generalized by recent research (cf. [10] and [6]; cf. also [9]).

The present chapter will discuss the "pure" part of fine structure theory, the one which is not linked to any particular kind of constructible model one might have in mind. We shall discuss Jensen's classical version of this theory. We shall not, however, deal with Jensen's Σ^* theory (which may be found in [15, Sections 1.6–1.8] or in [14]), and we shall also ignore other variants of the fine structure theory which have been created. What we shall deal with here is tantamount to what is presented in (parts of) [8, §§2 and 4].

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1. Acceptable J-structures

An inner model is a transitive proper class model of ZF. If A is a set or a proper class, then $L[A]$ is the least inner model which is closed under the operation $x \mapsto A \cap x$. An important example is $L = L[\emptyset]$, Gödel's constructible universe. V itself, the universe of all sets, is of the form $L[A]$ in a class forcing extension which does not add any new sets.[1]

Any model of the form $L[A]$ may be stratified in two ways: into levels of the L-hierarchy and into levels of the J-hierarchy. The former approach was Gödel's original one, but it turned out that the latter one (which was introduced by Jensen in [3]) is more useful.

In order to define the J-hierarchy we need the concept of rudimentary functions (cf. [3, p. 233]).

1.1 Definition. Let A be a set or a proper class. A function $f\colon V^k \to V$, where $k < \omega$, is called *rudimentary in A* (or, rud_A) if it is generated by the following schemata:

$$\begin{array}{rcl} f(\langle x_1, ..., x_k \rangle) & = & x_i \\ f(\langle x_1, ..., x_k \rangle) & = & x_i \setminus x_j \\ f(\langle x_1, ..., x_k \rangle) & = & \{x_i, x_j\} \\ f(\langle x_1, ..., x_k \rangle) & = & h(g_1(\langle x_1, ..., x_k \rangle), ..., g_\ell(\langle x_1, ..., x_k \rangle)) \\ f(\langle x_1, ..., x_k \rangle) & = & \bigcup_{y \in x_1} g(\langle y, x_2, ..., x_k \rangle) \\ f(x) & = & x \cap A \end{array}$$

f is called *rudimentary* (or, rud) if f is $\mathrm{rud}_\emptyset$.

Let us write $\vec{x}$ for $\langle x_1, ..., x_k \rangle$. It is easy to verify that for instance the following functions are rudimentary: $f(\vec{x}) = \bigcup x_i$, $f(\vec{x}) = x_i \cup x_j$, $f(\vec{x}) = \{x_1, ..., x_k\}$, and $f(\vec{x}) = \langle x_1, ..., x_k \rangle$. Proposition 1.3 below will provide more information.

[1] This class forcing extension is obtained simply by forcing with enumerations $p\colon \alpha \to V$, ordered by end-extension.

If U is a set and A is a set or a proper class then we shall denote by $\mathrm{rud}_A(U)$ the rud_A closure of U,[2] i.e., the set

$$U \cup \{f(\langle x_1, ..., x_k\rangle) \ ; \ f \text{ is } \mathrm{rud}_A \text{ and } x_1, ..., x_k \in U\}.$$

It is not hard to verify that if U is transitive, then so is $\mathrm{rud}_A(U \cup \{U\})$. We shall now be interested in $\mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\})$ (cf. Lemma 1.4 below).

1.2 Definition. Let A be a set or a proper class. A relation $R \subseteq V^k$, where $k < \omega$, is called *rudimentary in A* (or, rud_A) if there is a rud_A function $f\colon V^k \to V$ such that $R = \{\vec{x}; f(\vec{x}) \neq \emptyset\}$. R is called *rudimentary* (or, rud) if R is $\mathrm{rud}_\emptyset$.

1.3 Proposition. *Let A be a set or a proper class.*

(a) The relation $\notin$ is rud.

(b) Let f, R be rud_A. *Let $g(\vec{x}) = f(\vec{x})$ if $R(\vec{x})$ holds, and $g(\vec{x}) = \emptyset$ if not. Then g is* rud_A.

(c) If R, S are rud_A, *then so is $R \cap S$.*

(d) Membership in A is rud_A.

(e) If R is rud_A, *then so is its characteristic function χ_R.*

(f) R is rud_A *iff $\neg R$ is* rud_A.

(g) Let R be rud_A. *Let $f(\langle y, \vec{x}\rangle) = y \cap \{z; R(\langle z, \vec{x}\rangle)\}$. Then f is* rud_A.

(h) If $R(\langle y, \vec{x}\rangle)$ is rud_A *then so is $\exists z \in y R(\langle z, \vec{x}\rangle)$.*

Proof. (a) $x \notin y$ iff $\{x\} \setminus y \neq \emptyset$. (b) If $R(\vec{x}) \leftrightarrow r(\vec{x}) \neq \emptyset$, where r is rud_A, then $g(\vec{x}) = \bigcup_{y \in r(\vec{x})} f(\vec{x})$. (c) Let $R(\vec{x}) \leftrightarrow f(\vec{x}) \neq \emptyset$, where f is rud_A. Let $g(\vec{x}) = f(\vec{x})$ if $S(\vec{x})$ holds, and $g(\vec{x}) = \emptyset$ if not. g is rud_A by (b), and thus g witnesses that $R \cap S$ is rud_A. (d) $x \in A$ iff $\{x\} \cap A \neq \emptyset$. (e): by (b). (f) $\chi_{\neg R}(\vec{x}) = 1 \setminus \chi_R(\vec{x})$. (g) Let $g(\langle z, \vec{x}\rangle) = \{z\}$ if $R(\langle z, \vec{x}\rangle)$ holds, and $g(\langle z, \vec{x}\rangle) = \emptyset$ if not. We have that g is rud_A by (b), and $f(\langle y, \vec{x}\rangle) = \bigcup_{z \in y} g(z, \vec{x})$. (h) Set $f(y, \vec{x}) = y \cap \{z; R(\langle z, \vec{x}\rangle)\}$. f is rud_A by (g), and thus f witnesses that $\exists z \in y R(\langle z, \vec{x}\rangle)$ is rud_A. $\dashv$

We shall be concerned here with structures of the form $\langle U, \in, A_0, ..., A_m\rangle$, where U is transitive. (By $\langle U, \in, A_0, ..., A_m\rangle$ we shall mean the structure $\langle U, \in\restriction U, A_0 \cap U, ..., A_m \cap U\rangle$.) Each such structure comes with a language $\mathcal{L}_{\dot{A}_0, ..., \dot{A}_m}$ with predicates $\dot{\in}$, $\dot{A}_0$, ..., $\dot{A}_m$. We shall restrict ourselves to the case where $m = 0$ or $m = 1$.

If $M = \langle |M|, ...\rangle$ is a structure, $X \subseteq |M|$, and $n < \omega$ then we let $\Sigma_n^M(X)$ denote the set of all relations which are Σ_n definable over M from parameters in X. We shall also write $\boldsymbol{\Sigma}_n^M$ for $\Sigma_n^M(M)$, and we shall write $\boldsymbol{\Sigma}_\omega^M$ for $\bigcup_{n<\omega} \boldsymbol{\Sigma}_n^M$. Further, we'll write Σ_n^M for $\Sigma_n^M(\emptyset)$, where $n \leq \omega$.

The following lemma says that $\mathrm{rud}_A(U \cup \{U\})$ is just the result of "stretching" $\boldsymbol{\Sigma}_\omega^{\langle U, \in, A\rangle}$ without introducing additional elements of $\mathcal{P}(U)$.

[2]This is in contrast to [3, p. 238], where $\mathrm{rud}_A(U)$ stands for the rud_A closure of $U \cup \{U\}$.

1.4 Lemma. *Let U be a transitive set, and let A be a set or proper class such that $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$. Then $\mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\}) = \mathcal{P}(U) \cap \mathbf{\Sigma}_\omega^{\langle U,\in,A\rangle}$.*

Proof. Notice that $\mathcal{P}(U) \cap \mathbf{\Sigma}_\omega^{\langle U,\in,A\rangle} = \mathcal{P}(U) \cap \mathbf{\Sigma}_0^{\langle U\cup\{U\},\in,A\cap U\rangle}$, so that we have to prove that

$$\mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\}) = \mathcal{P}(U) \cap \mathbf{\Sigma}_0^{\langle U\cup\{U\},\in,A\rangle}$$

"$\supseteq$": By Proposition 1.3 (a) and (d), $\notin$ and membership in A are both rud_A. By Proposition 1.3 (f), (c), and (h), the collection of rud_A relations is closed under complement, intersection, and bounded quantification. Therefore we get inductively that every relation which is Σ_0 in the language $\mathcal{L}_{\dot{A}}$ with $\dot{\in}$ and $\dot{A}$ is also rud_A.

Now let $x \in \mathcal{P}(U) \cap \mathbf{\Sigma}_0^{\langle U\cup\{U\},\in,A\rangle}$. There is then some rud_A relation R and there are x_1, ..., $x_k \in U \cup \{U\}$ such that $y \in x$ iff $y \in U$ and $R(\langle y, x_1, ..., x_k\rangle)$ holds. But then $x = U \cap \{y; R(\langle y, x_1, ..., x_k\rangle)\} \in \mathrm{rud}_A(U \cup \{U\})$ by Proposition 1.3 (g).

"$\subseteq$": Call a function $f\colon V^k \to V$, where $k < \omega$, *simple* iff the following holds true: if $\varphi(v_0, v_1, ..., v_m)$ is Σ_0 in the language $\mathcal{L}_{\dot{A}}$ with $\dot{\in}$ and $\dot{A}$, then $\varphi(f(v_1', ..., v_k'), v_1, ..., v_m)$ is equivalent over transitive rud_A closed structures to a Σ_0 formula in the same language. It is not hard to verify inductively that every rud_A function is simple. (Here we use the hypothesis that $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$ which ensures that in this situation quantifying over A is tantamount to quantifying over $A \cap U$.)

Now let $x \in \mathcal{P}(U) \cap \mathrm{rud}_A(U \cup \{U\})$, say $x = f(\langle x_1, ..., x_k\rangle)$, where x_1, ..., $x_k \in U \cup \{U\}$ and f is rud_A. Then "$v_0 \in f(\langle v_1, ..., v_k\rangle)$" is (equivalent over $\mathrm{rud}_A(U \cup \{U\})$ to) a Σ_0 formula in the language $\mathcal{L}_{\dot{A}}$, and hence $x = \{y \in U; y \in f(\langle x_1, ..., x_n\rangle)\}$ is in $\Sigma_0^{\langle U\cup\{U\},\in,A\rangle}(\{x_1, ..., x_n\})$. ⊣

Of course Lemma 1.4 also holds with $\mathcal{P}(U)$ being replaced by the set of all relations on U. The hypothesis that $A \cap V_{\mathrm{rk}(U)+\omega} \subseteq U$ in Lemma 1.4 is needed to avoid pathologies; it is always met in the construction of fine structural inner models.

Let U be rud_A closed, and let $x \in U$ be transitive. Suppose that $B \in \Sigma_0^{\langle U,\in,A\rangle}(\{x_1, ..., x_k\})$, where x_1,...,$x_k \in x$. Then $B \cap x \in \mathbf{\Sigma}_0^{\langle x,\in,A\rangle}$, and hence $B \cap x \in \mathrm{rud}_A(x \cup \{x\})$ by Lemma 1.4. But $\mathrm{rud}_A(x \cup \{x\}) \subseteq U$, and therefore $B \cap x \in U$. We have shown the following.

1.5 Lemma. *Let U be a transitive set such that for every $x \in U$ there is some transitive $y \in U$ with $x \in y$, let A be a set or a proper class, and suppose that U is rud_A closed. Then $\langle U, \in, A\rangle$ is a model of Σ_0 comprehension in the sense that if $B \in \mathbf{\Sigma}_0^{\langle U,\in,A\rangle}$ and $x \in U$ is transitive then $B \cap x \in U$.*

In the next section we shall start with studying possible failures of Σ_1 comprehension in rud_A closed structures. Lemma 1.5 provides the key element for proving that (all but two of) the structures we are now about to define are models of "basic set theory" (cf. [1, p. 36]), a theory which consists of Σ_0 comprehension together with extensionality, foundation, pairing, union, infinity, and the statement that Cartesian products exist.[3]

We may now define the J^A_α hierarchy as follows. For later purposes it is convenient to index this hierarchy by limit ordinals.[4]

1.6 Definition. Let A be a set or a proper class.

$$\begin{aligned} J^A_0 &= \emptyset \\ J^A_{\alpha+\omega} &= \mathrm{rud}_A(J^A_\alpha \cup \{J^A_\alpha\}) \\ J^A_{\omega\lambda} &= \bigcup_{\alpha<\lambda} J^A_{\omega\alpha} \text{ for limit } \lambda \\ L[A] &= \bigcup_{\alpha\in\mathrm{On}} J^A_{\omega\alpha} \end{aligned}$$

Every J^A_α is rud_A closed and transitive. We shall also denote by J^A_α the structure $\langle J^A_\alpha, \in\restriction J^A_\alpha, A\cap J^A_\alpha\rangle$.

An important special case is obtained by letting $A=\emptyset$ in Definition 1.6. We write J_α for $J^\emptyset_\alpha$, and L for $L[\emptyset]$. L is Gödel's constructible universe; it will be studied in the last section of this chapter. Other important examples are obtained by letting A code a (carefully chosen) sequence of extenders; such models are discussed in [5], [9], and [12].

The following is an immediate consequence of Lemma 1.4

1.7 Lemma. *Let A be a set or proper class such that $A\cap V_{\mathrm{rk}(U)+\omega}\subseteq U$, and let α be a limit ordinal. Then $\mathcal{P}(J^A_\alpha)\cap J^A_{\alpha+\omega} = \mathcal{P}(J^A_\alpha)\cap \mathbf{\Sigma}^{J^A_\alpha}_\omega$.*

It is often necessary to work with the auxiliary hierarchy S^A_α of [3, p. 244] which is defined as follows:

$$\begin{aligned} S^A_0 &= \emptyset \\ S^A_{\alpha+1} &= \mathbf{S}^A(S^A_\alpha) \\ S^A_\lambda &= \bigcup_{\xi<\lambda} S^A_\xi \quad \text{for limit } \lambda \end{aligned}$$

where $\mathbf{S}^A$ is an operator which, applied to a set U, adds images of members of $U\cup\{U\}$ under rud_A functions from a certain carefully chosen fixed *finite* list. We may set

$$\mathbf{S}^A(U) = \bigcup_{i=0}^{15} F_i\text{“}(U\cup\{U\})^2,$$

[3] Said structures will also be models of "the transitive closure of any set exists," a statement which - despite of a claim made in [1] - is not provable even in Zermelo's set theory.

[4] This is again in contrast with [3].

where

$$
\begin{array}{rcl}
F_0(x,y) & = & \{x,y\} \\
F_1(x,y) & = & x \setminus y \\
F_2(x,y) & = & x \times y \\
F_3(x,y) & = & \{\langle u,z,v\rangle ;\, z \in x \wedge \langle u,v\rangle \in y\} \\
F_4(x,y) & = & \{\langle u,v,z\rangle ;\, z \in x \wedge \langle u,v\rangle \in y\} \\
F_5(x,y) & = & \bigcup x \\
F_6(x,y) & = & \mathrm{dom}(x) \\
F_7(x,y) & = & \in \cap (x \times x) \\
F_8(x,y) & = & \{x``\{z\} ;\, z \in y\} \\
F_9(x,y) & = & \langle x,y\rangle \\
F_{10}(x,y) & = & x``\{y\} \\
F_{11}(x,y) & = & \langle \mathrm{left}(y), x, \mathrm{right}(y)\rangle \\
F_{12}(x,y) & = & \langle \mathrm{left}(y), \mathrm{right}(y), x\rangle \\
F_{13}(x,y) & = & \{\mathrm{left}(y), \langle \mathrm{right}(y), x\rangle\} \\
F_{14}(x,y) & = & \{\mathrm{left}(y), \langle x, \mathrm{right}(y)\rangle\} \\
F_{15}(x,y) & = & A \cap x.
\end{array}
$$

(Here, $\langle x_1, x_2, \ldots x_n\rangle = \langle x_1, \langle x_2, \ldots, x_n\rangle\rangle$, and $\mathrm{left}(y) = u$ and $\mathrm{right}(y) = v$ if $y = \langle u, v\rangle$ and $\mathrm{left}(y) = 0 = \mathrm{right}(y)$ if y is not an ordered pair.) It is not difficult to show that each F_i, $0 \leq i \leq 15$, is rud_A. A little bit more work is necessary to show that every rud_A function can be generated by using functions from this list. The functions F_i, $0 \leq i \leq 15$, are therefore a *basis* for the set of rud_A functions (cf. [3, Lemma 1.8]).

Every S^A_α is transitive,[5] and moreover

$$J^A_\alpha = S^A_\alpha \tag{I.1}$$

for all limit ordinals α. It is easy to see that there is only a finite jump in rank from S^A_α to $S^A_{\alpha+1}$. A straightforward induction shows that $J^A_\alpha \cap \mathrm{On} = \alpha$ for all limit ordinals α.

Recall that a structure $\langle U, \in, A_1, ..., A_m\rangle$ is called *amenable* if and only if $A_i \cap x \in U$ whenever $1 \leq i \leq m$ and $x \in U$. Lemma 1.5 together with (I.1) readily gives the following.

1.8 Lemma. *Let A be a set or proper class, and let α be a limit ordinal. Let $B \in \boldsymbol{\Sigma}_0^{J^A_\alpha}$. Then $\langle J^A_\alpha, B\rangle$ is amenable, i.e., J^A_α is a model of Σ_0 comprehension in the language $\mathcal{L}_{\dot{A},\dot{B}}$ with $\dot{\in}$, $\dot{A}$ and $\dot{B}$.*

[5] The above list in fact contains more functions than the list from [3, Lemma 1.8]; this enlargement yields the transitivity of each S^A_α.

1.9 Definition. A *J-structure* is an amenable structure of the form $\langle J_\alpha^A, B\rangle$ for a limit ordinal α and predicates A, B.

Here, $\langle J_\alpha^A, B\rangle$ denotes the structure $\langle J_\alpha^A, \in\restriction J_\alpha^A, A\cap J_\alpha^A, B\cap J_\alpha^A\rangle$. Any J_α^A is a J-structure.

1.10 Lemma. *Let J_α^A be a J-structure.*

(1) For all $\beta<\alpha$, $\langle S_\gamma^A;\gamma<\beta\rangle\in J_\alpha^A$. In particular, $S_\beta^A\in J_\alpha^A$ for all $\beta<\alpha$.

(2) $\langle S_\gamma^A;\gamma<\alpha\rangle$ is uniformly $\Sigma_1^{J_\alpha^A}$. That is, "$x=S_\gamma^A$" is Σ_1 over J_α^A as witnessed by a formula which does not depend on α.

Proof. (1) and (2) are shown simultaneously by induction on $\langle\alpha,\beta\rangle$, ordered lexicographically. Fix α and $\beta<\alpha$. If β is a limit ordinal then inductively by (2), $\langle S_\gamma^A;\gamma<\beta\rangle$ is $\Sigma_1^{J_\beta^A}$, and hence $\langle S_\gamma^A;\gamma<\beta\rangle\in J_\alpha^A$ by Lemma 1.7. If $\beta=\delta+1$ then inductively by (1), $\langle S_\gamma^A;\gamma<\delta\rangle\in J_\alpha^A$. If δ is a limit ordinal then $S_\delta^A=\bigcup_{\gamma<\delta}S_\gamma^A\in J_\alpha^A$, and if $\delta=\bar\delta+1$ then $S_\delta^A=S_{\bar\delta}^A\cup\mathbf{S}^A(S_{\bar\delta}^A)\in J_\alpha^A$ as well. It follows that $\langle S_\gamma^A;\gamma<\beta\rangle\in J_\alpha^A$. (2) is then not hard to verify. ⊣

We may recursively define a well-ordering $<_\beta^A$ of S_β^A as follows. If β is a limit ordinal then we let $<_\beta^A=\bigcup_{\gamma<\beta}<_\gamma^A$. Now suppose that $\beta=\bar\beta+1$. The order $<_{\bar\beta}^A$ induces a lexicographical order, call it $<_{\bar\beta,\mathrm{lex}}^A$, of $16\times S_{\bar\beta}^A\times S_{\bar\beta}^A$. We may then set

$$x<_\beta^A y\iff\begin{cases}x,y\in S_{\bar\beta}^A\text{ and }x<_{\bar\beta}^A y,\text{ or else}\\ x\in S_{\bar\beta}^A\wedge y\notin S_{\bar\beta}^A,\text{ or else}\\ x,y\notin S_{\bar\beta}^A\text{ and }(i,u_x,v_x)<_{\bar\beta,\mathrm{lex}}^A(j,u_y,v_y)\\ \qquad\text{where }(i,u_x,v_x)\text{ is }<_{\bar\beta,\mathrm{lex}}^A-\text{minimal with }x=F_i(u_x,v_x)\\ \qquad\text{and }(j,u_y,v_y)\text{ is }<_{\bar\beta,\mathrm{lex}}^A-\text{minimal with }y=F_j(u_y,v_y).\end{cases}$$

The following is easy to prove.

1.11 Lemma. *Let J_α^A be a J-structure.*

(1) For all $\beta<\alpha$, $\langle<_\gamma^A;\gamma<\beta\rangle\in J_\alpha^A$. In particular, $<_\beta^A\in J_\alpha^A$ for all $\beta<\alpha$.

(2) $\langle<_\gamma^A;\gamma<\alpha\rangle$ is uniformly $\Sigma_1^{J_\alpha^A}$. That is, "$x=<_\gamma^A$" is Σ_1 over J_α^A as witnessed by a formula which does not depend on α.

If $M=J_\alpha^A$, then we shall also write $<_M$ for $<_\alpha^A$.

We shall now start working towards showing that J-structures have Σ_1 definable Σ_1 Skolem functions.

In what follows we shall fix a recursive enumeration $\langle\varphi_i;i\in\omega\rangle$ of all Σ_1 formulae of the language $\mathcal{L}_{\dot A}$. (What we shall say easily generalizes to

$\mathcal{L}_{\dot{A}_1,\ldots,\dot{A}_m}$.) We shall denote by $\ulcorner\varphi\urcorner$ the Gödel number of φ, i.e., $\ulcorner\varphi\urcorner = i$ iff $\varphi = \varphi_i$. We may and shall assume that if $\bar{\varphi}$ is a proper subformula of φ then $\ulcorner\bar{\varphi}\urcorner < \ulcorner\varphi\urcorner$. We shall write $v(i)$ for the set of free variables of φ_i. Recall that all the relevant syntactical concepts are representable in (weak fragments of) Peano arithmetic, so that the representability of these concepts is immediate.

Let M be a structure for $\mathcal{L}_{\dot{A}}$. We shall express by $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$ the fact that $\mathsf{a}\colon v(i) \to M$, i.e., a assigns elements of M to the free variables of φ_i, and φ_i holds true in M under this assignment. We shall also write $\models^{\Sigma_1}_M$ for the set of $\langle i, \mathsf{a}\rangle$ such that $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$. We shall express by $\models^{\Sigma_0}_M \varphi_i[\mathsf{a}]$ the fact that $\models^{\Sigma_1}_M \varphi_i[\mathsf{a}]$ holds, but with φ_i being a Σ_0 formula, and we shall write $\models^{\Sigma_0}_M$ for the set of $\langle i, \mathsf{a}\rangle$ such that $\models^{\Sigma_0}_M \varphi_i[\mathsf{a}]$.

It turns out that once we have verified that $\models^{\Sigma_0}_M$ is uniformly Δ_1 over J-structures M (which are structures of $\mathcal{L}_{\dot{A}}$), we easily get that $\models^{\Sigma_1}_M$ is Σ_1 definable over such structures and that these structures admit Σ_1 definable Σ_1 Skolem functions. R on M is Δ_1 iff R, $\neg R$ are both Σ_1.

Let us fix a J-structure $M = J^A_\alpha$, a structure for $\mathcal{L}_{\dot{A}}$.

1.12 Proposition. *Let $N \in M$ be transitive. For each $n < \omega$, there is a unique $f = f^N_n \in M$ such that* $\mathrm{dom}(f) = n$ *and for all $i < n$, if φ_i is not a Σ_0 formula then $f(i) = \emptyset$, and if φ_i is a Σ_0 formula then*

$$f(i) = \{\mathsf{a} \in {}^{v(i)}N \ ; \ \models^{\Sigma_0}_N \varphi_i[\mathsf{a}]\}.$$

Proof. As uniqueness is clear, let us verify inductively that $f^N_n \in M$. Well, $f^N_0 = \emptyset \in M$. Now suppose that $f^N_n \in M$. If φ_n is not Σ_0, then $f^N_{n+1} = f^N_n \cup \{\langle n, \emptyset\rangle\} \in M$. Now let φ_i be Σ_0. We have that ${}^{v(n)}N \in M$, and if

$$T = \{\mathsf{a} \in {}^{v(n)}N \ ; \ \models^{\Sigma_0}_N \varphi_n[\mathsf{a}]\}$$

then $T \in \mathcal{P}({}^{v(n)}N) \cap \boldsymbol{\Sigma}^M_0$, and thus $T \in M$ by Lemma 1.8. Therefore,

$$f^N_{n+1} = f^N_n \cup \{\langle n, T\rangle\} \in M.$$

$\dashv$

Now let $\Theta(f, N, n)$ denote the following formula.

$$\begin{array}{lcl}
 & & N \text{ is transitive } \wedge f\colon n \to N \wedge \forall i < n \ [\\
(\ i & = & \ulcorner v_{i_0} \in v_{i_1}\urcorner, \text{ some } v_{i_0}, v_{i_1} \ \to f(i) = \{\mathsf{a} \in {}^{v(i)}N; \mathsf{a}(v_{i_0}) \in \mathsf{a}(v_{i_1})\}) \wedge \\
(\ i & = & \ulcorner \dot{A}(v_{i_0})\urcorner, \text{ some } v_{i_0} \ \to f(i) = \{\mathsf{a} \in {}^{v(i)}N; \mathsf{a}(v_{i_0}) \in A\}) \wedge \\
(\ i & = & \ulcorner \psi_0 \wedge \psi_1\urcorner, \text{ some } \psi_0, \psi_1 \ \to \\
 & & f(i) = \{\mathsf{a} \in {}^{v(i)}N; \mathsf{a} \restriction v(\ulcorner\psi_0\urcorner) \in f(\ulcorner\psi_0\urcorner) \wedge \mathsf{a} \restriction v(\ulcorner\psi_1\urcorner) \in f(\ulcorner\psi_1\urcorner)\}) \wedge \\
(\ i & = & \ulcorner \exists v_{i_0} \in v_{i_1} \psi\urcorner, \text{ some } v_{i_0}, v_{i_1}, \psi, \text{ where } \psi \text{ is } \Sigma_0 \ \to
\end{array}$$

$$f(i) = \{\mathsf{a} \in {}^{v(i)}N; \exists x \in \mathsf{a}(v_{i_1})$$
$$(\mathsf{a} \cup \{\langle v_{i_0}, x\rangle\}) \upharpoonright v(\ulcorner\psi\urcorner) \in f(\ulcorner\psi\urcorner)\}) \wedge$$
$$(i = \ulcorner\varphi\urcorner, \text{ some } \varphi, \text{ where } \varphi \text{ is not } \Sigma_0 \to f(i) = \emptyset\ ].$$

It is straightforward to check that $\Theta(f, N, n)$ holds (in M) if and only if $f = f_n^N$. Now Proposition 1.12 and the fact that every element of M is contained in a transitive element of M (for instance in some S_β^A; cf. Lemma 1.10) immediately gives the following.

1.13 Proposition. *Let φ_i be Σ_0, and let $\mathsf{a}\colon v(i) \to M$. Then $\models_M^{\Sigma_0} \varphi_i[\mathsf{a}]$ if and only if*

$$M \models \exists f\ \exists N\ (\operatorname{ran}(\mathsf{a}) \subseteq N \wedge \Theta(f, N, i+1) \wedge \mathsf{a} \in f(i)),$$

if and only if

$$M \models \forall f\ \forall N\ ((\operatorname{ran}(\mathsf{a}) \subseteq N \wedge \Theta(f, N, i+1)) \to \mathsf{a} \in f(i)).$$

In particular, the relation $\models_M^{\Sigma_0}$ is Δ_1^M.

We are now ready to prove two important results.

1.14 Theorem. *Let $M = J_\alpha^A$ be a J-structure. The Σ_1-satisfaction relation $\models_M^{\Sigma_1}$ is then uniformly Σ_1^M.*

If M is a structure then h is a Σ_1 Skolem function for M if

$$h\colon \bigcup_{i<\omega} \{i\} \times {}^{v(i)}|M| \to |M|,$$

where h may be partial, and whenever $\varphi_i = \exists v_{i_0} \varphi_j$ and $\mathsf{a}\colon v(i) \to |M|$ then

$$\begin{aligned} \exists y \in |M| \quad & \models_M^{\Sigma_1} \quad \varphi_j[\mathsf{a} \cup \{\langle v_{i_0}, y\rangle\} \upharpoonright v(j)] \\ \Longrightarrow \quad & \models_M^{\Sigma_1} \quad \varphi_j[\mathsf{a} \cup \{\langle v_{i_0}, h(i, \mathsf{a})\rangle\} \upharpoonright v(j)]. \end{aligned}$$

1.15 Theorem. *Let M be a J-structure. There is a Σ_1 Skolem function h_M which is uniformly Σ_1^M.*

The above two theorems are to be understood as follows. There are Σ_1-formulae Φ, Ψ such that whenever M is a J-structure,

a) Φ defines $\models_M^{\Sigma_1}$, i.e. $\models_M^{\Sigma_1} \varphi_i[\mathsf{a}] \leftrightarrow M \models \Phi(i, \mathsf{a})$, and

b) Ψ defines h_M, i.e. $y = h_M(i, \mathsf{a}) \leftrightarrow M \models \Psi(i, \mathsf{a}, y)$.

Proof of Theorem 1.14. We have that $\models_M^{\Sigma_1} \varphi_i[\mathsf{a}]$ iff

$$\begin{aligned} \exists \mathsf{b} \in M\ \exists\langle v_{i_0},...,v_{i_k},j\rangle,\ \text{some } v_{i_0},...,v_{i_k},j \quad [\quad & i = \ulcorner \exists v_{i_0} \cdots \exists v_{i_k} \varphi_j \urcorner \wedge \\ \varphi_j \text{ is } \Sigma_0 \wedge \mathsf{a},\mathsf{b} \text{ are functions} \quad \wedge \quad & \operatorname{dom}(\mathsf{a}) = v(i) \wedge \\ \operatorname{dom}(\mathsf{b}) = v(j) \wedge \mathsf{a} = \mathsf{b} \restriction v(i) \quad \wedge \quad & \models_M^{\Sigma_0} \varphi_j[\mathsf{b}]\ ]. \end{aligned}$$

Here, $\models_M^{\Sigma_0}$ is uniformly Δ_1^M by Proposition 1.13. The rest follows. $\dashv$

Proof of Theorem 1.15. The idea here is to let $y = h_M(i,\mathsf{a})$ be the "first component" of a minimal witness to the Σ_1 statement in question (rather than letting y be minimal itself). We may let $y = h_M(i,\mathsf{a})$ iff

$$\begin{aligned} \exists N\ \exists\beta\ \exists R\ \exists\mathsf{b},\ \text{all in } M,\ \exists\langle v_{i_0},...,v_{i_k},j\rangle,\ \text{some } v_{i_0},...,v_{i_k},j\ [\\ N = S_\beta^A \wedge R = <_\beta^A \wedge i = \ulcorner \exists v_{i_0} \cdots \exists v_{i_k} \varphi_j \urcorner \wedge \varphi_j \text{ is } \Sigma_0 \wedge \mathsf{a},\mathsf{b} \text{ are functions } \wedge \\ \operatorname{dom}(\mathsf{a}) = v(i) \wedge \operatorname{dom}(\mathsf{b}) = v(j) \wedge \mathsf{a} = \mathsf{b}\restriction v(i) \wedge \operatorname{ran}(\mathsf{b}) \subseteq N \wedge \\ \models_M^{\Sigma_0} \varphi_j[\mathsf{b}] \wedge \forall \bar{\mathsf{b}} \in N((\bar{\mathsf{b}} \text{ is a function } \wedge \operatorname{dom}(\bar{\mathsf{b}}) = v(j) \wedge \\ \mathsf{a} = \bar{\mathsf{b}} \restriction v(i) \wedge \operatorname{ran}(\bar{\mathsf{b}}) \subseteq N \wedge \bar{\mathsf{b}}\ R\ \mathsf{b}) \rightarrow \neg \models_M^{\Sigma_0} \varphi_j[\bar{\mathsf{b}}]) \wedge \\ y = \mathsf{b}(v_{i_0})\ ]. \end{aligned}$$

Here, "$N = S_\beta^A$" and "$R = <_\beta^A$" are uniformly Σ_1^M by Lemmata 1.10 (2) and 1.11 (2), and $\models_M^{\Sigma_0}$ is uniformly Δ_1^M by Proposition 1.13. Therefore, the rest follows. $\dashv$

If we were to define a Σ_2 Skolem function for M in the same manner then we would end up with a Σ_3 definition. Jensen solved this problem by showing that under favourable circumstances Σ_n over M can be viewed as Σ_1 over a "reduct" of M. Reducts will be introduced in the fifth section of this chapter.

Another useful fact is the so-called *condensation lemma*.[6]

1.16 Theorem. *Let $M = \langle J_\alpha^A, B\rangle$ be a J-structure, and let $\pi\colon \bar{M} \xrightarrow[\Sigma_1]{} M$ where $\bar{M}$ is transitive. Then $\bar{M}$ is a J-structure, i.e., there are $\bar{\alpha} \leq \alpha$, $\bar{A}$, and $\bar{B}$ such that $\bar{M} = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{B}\rangle$.*

Proof. Set $\bar{\alpha} = \mathrm{On} \cap \bar{M} \leq \alpha$, $\bar{A} = \pi^{-1}{}''A$, and $\bar{B} = \pi^{-1}{}''B$. We claim that $\bar{M} = \langle J_{\bar{\alpha}}^{\bar{A}}, \bar{B}\rangle$.

Well, Lemma 1.10 easily gives that $S_\beta^{\bar{A}} \in \bar{M}$ whenever $\beta < \bar{\alpha}$. Therefore, $J_{\bar{\alpha}}^{A} \subseteq \bar{M}$. On the other hand, let $x \in \bar{M}$. Then $\pi(x) \in S_\beta^A$ for some $\beta < \alpha$, and hence $x \in S_\beta^{\bar{A}}$ for some $\beta < \bar{\alpha}$. $\dashv$

[6]For $n < \omega$, $X \prec_{\Sigma_n} M$ means that Σ_n formulae with parameters taken from X are absolute between X and M. To have $\pi\colon \bar{M} \xrightarrow[\Sigma_n]{} M$ means that $\operatorname{ran}(\pi) \prec_{\Sigma_n} M$.

We also want to write $h_M(X)$ for the closure of X under h_M, more precisely:

Convention. Let $M = J^A_\alpha$ be a J-structure, and let $X \subseteq |M|$. We shall write $h_M(X)$ for h_M"$(\bigcup_{i<\omega}(\{i\} \times {}^{v(i)}X))$.

Using Theorem 1.15, it is easy to verify that $h_M(X) \prec_{\Sigma_1} M$. There will be no danger of confusing the two usages of "h_M." $[X]^{<\omega}$ denotes the set of all finite subsets of X.

1.17 Lemma. *Let $M = J^A_\alpha$ be a J-structure. There is then some surjective $f\colon [\alpha]^{<\omega} \to M$ which is Σ^M_1.*

If α is closed under the Gödel pairing function, then there is a surjection $g\colon \alpha \to J^A_\alpha$ which is Σ^M_1. For an arbitrary α, there is a surjection $h\colon \alpha \to J^A_\alpha$ which is $\boldsymbol{\Sigma}^M_1$.

Proof. We have that $h_M(\alpha) \prec_{\Sigma_1} M$, and hence $h_M(\alpha) = M$. But it is straightforward to construct a surjective $g'\colon [\alpha]^{<\omega} \to \bigcup_{i<\omega}(\{i\} \times {}^{v(i)}\alpha)$ which is Σ^M_1. We may then set $f = h_M \circ g'$.

As to the existence of g, let $\Phi\colon \mathrm{otp}(<^A_\alpha) \to J^A_\alpha$ denote the enumeration of J^A_α according to $<^A_\alpha$. It is not hard to verify by a simultaneous induction (cf. the proof of Lemma 1.10) that for all limit ordinals $\beta \leq \alpha$, $\Phi \restriction \beta$ is $\Sigma^{J^A_\beta}_1$ and for all $\gamma < \beta$, $\Phi \restriction \gamma \in J^A_\beta$. But now if α is closed under the Gödel pairing function then $\mathrm{otp}(<^A_\alpha) = \alpha$.

The existence of h is established by [3, Lemma 2.10]. $\dashv$

In the following we describe a useful class of formulae, which lies somewhere between Σ_1 and Π_2. It turns out that many notions can be expressed by statements belonging to this class.

1.18 Definition. We say that φ is a *Q-formula* iff φ is of the form

$$\forall u \, \exists v \supseteq u \; \psi(v),$$

where ψ is Σ_1 and does not contain u. Instead of $\forall u \, \exists v \supseteq u$ we write briefly Qv. The above formula then has the form $Qv \, \psi(v)$ and we read "for cofinally many v, $\psi(v)$". A map π which preserves Q formulae is called *Q-preserving* and we write

$$\pi : \bar{M} \underset{Q}{\longrightarrow} M.$$

A property which is characterized by a Q-formula is also called a *Q-condition*.

1.19 Definition. Let U, V be transitive structures. A map $\sigma : U \to V$ is *cofinal* iff for all $y \in V$ there is some $x \in U$ such that $y \subseteq \sigma(x)$.

Let $\sigma : U \underset{\Sigma_1}{\longrightarrow} V$, where U, V are transitive structures, and let φ be a Q-formula. It is easy to see that

a) φ is preserved downwards,

b) if σ is cofinal, then φ is preserved upwards.

Note also that Q-formulae are closed under $\wedge$ and $\vee$ (modulo the "basic set theory" of [1, p. 36]).

We now introduce the notion of acceptability which is fundamental for the general fine structure theory. As will follow from the definition, acceptability can be considered as a strong version of GCH.

1.20 Definition. A J-structure $M = \langle J^A_\alpha, B\rangle$ is *acceptable* iff the following holds: Whenever $\xi < \alpha$ is a limit ordinal and $\mathcal{P}(\tau) \cap J^A_{\xi+\omega} \not\subseteq J^A_\xi$ for some $\tau < \xi$, there is a surjective map $f : \tau \to \xi$ in $J^A_{\xi+\omega}$. (This means that $\mathrm{Card}(\xi) \leq \tau$ in $J^A_{\xi+\omega}$.)

1.21 Lemma. *Being an acceptable J-structure is a Q-property. More precisely: There is a fixed Q-sentence Ψ such that for any $M = \langle |M|, A, B\rangle$ which is transitive and closed under pairing, M is an acceptable J-structure iff $M \models \Psi$.*

Proof. The statement $\langle |M|, A\rangle = J^A_\alpha$ is a Q-condition for $\langle |M|, A\rangle$, as we may write this as

$$Qu\ \exists\beta\ u = S^A_\beta.$$

Here, "$u = S^A_\beta$" is the Σ_1 formula from Lemma 1.10 (2). Amenability can be expressed by

$$Qu\ \exists z\ z = B \cap u.$$

It only remains to prove that the fact that we collapse ξ whenever we add a new bounded subset is expressible in a Q-fashion. We note first that a J-structure M is acceptable iff the following holds in M:

$$\begin{cases} \forall \text{ limit ordinals } \xi\ \exists n \in \omega\ \forall m \geq n\ \forall \tau < \xi \\ \qquad [\mathcal{P}(\tau) \cap S^A_{\xi+m} \not\subseteq S^A_\xi \Longrightarrow \exists f \in S^A_{\xi+m}\ f : \tau \xrightarrow{\text{onto}} \xi]. \end{cases} \tag{I.2}$$

Denote the sentence from (I.2) by ψ. It is easy to see that if M satisfies ψ then M is acceptable. To see the converse, fix a limit ordinal ξ and suppose there is a $\tau < \xi$ such that $\mathcal{P}(\tau) \cap S^A_{\xi+\omega} \not\subseteq S^A_\xi$. Let τ be minimal with this property. By acceptability of M, there is an $n \in \omega$ such that $S^A_{\xi+n}$ contains a function f mapping τ onto ξ. Using f, it is easy to construct a surjective map $f_{\tau'} : \tau' \to \xi$ for any $\tau' < \xi$ whose power set in $S^A_{\xi+\omega}$ is larger than that in S^A_ξ (since $\tau' \geq \tau$) and it follows immediately that $f_{\tau'} \in S^A_{\xi+n+k}$ for some $k < \omega$; so we have a uniform bound for all such functions.

If the height of M is $\omega\alpha$ for some limit α, then acceptability is equivalent to the statement

$$Q\xi\ S^A_\xi \models \psi.$$

Otherwise we have to state (I.2) explicitly for the last level. Hence, the desired condition is then

$$\begin{cases} \langle |M|, A, B\rangle \text{ is an amenable } J\text{-structure } \wedge \\ Q\zeta\ (\lim(\zeta) \Longrightarrow S^A_\zeta \models \psi)\ \wedge\ [Q\zeta\ (\zeta \text{ is a limit}) \vee \varphi] \end{cases} \tag{I.3}$$

where φ is the sentence

$$Q\zeta\ \exists\,\beta < \zeta\ [\lim(\beta)\ \wedge\ \forall\,\eta < \zeta\ (\eta > \beta \rightarrow \mathrm{succ}(\eta))\ \wedge\ \varphi'(\beta, \zeta)]$$

and $\varphi'(\beta, \zeta)$ is the formula

$$\forall\,\tau < \beta\ [\exists\,u \in S^A_\zeta\ (u \notin S^A_\beta\ \wedge\ u \subseteq \tau) \Longrightarrow \exists\,f \in S^A_\zeta\ f : \tau \xrightarrow{\text{onto}} \beta].$$

φ is clearly a Q-sentence, hence (I.3) is a Q-sentence as well. Denote this formula by Ψ. Then M is an acceptable J-structure iff $M \models \Psi$. ⊣

1.22 Corollary. *a) If $\pi : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$ and M is acceptable, then so is $\bar{M}$.*

b) If $\pi : \bar{M} \underset{Q}{\longrightarrow} M$ and $\bar{M}$ is acceptable, then so is M. This holds in particular if π is a Σ_0 preserving cofinal map.

1.23 Lemma. *Let $M = J^A_\alpha$ be acceptable and let $\rho \in M$ be an infinite cardinal in M. Given $u \in J^A_\rho$, any $a \in M$ which is a subset of u is in fact an element of J^A_ρ.*

Proof. Since $u \in J^A_\rho$, there is some $\tau < \rho$ and a surjective map $g : \tau \to u$ in J^A_ρ (cf. Lemma 1.17). Set $\bar{a} = g^{-1}$"a. Then $\bar{a} \subseteq \tau$ and $a \in J^A_\rho \iff \bar{a} \in J^A_\rho$. But if $\bar{a} \notin J^A_\rho$, then there is an $f : \tau \xrightarrow{\text{onto}} \xi$, where ξ is such that $\bar{a} \in J^A_{\xi+\omega} \setminus J^A_\xi$; hence $\xi \geq \rho$. This contradicts the fact that ρ is a cardinal in J^A_α. Consequently, $a \in J^A_\rho$. ⊣

1.24 Lemma. *Let M be as above and ρ an infinite successor cardinal in M. Let $a \subseteq J^A_\rho$ be such that $a \in M$ and $\mathrm{Card}(a) < \rho$ in M. Then $a \in J^A_\rho$.*

Proof. Let $\gamma < \rho$ and $g \in M$ be such that $g : \gamma \xrightarrow{\text{onto}} a$. Define $f : \gamma \to \rho$ by

$$\zeta = f(\xi) \iff g(\xi) \in S^A_{\zeta+\omega} \setminus S^A_\zeta.$$

Then $f \in M$.

Claim. f is bounded in ρ.

Suppose this Claim to hold true. Let $\delta < \rho$ be such that $\mathrm{rng}(f) \subseteq \delta$. Then $a \subseteq S^A_\delta \in J^A_\rho$ and, by Lemma 1.23, $a \in J^A_\rho$. Hence it suffices to prove the Claim.

Suppose f is unbounded in ρ. Define $G : \rho \to M$ by

$$G(\eta) = \text{ the } <^A_\alpha\text{-least function of } \gamma \text{ onto } \eta.$$

This is possible since ρ is a successor cardinal in M, so we can pick γ large enough that all we have done so far goes through. Clearly G is definable over J^A_ρ, hence $G \in M$. Now define $F : \gamma \times \gamma \xrightarrow{\text{onto}} \rho$ by

$$F(\xi, \eta) = G(f(\xi))(\eta).$$

Then $F \in M$. By Lemma 1.17, there is some surjection $g \colon \gamma \to \gamma \times \gamma$ which is $\mathbf{\Sigma}_1^{J^A_\gamma}$; hence $g \in M$. But then $F \circ g \in M$ witnesses that ρ is not a cardinal in M. This contradiction yields the Claim. $\dashv$

1.25 Corollary. *Let M, ρ be as in Lemma 1.24. Then* $J^A_\rho \models \mathrm{ZFC}^-$.

Proof. The point here is to verify the separation and replacement schemata in J^A_ρ, since the rest of the axioms hold in J^A_ρ automatically. The former follows from Lemma 1.23 and the latter from Lemma 1.24 in a straightforward way. $\dashv$

1.26 Corollary. *Let M be as above where $\rho > \omega$ is a limit cardinal in M. Then $J^A_\rho \models \mathrm{ZC}$ (where* ZC *is Zermelo set theory with choice).*

Proof. We only have to verify the power set axiom; the rest goes through as before. Let $a \in J^A_\rho$. Pick $\gamma < \rho$ such that γ is a cardinal in J^A_ρ and $a \in J^A_\gamma$. Then for every $x \in \mathcal{P}(a) \cap J^A_\rho$, $x \in J^A_\gamma$. Hence $\mathcal{P}(a) \cap J^A_\rho \in J^A_\rho$. $\dashv$

1.27 Corollary. *Let M, ρ be as above. Then*

$$|J^A_\rho| = H^M_\rho \overset{\text{def}}{=} \{x \in M\,;\, \mathrm{Card}\,(\mathrm{TC}(x)) < \rho \ \textit{ in } M\}.$$

Proof. Clearly $|J^A_\rho| \subseteq H^M_\rho$. So it is sufficient to prove the converse. Suppose it fails. Let ρ be the least counterexample. Then ρ is a successor cardinal in M. Let $x \in H^M_\rho$ be $\in$-minimal such that $x \notin J^A_\rho$. Then $x \subseteq J^A_\rho$. Since $\mathrm{card}(x) < \rho$ in M, $x \in J^A_\rho$ by Lemma 1.23. Contradiction. $\dashv$

2. The first projectum

We now introduce the central notions of fine structure theory – the notions *projectum, standard code* and *good parameter.* We stress that we are working with arbitrary J-structures and that these structures have, in general, very few closure properties. This means that there might be bounded definable subsets of these structures (in a precise sense) failing to be *elements.* However, each J-structure has an initial segment which is "firm" in the sense that it does contain all sets reasonably definable over the whole structure. The height of this "firm" segment is called the *projectum. Standard codes* are (boldface) definable relations computing truth and *good parameters* are parameters which occur in the definitions of standard codes.

2.1 Definition. The Σ_1-*projectum* (or, *first projectum*) $\rho(M)$ of an acceptable J-structure $M = J_\alpha^A$ is defined by

$$\rho(M) = \text{ the least } \rho \in \mathbf{On} \text{ such that } \mathcal{P}(\rho) \cap \mathbf{\Sigma}_1^M \not\subseteq M.$$

2.2 Lemma. *Let M be as above. If $\rho(M) \in M$, then $\rho(M)$ is a cardinal in M.*

Proof. Suppose not. Set $\rho = \rho(M)$. Let $f \in M$ be such that $f : \gamma \xrightarrow{\text{onto}} \rho$ for some $\gamma < \rho$ and $A \in \mathbf{\Sigma}_1^M$ be such that $a = A \cap \rho \notin M$. Let $\bar{a} = f^{-1}{}''a$. Then $\bar{a} \notin M$, since otherwise $a = f"\bar{a} \in M$. On the other hand, $\bar{a} \in M$ by the definition of ρ, since $\bar{a} \subseteq \gamma$ and $\bar{a} \in \mathbf{\Sigma}_1^M$. Contradiction. ⊣

2.3 Lemma. *Let M be as above and $\rho = \rho(M)$. Then ρ is a Σ_1-cardinal in M (i.e. there is no $\mathbf{\Sigma}_1^M$ partial map from some $\gamma < \rho$ onto ρ).*

Proof. Suppose there is such a map, say $f : \gamma \xrightarrow{\text{onto}} \rho$. We know that there is a $\mathbf{\Sigma}_1^{J_\rho^A}$ map of ρ onto J_ρ^A (cf. Lemma 1.17). Hence there is a $\mathbf{\Sigma}_1^M$ map $g : \gamma \xrightarrow{\text{onto}} J_\rho^A$. Define a set b by

$$\xi \in b \Longleftrightarrow \xi \notin g(\xi).$$

Then b is clearly $\mathbf{\Sigma}_1^M$ and $b \subseteq \gamma$. Moreover, $b \notin J_\rho^A$ by a diagonal argument: if $b \in J_\rho^A$ then $b = g(\xi_0)$ for some $\xi_0 < \gamma$ which would give $\xi_0 \in b = g(\xi_0)$ iff $\xi_0 \notin g(\xi_0)$. Hence $b \notin M$: this follows from Lemma 1.23 if $\rho \in M$ and is immediate otherwise. On the other hand, $\gamma < \rho$, and therefore $b \in M$. Contradiction! ⊣

Lemma 1.17 and Lemma 1.23 now immediately give the following.

2.4 Corollary. *Let M be acceptable and $\rho = \rho(M)$.*

(a) If $B \subseteq J_\rho^A$ is $\mathbf{\Sigma}_1^M$, then $\langle J_\rho^A, B\rangle$ is amenable.

(b) $|J_\rho^A| = H_\rho^M$.

Recall that we fixed a recursive enumeration $\langle \varphi_i; i < \omega \rangle$ of all Σ_1 formulae. In what follows it will be convenient to pretend that each φ_i has exactly *one* free variable. For instance, if $\varphi_i = \varphi_i(v_{i_1}, ..., v_{i_\ell})$ with all free variables shown then we might confuse φ_i with

$$\exists v_{i_1} ... \exists v_{i_\ell} (u = \langle v_{i_1}, ..., v_{i_\ell} \rangle \wedge \varphi_i(v_{i_1}, ..., v_{i_\ell})).$$

To make things even worse, we shall nevertheless frequently write $\varphi_i(x_1, ..., x_\ell)$ instead of $\varphi_i(\langle x_1, ..., x_\ell \rangle)$. If $\mathsf{a}\colon v(i) \to V$ assigns values to the free variable(s) $v_{i_1}, ..., v_{i_\ell}$ of φ_i then, setting $x_1 = \mathsf{a}(v_{i_1})$, ..., $x_\ell = \mathsf{a}(v_{i_\ell})$ we shall use the more suggestive $M \models \varphi_i(x_1, ..., x_\ell)$ rather than $\models_M^{\Sigma_1} \varphi_i[\mathsf{a}]$ in what follows. We shall also write $h_M(i, \vec{x})$ instead of $h_M(i, \mathsf{a})$.

2.5 Definition. Let $M = \langle J_\alpha^B, D \rangle$ be an acceptable J-structure, $\rho = \rho(M)$ and $p \in M$. We define

$$A_M^p = \{\langle i, x \rangle \in \omega \times H_\rho^M;\ M \models \varphi_i(x, p)\}.$$

A_M^p is called the *standard code* determined by p. Let us stress that A_M^p is the intersection of $\omega \times H_\rho^M$ with a set $\tilde{A}_M^p$ (defined in an obvious way) which is $\Sigma_1^M(\{p\})$. We shall often write $A_M^p(i, x)$ instead of $\langle i, x \rangle \in A_M^p$. The structure

$$M^p = \langle J_\rho^B, A_M^p \rangle$$

is called the *reduct* determined by p. If $\delta = \rho$ or $\delta < \rho$ where δ is a cardinal in M, we also set

$$\begin{array}{rcl} A_M^{p,\delta} & = & A^p \cap J_\delta^B \\ M^{p,\delta} & = & \langle J_\delta^B, A_M^{p,\delta} \rangle. \end{array}$$

We shall omit the subscript $_M$ whenever there is no danger of confusion.

2.6 Definition. Let M be acceptable and $\rho = \rho(M)$.

$$\begin{array}{rl} P_M = & \text{the set of all } p \in [\rho(M), \mathrm{On} \cap M)^{<\omega} \text{ for which} \\ & \text{there is a } B \in \Sigma_1^M(\{p\}) \text{ such that} \quad B \cap \rho \notin M. \end{array}$$

The elements of P_M are called *good parameters*.

2.7 Lemma. *Let M be as before, $p \in M$ and $A = A_M^p$. Then*

$$p \in P_M \leftrightarrow A \cap (\omega \times \rho(M)) \notin M.$$

Proof. ($\rightarrow$) Pick B which witnesses that $p \in P_M$. Suppose B is defined by φ_i. Hence $B(\xi) \leftrightarrow \langle i, \xi\rangle \in A$, which means that if $A \cap (\omega \times \rho(M))$ is in M, then so is $B \cap \rho(M)$. Hence the former is not an element of M.

($\leftarrow$) Suppose $A \cap (\omega \times \rho(M)) \notin M$. Let $f : \omega \times \rho(M) \longrightarrow \rho(M)$ be defined by $f(i, \omega\xi + j) = \omega\xi + 2^i \cdot 3^j$. Clearly f is Σ_1^M uniformly and if $\rho(M) \in M$ then $f \in M$. Let $B = f``A$. Then $B$ is $\Sigma_1^M(\{p\})$ and $B \cap \rho(M) \notin M$: if $\rho(M) \in M$, this follows from the fact that $f \in M$ and $A \cap (\omega \times \rho(M)) = f^{-1}``(B \cap \rho(M))$; if $\rho(M) \notin M$, it follows from the fact that B is cofinal in $\rho(M)$. $\dashv$

2.8 Definition. Let M be acceptable and $\rho = \rho(M)$. We set

$$R_M = \text{ the set of all } r \in [\rho(M), \mathrm{On} \cap M)^{<\omega} \text{such that } h_M(\rho \cup \{r\}) = |M|.$$

The elements of R_M are called *very good parameters*.

2.9 Lemma. $R_M \subseteq P_M \neq \emptyset$.

Proof. By the proof of Lemma 1.17, $h_M(\mathbf{On} \cap M) = M$ for any J-structure M. Given an acceptable structure M, if A is a relation which is $\Sigma_1^M(\{p\})$ for some $p \in M$ then A is therefore also $\Sigma_1^M(\{q\})$ for some $q \in [\mathbf{On} \cap M]^{<\omega}$. As $\rho(M)$ is closed under the Gödel pairing function (if $\rho(M) < \mathbf{On} \cap M$), the inequality easily follows. As to the inclusion, define the set $a \subseteq \omega \times \mathbf{On} \cap M$ by

$$\langle i, \xi\rangle \in a \Longleftrightarrow \langle i, \xi\rangle \notin h_M(i, \langle \xi, p\rangle)$$

for a $p \in R_M$. By a diagonal argument, $a \cap \omega \times \rho(M) \notin M$ and a is $\Sigma_1(M)$ in p. Using the map f from the proof of Lemma 2.7 it is easy to turn the set a into some $b \subseteq \mathbf{On} \cap M$ such that b is Σ_1^M in p and $b \cap \rho(M) \notin M$. $\dashv$

We remark that P_M and R_M are often defined differently so as to include arbitrary elements of M rather than just finite sequences of ordinals in the half-open interval $[\rho(M), \mathbf{On} \cap M)$.

3. Downward extension of embeddings

Given a Σ_0 preserving map between the reducts of two acceptable structures, the question naturally arises whether the map can be extended to a map between the original structures. It turns out that this is possible. The conjunction of the following three lemmas is called the *downward extension of embeddings lemma*.

3.1 Lemma. *Let $\pi : \bar{M}^{\bar{p}} \underset{\Sigma_0}{\longrightarrow} M^p$, where $\bar{p} \in R_{\bar{M}}$. Then there is a unique $\tilde{\pi} : \bar{M} \underset{\Sigma_0}{\longrightarrow} M$ such that $\tilde{\pi} \supseteq \pi$ and $\tilde{\pi}(\bar{p}) = p$. Moreover, $\tilde{\pi} : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$.*

Proof. Uniqueness: Assume that $\tilde{\pi}$ has the above properties. Let $x \in \bar{M}$. Then $x = h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)$ for some $i \in \omega$ and $\xi < \rho(M)$. Let $\bar{H}$ be $\Sigma_0^{\bar{M}}$ such that $\exists z \; \bar{H}(z, x, i, \xi, \bar{p})$ defines the Skolem function $h_{\bar{M}}$ (this involves a slight abuse of notation). $\bar{H}$ has a uniform definition, so let H have the same definition over M. Pick z such that $\bar{H}(z, x, i, \xi, \bar{p})$. Since $\tilde{\pi}$ is Σ_0 preserving, we have $H(\tilde{\pi}(z), \tilde{\pi}(x), i, \tilde{\pi}(\xi), p)$, i.e. $\tilde{\pi}(x) = h_M(i, \langle \tilde{\pi}(\xi), p \rangle) = h_M(i, \langle \pi(\xi), p \rangle)$. Hence, there is at most one such $\tilde{\pi}$.

Existence: The above proof of uniqueness suggests how to define the extension $\tilde{\pi}$. Here we show that such a definition is correct. We first observe:

Claim. Suppose that $\varphi(v_1, \ldots, v_\ell)$ is a Σ_1-formula. Let $\bar{x}_i = h_{\bar{M}}(j_i, \langle \bar{\xi}_i, \bar{p} \rangle)$ for some $j_i \in \omega$, $\bar{\xi}_i < \rho(M)$ and $x_i = h_M(j_i, \langle \xi_i, p \rangle)$ where $\xi_i = \pi(\bar{\xi}_i)$ $(i = 1, \ldots, \ell)$. Then

$$\bar{M} \models \varphi(\bar{x}_1, \ldots, \bar{x}_\ell) \quad \text{iff} \quad M \models \varphi(x_1, \ldots, x_\ell).$$

Proof. $\bar{M} \models \varphi(\bar{x}_1, \ldots, \bar{x}_\ell)$ is equivalent to

$$\bar{M} \models \varphi(h_{\bar{M}}(j_1, \langle \bar{\xi}_1, \bar{p} \rangle), \ldots, h_{\bar{M}}(j_\ell, \langle \bar{\xi}_\ell, \bar{p} \rangle)).$$

Since $h_{\bar{M}}$ has a uniform Σ_1-definition over $\bar{M}$, there is a Σ_1-formula ψ such that the above is equivalent to

$$\bar{M} \models \psi(\bar{\xi}_1, \ldots, \bar{\xi}_\ell, \bar{p}). \tag{I.4}$$

The formula ψ clearly does not depend on the actual structure in question, so the statement $M \models \varphi(x_1, \ldots, x_\ell)$ is equivalent to

$$M \models \psi(\xi_1, \ldots, \xi_\ell, p). \tag{I.5}$$

Now suppose $\psi(\xi_1, \ldots, \xi_\ell, p) \iff \varphi_k(\langle \xi_1, \ldots, \xi_\ell \rangle, p)$ in our fixed recursive enumeration of Σ_1 formulae, hence (I.4) is equivalent to

$$A_{\bar{M}}^{\bar{p}}(k, \langle \bar{\xi}_1, \ldots, \bar{\xi}_\ell \rangle) \tag{I.6}$$

and (I.5) is equivalent to

$$A_M^p(k, \langle \xi_1, \ldots, \xi_\ell \rangle). \tag{I.7}$$

Since π is Σ_0 preserving, (I.6) and (I.7) are equivalent. $\dashv$

Now define $\tilde{\pi}$ by

$$\tilde{\pi}(h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)) \simeq h_M(i, \langle \pi(\xi), p \rangle)$$

for $i \in \omega$ and $\xi < \rho(\bar{M})$. We have to verify several facts:

- $\tilde{\pi}$ is well defined.
  Let $h_{\bar{M}}(j_1, \langle \bar{\xi}_1, \bar{p} \rangle) = h_{\bar{M}}(j_2, \langle \bar{\xi}_2, \bar{p} \rangle)$ for some $j_1, j_2 \in \omega$ and $\bar{\xi}_1, \bar{\xi}_2 < \rho(\bar{M})$. We have to show that $h_M(j_1, \langle \xi_1, p \rangle) = h_M(j_2, \langle \xi_2, p \rangle)$, where $\xi_i = \pi(\bar{\xi}_i) \quad (i = 1, 2)$. By the above claim, this follows immediately.

- $\tilde{\pi}$ is Σ_1 preserving.
  Since $\tilde{\pi}$ is a well defined map, this follows immediately from the above claim.

- $\tilde{\pi} \supseteq \pi$.
  There is an $i \in \omega$ such that the equality $x = h_M(i, \langle x, q \rangle)$ holds uniformly and independently of q. In particular, for $\xi < \rho(\bar{M})$ we have $\xi = h_{\bar{M}}(i, \langle \xi, \bar{p} \rangle)$, so

$$\tilde{\pi}(\xi) = h_M(i, \langle \pi(\xi), p \rangle) = \pi(\xi).$$

- $\tilde{\pi}(\bar{p}) = p$.
  Similarly as above, there is an $i \in \omega$ such that $q = h(i, \langle x, q \rangle)$ uniformly. Hence $\bar{p} = h_{\bar{M}}(i, \langle 0, \bar{p} \rangle)$ and $\tilde{\pi}(\bar{p}) = h_M(i, \langle 0, p \rangle) = p$.

$\dashv$

3.2 Lemma. *Let $\bar{M}$, M, $\bar{p}$, p, π, $\tilde{\pi}$ be as above. Suppose moreover that $p \in R_M$. Let $\pi : \bar{M}^{\bar{p}} \underset{\Sigma_n}{\longrightarrow} M^p$. Then*

$$\tilde{\pi} : \bar{M} \underset{\Sigma_{n+1}}{\longrightarrow} M.$$

Proof. We shall proceed by induction. Suppose the lemma holds for n. Suppose we have a Σ_{n+1} formula φ which is of the form

$$\exists\, z_1\ \forall\, z_2 \ldots \exists/\forall\, z_{n+1}\ \bar{\varphi}(z_1, \ldots, z_{n+1}, x_1, \ldots, x_\ell), \tag{I.8}$$

where $\bar{\varphi}$ is Σ_0. For notational simplicity, assume that $n = 2$ and $\ell = 1$. Given a J-structure N and an arbitrary $q \in R_N$, the structure N satisfies (I.8) iff it satisfies the formula

$$\left\{\begin{array}{l} \exists \xi_1 < \rho(N)\ \exists i_1\ \forall \xi_2 < \rho(N)\ \forall i_2\ [\exists y\ y = h_N(i_2, \langle \xi_2, q \rangle) \implies \\ \exists y_1\ \exists y_2\ \exists x\ \exists z_3\ (y_1 = h_N(i_1, \langle \xi_1, q \rangle) \wedge y_2 = h_N(i_2, \langle \xi_2, q \rangle) \wedge \\ \qquad x = h_N(j_1, \langle \zeta_1, q \rangle) \wedge \bar{\varphi}(y_1, y_2, z_3, x)\ ] \end{array}\right. \tag{I.9}$$

where $x_1 = h_N(j_1, \langle \zeta_1, q \rangle)$. (For an arbitrary n, the analogous fact can be verified by a straightforward induction on n using that q is a very good parameter.) Notice that the matrix in (I.9) is of the form $\psi_1 \to \psi_2$, where both ψ_1 and ψ_2 are Σ_1 via a uniform transformation, i.e., there are k_1,

$k_2 \in \omega$ depending only on $\bar{\varphi}$ such that (I.9) can be expressed in a Σ_2-fashion over N^q in the following way:

$$\exists \xi_1\ \exists i_1\ \forall \xi_2\ \forall i_2\ [\ A^q_N(k_1, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle) \Longrightarrow A^q_N(k_2, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle)\ ].$$

Then, if in fact $x_1 = h_{\bar{M}}(j_1, \langle \zeta_1, \bar{p} \rangle)$,

$$\begin{aligned} \bar{M} \models \varphi(x_1) &\iff \\ \exists \xi_1\ \exists i_1\ \forall \xi_2\ \forall i_2\ [\ A^{\bar{p}}_{\bar{M}}(k_1, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle) &\Longrightarrow \\ A^{\bar{p}}_{\bar{M}}(k_2, \langle \xi_1, i_1, \xi_2, i_2, \zeta_1, j_1 \rangle)\ ] &\iff \\ \exists \xi_1\ \exists i_1\ \forall \xi_2\ \forall i_2\ [\ A^{p}_{M}(k_1, \langle \xi_1, i_1, \xi_2, i_2, \pi(\zeta_1), j_1 \rangle) &\Longrightarrow \\ A^{p}_{M}(k_2, \langle \xi_1, i_1, \xi_2, i_2, \pi(\zeta_1), j_1 \rangle)\ ]& \\ \iff M \models \varphi(\tilde{\pi}(x_1)).& \end{aligned}$$

Similar, but slightly more complicated reductions can be done for arbitrary n; we leave this to the reader. ⊣

3.3 Lemma. *Let $\pi : N \underset{\Sigma_0}{\longrightarrow} M^p$, where N is a J-structure and $p \in R_M$. Then there are unique $\bar{M}$, $\bar{p}$ such that $\bar{p} \in R_{\bar{M}}$ and $N = \bar{M}^p$.*

Proof. Let $M = \langle J^B_\alpha, D \rangle$, $M^p = \langle J^B_\rho, A \rangle$ and $N = \langle J^{\bar{B}'}_{\bar{\rho}}, \bar{A} \rangle$. (In particular, $A = A^p_M$.) Let also $X = \mathrm{rng}(\pi)$, $Y = h_M(X \cup \{p\})$, let $\bar{M}$ be the transitive collapse of Y and let $\tilde{\pi} : \bar{M} \to Y$ be the inverse of the Mostowski collapse. The map $\tilde{\pi}$ is clearly Σ_1-preserving, hence $\bar{M}$ is of the form $\langle J^{\bar{B}}_{\bar{\alpha}}, \bar{D} \rangle$. We shall show that $\bar{M}$ is the desired structure. We prove first

$$\tilde{\pi} \supseteq \pi \quad \text{and} \quad J^{\bar{B}'}_{\bar{\rho}} = J^{\bar{B}}_{\bar{\rho}}, \tag{I.10}$$

which follows immediately from

$$\text{if} \quad x \in X,\ y \in Y \text{ and } y \in x, \text{ then} \quad y \in X \tag{I.11}$$

since the latter tells us that the collapsing map for Y restricted to X coincides with π^{-1}.

So suppose $y = h_M(i, \langle z, p \rangle)$ for some $z \in X$. Since both $y, z \in M^p$, this can be equivalently expressed in the form $A(k, \langle y, z \rangle)$ for some $k \in \omega$. Thus we have

$$\exists\, v \in x\ A(k, \langle v, z \rangle)$$

which is preserved by a Σ_0 map, so

$$\exists\, v \in \bar{x}\ \bar{A}(k, \langle v, \bar{z} \rangle)$$

where $(\bar{x}, \bar{z}) = \pi^{-1}(x, z)$. Let $\bar{y} \in \bar{x}$ be such that $A(k, \langle \bar{y}, \bar{z} \rangle)$. Such a $\bar{y}$ is uniquely determined: if $\bar{y}_1$ were another one, we would have

$$A(k, \langle \pi(\bar{y}), z \rangle)\ \wedge\ A(k, \langle \pi(\bar{y}_1), z \rangle),$$

which means that $\pi(\bar{y}) = h_M(i, \langle z, p\rangle) = \pi(\bar{y}_1)$, hence $\bar{y} = \bar{y}_1$. This argument also shows $\pi(\bar{y}) = y$. Hence $y \in X$.

We have shown I.10 and I.11.

Now let

$$\bar{p} = \tilde{\pi}^{-1}(p).$$

Note that there is a $\mathbf{\Sigma}_1^{\bar{M}}$ map of $\bar{\rho}$ onto $\bar{M}$; this follows immediately from the fact that $\bar{M} = h_{\bar{M}}(J_{\bar{\rho}}^{\bar{B}} \cup \{\bar{p}\})$ and that there is a $\mathbf{\Sigma}_1^{\bar{M}}$ map of $\bar{\rho}$ onto $J_{\bar{\rho}}^{\bar{B}}$. So we have

$$\rho(\bar{M}) \leq \bar{\rho}. \tag{I.12}$$

Note also that if $i \in \omega$ and $x \in N$, then

$$\bar{A}(i, x) \iff A(i, \tilde{\pi}(x)) \iff M \models \varphi_i(\tilde{\pi}(x), p) \iff \bar{M} \models \varphi_i(x, \bar{p}) \tag{I.13}$$

where $\langle \varphi_i \rangle_i$ is the fixed recursive enumeration of the Σ_1-formulae. Our aim is to show

$$\rho(\bar{M}) = \bar{\rho} \tag{I.14}$$

which reduces to proving the inequality $\bar{\rho} \leq \rho(\bar{M})$. Let P be $\mathbf{\Sigma}_1^{\bar{M}}(\{\bar{q}\})$. By the fact that $h_{\bar{M}}(J_{\bar{\rho}}^{\bar{B}} \cup \{\bar{p}\}) = \bar{M}$ we can find an R which is $\Sigma_1^{\bar{M}}$ and such that

$$P(z) \iff R(z, x, \bar{p})$$

for some fixed $x \in N$. By (I.13), there is an $i \in \omega$ such that

$$P(z) \iff \bar{A}(i, \langle z, x\rangle).$$

Hence if $\gamma < \bar{\rho}$ then $P \cap \gamma$ is a projection of $\bar{A} \cap (\{i\} \times \gamma \times \{x\}) \in N \subseteq \bar{M}$, thus it is itself in $\bar{M}$. This proves (I.14).

As an immediate consequence of (I.13) and (I.14) we get

$$\bar{A} = A_{\bar{M}}^{\bar{p}}. \tag{I.15}$$

It only remains to prove that

$$\bar{p} \in R_{\bar{M}}. \tag{I.16}$$

If $\bar{M} = N$ this is trivial. Otherwise there is a Σ_1^M map of $\bar{\rho}$ onto $J_{\bar{\rho}}^{\bar{B}}$. Since $h_{\bar{M}}(J_{\bar{\rho}}^{\bar{B}} \cup \{\bar{p}\}) = \bar{M}$, the proof is complete. $\dashv$

4. Upward extension of embeddings

In this section we present a method which gives a solution to the problem dual to that from the previous section, where we formed the extension of an embedding from the reduct of a J-structure to the whole structure: namely, we now aim to build a target model which can serve as the codomain of an extended embedding. This problem is a bit more delicate than the previous one, since such an extension need not always exist; therefore we have to strengthen our requirements on the embeddings we intend to extend. The difference between forming downward and upward extensions lies in the fact that the former ones are related to taking hulls and collapsing them, which is always possible, whilst the latter ones are related to forming ultrapowers, which have transitive isomorphs only if they are well-founded.

4.1 Definition. $\pi : \bar{M} \to M$ is a *strong embedding* iff

a) $\pi : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$

b) For any $\bar{R}$, R such that $\bar{R}$ is rudimentary over $\bar{M}$ and R is rudimentary over M by the same definition the following holds:

If $\bar{R}$ is well-founded, then so is R.

The *upward extension of embeddings lemma* is the conjunction of the following lemma together with Lemmata 3.1 and 3.2.

4.2 Lemma. *Let $\pi : \bar{M}^{\bar{p}} \to N$ be a strong embedding, where N is acceptable and $\bar{p} \in R_{\bar{M}}$. Then there are unique M, p such that $N = M^p$ and $p \in R_M$. Moreover, $\tilde{\pi}$ is strong, where $\tilde{\pi} \supseteq \pi$, $\tilde{\pi} : \bar{M} \underset{\Sigma_1}{\longrightarrow} M$ and $\tilde{\pi}(\bar{p}) = p$.*

Proof. Uniqueness. Suppose $\tilde{\pi}_1 : \bar{M} \to M_1$ and $\tilde{\pi}_2 : \bar{M} \to M_2$ are two extensions of π satisfying the conclusions of the lemma and that p_1, p_2 are the corresponding parameters. Then $A^{p_1}_{M_1} = A^{p_2}_{M_2}$, call it A, and every $x \in M_k$ is of the form $h_{M_k}(i, \langle \xi, p_k \rangle)$ for some $i \in \omega$ and $\xi \in \mathrm{On} \cap N$ $(k = 1, 2)$. Let $\sigma : M_1 \to M_2$ be the map sending $h_{M_1}(i, \langle \xi, p_1 \rangle)$ to $h_{M_2}(i, \langle \xi, p_2 \rangle)$. Then σ is well defined since

$$\exists z \; z = h_{M_1}(i, \langle \xi, p_1 \rangle) \iff A(j, \langle i, \xi \rangle) \iff \exists z \; z = h_{M_2}(i, \langle \xi, p_2 \rangle) \qquad \text{(I.17)}$$

for an appropriate j (i.e., j is such that $\exists z \; z = h_{M_k}(i, \langle \xi, p_k \rangle)$ can be expressed as $M_k \models \varphi_j(\langle i, \xi \rangle)$ for $k = 1, 2$). Also, σ is Σ_1-preserving, since given any Σ_1-formula $\psi(v_1, \ldots, v_\ell)$ and $x_s = h_{M_1}(i_s, \langle \xi_s, p_1 \rangle)$ $(s = 1, \ldots, \ell)$,

$$\begin{aligned} M_1 \models \psi(x_1, \ldots, x_\ell) &\iff M_1 \models \psi(h_{M_1}(i_1, \langle \xi_1, p_1 \rangle), \ldots, h_{M_1}(i_\ell, \langle \xi_\ell, p_1 \rangle)) \\ &\iff A(j, \langle \langle i_1, \xi_1 \rangle, \ldots, \langle i_\ell, \xi_\ell \rangle \rangle) \\ &\iff M_2 \models \psi(h_{M_2}(i_1, \langle \xi_1, p_2 \rangle), \ldots, h_{M_2}(i_\ell, \langle \xi_\ell, p_2 \rangle)) \\ &\iff M_2 \models \psi(\sigma(x_1), \ldots, \sigma(x_\ell)) \end{aligned}$$

for a suitable j (which only depends on ψ). It is then easy to see that σ is structure preserving and $\sigma \circ \tilde{\pi}_1 = \tilde{\pi}_2$. Furthemore, (I.17) implies that $\mathrm{ran}(\sigma) = M_2$. Thus, $M_1 = M_2$, $\tilde{\pi}_1 = \tilde{\pi}_2$ and $p_1 = p_2$.

Existence. The idea of the construction is simple: using the fact that $\bar{p} \in R_{\bar{M}}$, we encode the whole structure $\bar{M}$ and its satisfaction relation in a rudimentary fashion over $\bar{M}^{\bar{p}}$. The preservation properties of π then guarantee that the corresponding relations with the same rudimentary definitions over N encode the required structure M; the process of decoding will also yield the extension $\tilde{\pi}$. However, the verification of all details is somewhat technical.

Suppose $\bar{M} = \langle J^{\bar{B}}_{\bar{\alpha}}, \bar{D}\rangle$. Let $\bar{k}(\langle i,z\rangle) \simeq h_{\bar{M}}(i, \langle z, \bar{p}\rangle)$ and $\bar{d} = \mathrm{dom}(\bar{k})$. Then membership in $\bar{d}$ is expressible by a Σ_1-statement over $\bar{M}$ in $\bar{p}$ as

$$x \in \bar{d} \Longleftrightarrow \exists i\ \exists z\ (i \in \omega\ \wedge\ x = \langle i, z\rangle\ \wedge\ \exists y\ y = h_{\bar{M}}(i, \langle z, \bar{p}\rangle)).$$

So there is an $i_0 \in \omega$ such that for every $x \in \bar{M}^{\bar{p}}$ we have $x \in \bar{d}$ iff $A^{\bar{p}}_{\bar{M}}(i_0, x)$. Note that the latter is a rudimentary relation over $\bar{M}^{\bar{p}}$. Similarly, the identity and membership relations as well as the membership in $\bar{B}$ and $\bar{D}$ can be expressed in a Σ_1-fashion over $\bar{M}$ in $\bar{p}$, and therefore in a rudimentary fashion over $\bar{M}^{\bar{p}}$. More precisely, we introduce relations $\bar{I}, \bar{E}, \bar{B}^*$ and $\bar{D}^*$ over $\bar{d}$ as follows

$$\begin{array}{rcl} x\bar{I}y & \Longleftrightarrow & \bar{k}(x) = \bar{k}(y) \\ x\bar{E}y & \Longleftrightarrow & \bar{k}(x) \in \bar{k}(y) \\ \bar{B}^*(x) & \Longleftrightarrow & \bar{B}(\bar{k}(x)) \\ \bar{D}^*(x) & \Longleftrightarrow & \bar{D}(\bar{k}(x)) \end{array}$$

and set

$$\bar{\mathbb{D}} = \langle \bar{d}, \bar{I}, \bar{E}, \bar{B}^*, \bar{D}^*\rangle.$$

The symbol for $=$ is interpreted in $\bar{\mathbb{D}}$ as $\bar{I}$, the symbol for $\in$ as $\bar{E}$, and the symbols for $\bar{B}, \bar{D}$ as $\bar{B}^*$ and $\bar{D}^*$, respectively. Thus $\bar{\mathbb{D}}$ encodes the structure $\bar{M}$: $\bar{I}$ is a congruence relation on $\bar{\mathbb{D}}$, $\bar{E}$ represents the membership relation and $\bar{k}$ is the Mostowski collapsing isomorphism between $\bar{\mathbb{D}}/\bar{I}$ and $\bar{M}$. We denote the Σ_1-satisfaction relation for $\bar{\mathbb{D}}$ by $\bar{T}$. More precisely, for $x_1, \ldots, x_\ell \in \bar{d}$ and $i \in \omega$ we have

$$\bar{T}(i, \langle x_1, \ldots, x_\ell\rangle) \Longleftrightarrow \bar{\mathbb{D}} \models \varphi_i(x_1, \ldots, x_\ell)$$

where $\langle \varphi_i;\ i \in \omega\rangle$ is our fixed recursive enumeration of Σ_1 formulae (remember that $\langle x_1, \ldots, x_\ell\rangle = \langle x_1, \langle x_2 \ldots, x_\ell\rangle\rangle$ and that we write $\varphi_i(x_1, \ldots, x_\ell)$ instead of $\varphi_i(\langle x_1, \ldots, x_\ell\rangle)$). One can easily show the following equivalence by induction:

$$\bar{T}(i, \langle x_1, \ldots, x_\ell\rangle) \Longleftrightarrow \bar{M} \models \varphi_i(\bar{k}(x_1), \ldots, \bar{k}(x_\ell)).$$

For atomic formulae this follows immediately from the definitions of the relations $\bar{I}, \bar{E}, \bar{B}^*$ and $\bar{D}^*$. To illustrate how the induction steps go we show the induction step for the formula $\varphi_j(v)$ of the form $\exists z \in \mathrm{left}(v)\ \varphi_i(z,v)$ (recall that $\mathrm{left}(v) = v_1$ and $\mathrm{right}(v) = v_2$ if $v = \langle v_1, v_2\rangle$ and undefined otherwise). Then

$$\begin{aligned} \bar{\mathbb{D}} \models \varphi_j(x_1,\dots,x_\ell) &\iff \exists w \in \bar{d}\ [w\bar{E}x_1 \ \wedge\ \bar{\mathbb{D}} \models \varphi_i(w,x_1,x_2,\dots,x_\ell)] \\ &\iff \exists w\ [\bar{k}(w) \in \bar{k}(x_1)\ \wedge \\ &\qquad \wedge\ \bar{M} \models \varphi_i(\bar{k}(w),\bar{k}(x_1),\bar{k}(x_2),\dots,\bar{k}(x_\ell))] \\ &\iff \bar{M} \models \varphi_j(\bar{k}(x_1),\dots,\bar{k}(x_\ell)). \end{aligned}$$

The middle equivalence follows by the induction hypothesis, the last one by the fact that if there is a z witnessing the bottom formula, then such a z is always of the form $\bar{k}(w)$ for some w.

Let d, I, E, B^* and D^* be rudimentary over N by the same rudimentary definitions as their counterparts over $\bar{M}^{\bar{p}}$. It follows from the above that $\bar{T}$ is $\Sigma_1^{\bar{M}}$ in $\bar{p}$ and therefore rudimentary over $\bar{M}^{\bar{p}}$. Let

$$\mathbb{D} = \langle d, I, E, B^*, D^*\rangle$$

and T be a relation which is rudimentary over N by the same rudimentary definition as $\bar{T}$. We show that T is the Σ_1-satisfaction predicate for $\mathbb{D}$. Strictly speaking, we must show that the following equivalences hold, where $\varphi_i(v)$ has the form indicated on the left hand side:

$$\begin{array}{lrcl}
\mathrm{left}(v) = \mathrm{right}(v) & T(i,\langle x,y\rangle) & \iff & xIy \\
\mathrm{left}(v) \in \mathrm{right}(v) & T(i,\langle x,y\rangle) & \iff & xEy \\
B(v) & T(i,x) & \iff & B^*(x) \\
D(v) & T(i,x) & \iff & D^*(x) \\
 & & & \\
\varphi_{i_1}(v)\ \wedge\ \varphi_{i_2}(v) & T(i,\vec{x}) & \iff & T(i_1,\vec{x})\ \wedge\ T(i_2,\vec{x}) \\
\neg\varphi_j(v) & T(i,\vec{x}) & \iff & \neg T(j,\vec{x}) \\
 & & & \\
\exists z \in \mathrm{left}(v)\ \varphi_j(z,v) & T(i,\langle x,\vec{y}\rangle) & \iff & \exists z \in d\ (zEx\ \wedge\ T(j,\langle z,x,\vec{y}\rangle)) \\
\forall z \in \mathrm{left}(v)\ \varphi_j(z,v) & T(i,\langle x,\vec{y}\rangle) & \iff & \forall z \in d\ (zEx\ \wedge\ T(j,\langle z,x,\vec{y}\rangle)) \\
 & & & \\
\exists z\ \varphi_j(z,v) & T(i,\vec{x}) & \iff & \exists z \in d\ T(j,\langle z,\vec{x}\rangle).
\end{array}$$

Here $\vec{x}$ stands for $\langle x_1,\dots,x_\ell\rangle$. We again proceed by induction on formulae. Suppose first that φ_i is an atomic formula, say the formula $\mathrm{left}(v) = \mathrm{right}(v)$. Then

$$\forall x, y \in \bar{d}\ (\bar{T}(i,\langle x,y\rangle) \iff x\bar{I}y).$$

This is a Π_1 statement over $\bar{M}^{\bar{p}}$, since the predicates $\bar{d}, \bar{T}$ and $\bar{I}$ are rudimentary. Since π is Σ_1 preserving, $T(i,\langle x,y\rangle)$ iff xIy iff $\mathbb{D} \models \varphi_i(x,y)$ for all $x, y \in d$.

Now suppose $\varphi_i(v)$ is the formula $\varphi_{i_1}(v) \wedge \varphi_{i_2}(v)$. Then

$$\forall x_1, \ldots, x_\ell \in \bar{d}\,[\bar{T}(i, \langle x_1, \ldots, x_\ell \rangle) \Longleftrightarrow \bar{T}(i_1, \langle x_1, \ldots, x_\ell \rangle) \wedge \bar{T}(i_2, \langle x_1, \ldots, x_\ell \rangle)].$$

This is again a Π_1-statement over $\bar{M}^{\bar{p}}$, so we obtain

$$\begin{aligned} T(i, \langle x_1, \ldots, x_\ell \rangle) &\iff T(i_1, \langle x_1, \ldots, x_\ell \rangle) \ \wedge \ T(i_2, \langle x_1, \ldots, x_\ell \rangle) \\ &\iff \mathbb{D} \models \varphi_{i_1}(x_1, \ldots, x_\ell) \ \wedge \ \varphi_{i_2}(x_1, \ldots, x_\ell) \\ &\iff \mathbb{D} \models \varphi_i(x_1, \ldots, x_\ell); \end{aligned}$$

the second equivalence follows from the induction hypothesis. We proceed similarly if $\varphi_i(v)$ is of the form $\neg\varphi_j(v)$.

Finally, suppose that $\varphi_i(v)$ introduces a quantifier; say $\varphi_i(v)$ is of the form $\exists z \in \mathrm{left}(v)\ \varphi_j(z, v)$. The implication ($\Longleftarrow$) follows easily: Since $\bar{T}$ is a satisfaction relation for $\bar{\mathbb{D}}$, we have

$$\forall x, y\ [\exists z\ (z\bar{E}x \ \wedge \ \bar{T}(j, \langle z, x, y \rangle)) \implies \bar{T}(i, \langle x, y \rangle)].$$

This is a Π_1-statement over $\bar{M}^{\bar{p}}$ and is therefore preserved upwards by π. To see the converse, let $\bar{g}$ be a $\Sigma_1^{\bar{M}}$-function in $\bar{p}$ uniformizing the relation

$$z \in \bar{k}(x) \ \wedge \ \varphi_j(z, \bar{k}(x), \bar{k}(y)).$$

Then $\bar{g}(u) \simeq h_{\bar{M}}(m, \langle u, \bar{p} \rangle) \simeq \bar{k}(\langle m, u \rangle)$ for an appropriate $m \in \omega$, hence

$$\begin{aligned} \varphi_i(\bar{k}(x), \bar{k}(y)) \implies & \langle m, x, y \rangle \in \mathrm{dom}(\bar{k}) \ \wedge \ \bar{k}(\langle m, x, y \rangle) \in \bar{k}(x) \ \wedge \\ & \wedge \ \varphi_j(\bar{k}(\langle m, x, y \rangle), \bar{k}(x), \bar{k}(y)) \end{aligned}$$

holds in $\bar{\mathbb{D}}$ for all x, y in $\bar{M}^{\bar{p}}$. Translating this into the language of $\bar{\mathbb{D}}$ we obtain

$$\forall x, y\ [\bar{T}(i, \langle x, y \rangle) \implies \langle m, x, y \rangle \in \bar{d} \ \wedge \ \langle m, x, y \rangle \bar{E} x \ \wedge \ \bar{T}(j, \langle \langle m, x, y \rangle, x, y \rangle)],$$

which is again a Π_1-statement over $\bar{M}^{\bar{p}}$. The required implication for $\mathbb{D}$ then follows immediately.

One consequence of the fact that T is a satisfaction relation for $\mathbb{D}$ is:

$$\pi : \bar{\mathbb{D}} \xrightarrow[\Sigma_2]{} \mathbb{D},$$

as follows immediately by the fact that $\bar{T}, T$ are rudimentary over $\bar{M}^{\bar{p}}, N$ respectively by the same rudimentary definition. This implies that I is a congruence relation on $\mathbb{D}$ and E is extensional modulo this congruence relation; the map π simply carries both properties from $\bar{\mathbb{D}}$ over to $\mathbb{D}$ (extensionality being Π_2). Note also that $\bar{E}$ is well-founded modulo the congruence relation $\bar{I}$ (in other words, the relation $x\bar{E}y \ \wedge \ \neg(x\bar{I}y)$ is well-founded). Hence E is well-founded modulo the congruence relation I by the strongness of π.

Let M be the transitive collapse of $\mathbb{D}$ and k be the collapsing map. Define $\tilde{\pi} : \bar{M} \to M$ by

$$\tilde{\pi}(\bar{k}(x)) = k(\pi(x)) \quad \text{for all } x \in \bar{d}.$$

It follows immediately that

$$\tilde{\pi} : \bar{M} \underset{\Sigma_2}{\longrightarrow} M.$$

In the following we show that M has all the required properties. Note first that M is a J-structure, say $M = \langle J_\alpha^B, D\rangle$. For the rest of the proof fix $i, i^* \in \omega$ so that

$$\begin{array}{rcl} x & = & h_Q(i, \langle x, q\rangle) \quad \text{holds uniformly over any } J\text{-structure } Q \\ \bar{p} & = & h_{\bar{M}}(i^*, \langle 0, \bar{p}\rangle). \end{array}$$

Then

$$x = \bar{k}(\langle i, x\rangle) \text{ for all } x \in \bar{M}^{\bar{p}} \text{ and } \bar{p} = \bar{k}(\langle i^*, 0\rangle).$$

We first observe that $|N| \subseteq M$. Given any $x \in \bar{M}^{\bar{p}}$, $\bar{k}(\langle j, z\rangle) \in x = \bar{k}(\langle i, x\rangle)$ iff $\bar{k}(\langle j, z\rangle) = w = \bar{k}(\langle i, w\rangle)$ for some $w \in x$; in other words,

$$\langle j, z\rangle \in \bar{d} \implies [\langle j, z\rangle \bar{E} \langle i, x\rangle \iff \exists w \in x \ (\langle j, z\rangle \bar{I} \langle i, w\rangle)]$$

holds for all $x, y \in \bar{M}^{\bar{p}}$ and $j \in \omega$. Quantifying over $\bar{M}^{\bar{p}}$ we obtain a Π_1-statement, which is preserved under π. This allows us to conclude: Whenever $x \in N$ (note: $\langle i, x\rangle \in d$ by the fact that π is Σ_1-preserving and the "same" holds for $\bar{M}^{\bar{p}}$),

$$k(\langle i, x\rangle) = \{k(\langle i, w\rangle);\ w \in x\}.$$

It follows by $\in$-induction that $k(\langle i, x\rangle) = x$ for all $x \in N$. Furthermore, given any $x \in \bar{M}^{\bar{p}}$,

$$\tilde{\pi}(x) = \tilde{\pi}(\bar{k}(\langle i, x\rangle)) = k(\pi(\langle i, x\rangle)) = k(\langle i, \pi(x)\rangle) = \pi(x),$$

and hence $\tilde{\pi} \supseteq \pi$.

Let B', A and ρ be such that $N = \langle J_\rho^{B'}, A\rangle$ (recall that, by our assumption, N is an acceptable structure). Set $p = k(\langle i^*, 0\rangle)$. It is easy to check that $\tilde{\pi}(\bar{p}) = p$. We now prove that $A = A_M^p$. Given an $x \in \bar{M}^{\bar{p}}$ and $j \in \omega$, we have $A_{\bar{M}}^{\bar{p}}(j, x) \iff \bar{M} \models \varphi_j(x, \bar{p}) \iff \bar{M} \models \varphi_j(\bar{k}(\langle i, x\rangle), \bar{k}(\langle i^*, 0\rangle)) \iff \bar{T}(j, \langle\langle i, x\rangle, \langle i^*, 0\rangle\rangle)$. Hence, the Π_1-statement

$$\forall x\ \forall j \in \omega\ [A_{\bar{M}}^{\bar{p}}(j, x) \iff \bar{T}(j, \langle\langle i, x\rangle, \langle i^*, 0\rangle\rangle)]$$

is preserved by π, which means that

$$A(j, x) \iff T(j, \langle\langle i, x\rangle, \langle i^*, 0\rangle\rangle) \iff M \models \varphi_j(x, p) \tag{I.18}$$

for every $x \in N$ and $j \in \omega$.

This equivalence easily yields that $B' = B \cap N$ and $N = \langle J^B_\rho, A\rangle$ for ρ as above. Pick a j such that $\varphi_j(u,v)$ is the formula "$B^*(u)$" for any acceptable structure of the form $\langle J^{B^*}_{\alpha^*}, D^*\rangle$. Since $\bar{B}(x) \iff A^{\bar{p}}_{\bar{M}}(j,x)$ holds for every $x \in \bar{M}^{\bar{p}}$, the preservation properties of π guarantee that $A(j,x) \iff B'(x)$. However, $A(j,x)$ is equivalent to $B(x)$ for all $x \in N$, as follows from (I.18).

To see that $A = A^p_M$ it suffices to show that $\rho = \rho(M)$, as $|N| = H^M_\rho$ then follows immediately (recall that $\rho = \mathrm{On} \cap N$). The computation below will also yield that $p \in R_M$. Notice that

$$\bar{k}(\langle j,x\rangle) = h_{\bar{M}}(j,\langle x,\bar{p}\rangle) = h_{\bar{M}}(\bar{k}(\langle i,j\rangle),\langle \bar{k}(\langle i,x\rangle),\bar{k}(\langle i^*,0\rangle)\rangle),$$

where the left equality is simply the definition of $\bar{k}$. Leaving out the middle term we obtain a Σ_1-statement, so it can be represented by some $m \in \omega$. More precisely, $\bar{T}(m,\langle\langle j,x\rangle,\langle i,j\rangle,\langle i,x\rangle,\langle i^*,0\rangle\rangle)$ holds for all $x \in \bar{M}^{\bar{p}}$ and $j \in \omega$. As above we apply π to obtain the corresponding statement for T and all $x \in N$. Using the fact that the Σ_1-Skolem functions have uniform definitions we infer that $k(\langle j,x\rangle) = h_M(k(\langle i,j\rangle,\langle k(\langle i,x\rangle),k(\langle i^*,0\rangle)\rangle)) = h_M(j,\langle x,p\rangle)$. Note that there is a *lightface* Σ^M_1 map[7] from ρ onto $|N|$; since the values $k(\langle j,x\rangle)$ range over all of M, we have

$$M = h_M(\rho \cup \{p\}) \quad \text{and} \quad \rho(M) \leq \rho.$$

The latter is obviously a consequence of the former. On the other hand, given any $r \in M$ we can pick a $\xi \in \mathrm{On} \cap N$ such that $r = h_M(j,\langle \xi,p\rangle)$ for some j. For any Σ_1 formula ψ, $M \models \psi(\eta,r)$ iff $M \models \varphi_m(\eta,\xi,p)$ for a suitable m; the latter is equivalent to $A(m,\langle\eta,\xi\rangle)$ by (I.18). Taken together, $M \models \psi(\eta,r)$ can be expressed in a rudimentary fashion over N. Since N is amenable, every bounded $\mathbf{\Sigma}^M_1$ subset of ρ is an element of N, which means that $\rho \leq \rho(M)$. Thus,

$$\rho = \rho(M) \quad \text{and} \quad p \in R_M.$$

It only remains to show that $\tilde{\pi}$ is strong. Let $\bar{R}, R$ be binary relations which are rudimentary over $\bar{M}, M$ respectively by the same rudimentary definition. Define $\bar{R}^*, R^*$ as follows

$$\begin{aligned} x\bar{R}^*y &\iff x,y \in \bar{d} \wedge \bar{k}(x)\bar{R}\bar{k}(y) \\ xR^*y &\iff x,y \in d \wedge k(x)Rk(y). \end{aligned}$$

Then $\bar{R}^*$ is well-founded since $\bar{R}$ is – any decreasing chain $x_0, x_1, \ldots, x_n, \ldots$ in $\bar{R}^*$ yields a decreasing chain $\bar{k}(x_0), \bar{k}(x_1), \ldots, \bar{k}(x_n), \ldots$ in $\bar{R}$, hence no such chain can be infinite. Furthermore, $\bar{R}^*, R^*$ are rudimentary over $\bar{M}^{\bar{p}}, N$

[7] "Lightface" means that no parameters are needed in the definition of such a function.

respectively by the same rudimentary definition. As π is strong, R^* must be well-founded. Hence R must be well-founded as well, since every infinite decreasing chain $x_0, x_1, \ldots, x_n, \ldots$ in R is of the form $k(z_0), k(z_1), \ldots, k(z_n), \ldots$ for some $z_0, z_1, \ldots, z_n, \ldots$ and the latter would be an infinite decreasing chain in R^*. $\dashv$

5. Iterated projecta

In this chapter we shall show how to iterate the process of defining a projectum and forming a standard code, and we shall introduce the notion of n-th projectum, n-th standard code, and n-th reduct.

5.1 Definition. Let $M = \langle J_\beta^B, D\rangle$ be an acceptable J-structure. For $n < \omega$ we recursively define the *n-th projectum* $\rho_n(M)$, the *n-th standard code* $A_M^{n,p}$ and the *n-th reduct* $M^{n,p}$ as follows:

$$\begin{aligned}
&\rho_0(M) = \beta, \qquad \Gamma_M^0 = \{\emptyset\}, \qquad A_M^{0,\emptyset} = \emptyset, \qquad M^{0,\emptyset} = M,\\
&\rho_{n+1}(M) = \min\{\rho(M^{n,p});\, p \in \Gamma_M^n\},\\
&\Gamma_M^{n+1} = \prod_{i\in n+1} [\rho_{i+1}(M), \rho_i(M))^{<\omega},
\end{aligned}$$

and for $p \in \Gamma_M^{n+1}$,

$$\begin{aligned}
A_M^{n+1,p} &= A_{M^{n,p\restriction n}}^{p(n),\rho_{n+1}(M)} \text{ and}\\
M^{n+1,p} &= (M^{n,p\restriction n})^{p(n),\rho_{n+1}(M)}.
\end{aligned}$$

We also set $\rho_\omega(M) = \min\{\rho_n(M); n < \omega\}$. The ordinal $\rho_\omega(M)$ is called the *ultimate projectum* of M.

The reader will gladly verify that $\rho_1(M) = \rho(M)$. On the other hand, if M is not 1-sound (cf. Definition 5.7 below) then it need not be the case that $\rho_2(M)$ is the least ρ such that $\mathcal{P}(\rho) \cap \mathbf{\Sigma}_2^M \not\subseteq M$.

Supposing that we know $\rho_n(M) \le \cdots \le \rho_1(M)$ we may identify $p = \langle p(0), \ldots, p(n)\rangle \in \Gamma_M^{n+1}$ with the (finite) set $\bigcup \mathrm{ran}(p)$ of ordinals; this will play a rôle in the next section.

5.2 Definition. We define $P_M^n, R_M^n \subseteq \Gamma_M^n$ as follows:

$$\begin{aligned}
&P_M^0 = \{\emptyset\}\\
&P_M^{n+1} = \{p \in \Gamma_M^{n+1};\, p \restriction n \in P_M^n \ \wedge\\
&\rho(M^{n,p\restriction n}) = \rho_{n+1}(M) \ \wedge\ p(n) \in P_{M^{n,p\restriction n}}\}
\end{aligned}$$

R_M^n is defined in the same way but with $R_{M^{n,p\restriction n}}$ in place of $P_{M^{n,p\restriction n}}$.

As before, we call the elements of P_M^n good parameters and the elements of R_M^n very good parameters.

5.3 Lemma. *Let M be acceptable.*

(a) $R_M^n \subseteq P_M^n \neq \emptyset$

(b) Let $p \in R_M^n$. If $q \in \Gamma_M^n$ then $A^{n,q}$ is $\mathrm{rud}_{M^{n,p}}$ in parameters from $M^{n,p}$.

(c) Let $p \in R_M^n$. Then $\rho(M^{n,p}) = \rho_{n+1}(M)$.

(d) $p \in P_M^n \Longrightarrow \forall\, i \in n\; p(i) \in P_{M^{i,p\restriction i}}$, and similarly for R_M^n. Moreover, if $p \restriction (n-1) \in R_M^{n-1}$ then equivalence holds.

Proof. (a) This is easily shown inductively by using Lemma 2.9 and by amalgamating parameters.

(b) By induction on $n < \omega$. The case $n = 0$ is trivial. Now let $n > 0$, and suppose (b) holds for $n-1$. Write $m = n-1$. Let $p \in R_M^n$ and $q \in \Gamma_M^n$. We have to show that $A_{M^{m,q\restriction m}}^{q(m),\rho_n(M)}$ is $\mathrm{rud}_{M^{n,p}}$ in parameters from $M^{n,p}$. Inductively, $M^{m,q\restriction m}$ is $\mathrm{rud}_{M^{m,p\restriction m}}$ in a parameter $t \in M^{n,p}$. As $p(m) \in R_{M^{m,p\restriction m}}$, there are e_0 and e_1 and $z \in M^{n,p}$ such that

$$q(m) = h_{M^{m,p\restriction m}}(e_0, \langle z, p(m)\rangle)$$

and

$$t = h_{M^{m,p\restriction m}}(e_1, \langle z, p(m)\rangle).$$

For $i < \omega$ and $x \in M^{n,p}$, we have that

$$\begin{aligned}
\langle i, x\rangle \in A_{M^{m,q\restriction m}}^{q(m),\rho_n(M)} \quad &\Longleftrightarrow \quad M^{m,q\restriction m} \models \varphi_i(x, q(m)) \\
&\Longleftrightarrow \quad M^{m,q\restriction m} \models \varphi_i(x, h_{M^{m,p\restriction m}}(e_0, \langle z, p(m)\rangle)) \\
&\Longleftrightarrow \quad M^{m,p\restriction m} \models \varphi_j(\langle x, z\rangle, p(m)) \\
&\Longleftrightarrow \quad \langle j, \langle x, z\rangle\rangle \in A_{M^{m,p\restriction m}}^{p(m)},
\end{aligned}$$

for some j which is recursively computable from i, as $M^{m,q\restriction m}$ is $\mathrm{rud}_{M^{m,p\restriction m}}$ in the parameter $t = h_{M^{m,p\restriction m}}(e_1, \langle z, p(m)\rangle)$. Therefore, $A_{M^{m,q\restriction m}}^{q(m),\rho_n(M)}$ is $\mathrm{rud}_{A_{M^{m,p\restriction m}}^{p(m)}}$ in the parameter z.

(c) Let $\rho_{n+1}(M) = \rho(M^{n,q})$, where $q \in \Gamma_M^n$. By (b), $M^{n,q}$ is $\mathrm{rud}_{M^{n,p}}$ in parameters from $M^{n,p}$, which implies that $\mathbf{\Sigma}_1^{M^{n,q}} \subseteq \mathbf{\Sigma}_1^{M^{n,p}}$. But then $\rho(M^{n,p}) \leq \rho(M^{n,q}) = \rho_{n+1}(M)$, and hence $\rho(M^{n,p}) = \rho_{n+1}(M)$.

(d) This is shown inductively by using (c). $\dashv$

The following is given just by the definition of R_M^{n+1}. Let M be acceptable, and let $p \in R_M^{n+1}$. Then

$$M = h_M(h_{M^{1,p\restriction 1}}(\cdots h_{M^{n,p\restriction n}}(\rho_{n+1}(M) \cup \{p(n)\}) \cdots) \cup \{p(0)\}).$$

We thus can, uniformly over M, define a function h_M^{n+1} basically as the iterated composition of the Σ_1 Skolem functions of the i^{th} reducts of M, $0 \leq i \leq n$, such that M is the h_M^{n+1}-hull of $\rho_{n+1}(M) \cup \{p\}$ whenever $p \in R_M^{n+1}$. The precise definition of the (partial) function $h_M^{n+1} \colon {}^{<\omega}\omega \times {}^{<\omega}|M^{n+1,p}| \to |M|$ is by recursion on $n < \omega$; we set $h_M^1 = h_M$ and

$$h_M^{n+1}(\langle \vec{i}, i_0, \cdots i_k \rangle, \langle \vec{x}_{i_0}, \cdots \vec{x}_{i_k} \rangle) =$$

$$h_M^n(\vec{i}, \langle h_{M^{n,p \restriction n}}(i_0, \vec{x}_{i_0}), \cdots, h_{M^{n,p \restriction n}}(i_k, \vec{x}_{i_k}) \rangle).$$

5.4 Lemma. *Let $n < \omega$, and let M be acceptable. Then h_M^{n+1} is in Σ_ω^M, and*

$$M = h_M^{n+1}{}''(\rho_{n+1}(M) \cup \{p\}),$$

whenever $p \in R_M^{n+1}$.

5.5 Lemma. *Let $0 < n < \omega$. Let M be acceptable, and let $p \in R_M^n$. Then $\boldsymbol{\Sigma}_\omega^M \cap \mathcal{P}(M^{n,p}) = \boldsymbol{\Sigma}_\omega^{M^{n,p}}$.*

Proof. It is easy to verify that $\boldsymbol{\Sigma}_\omega^{M^{n,p}} \subseteq \boldsymbol{\Sigma}_\omega^M \cap \mathcal{P}(M^{n,p})$. Let us prove the other direction.

It is straightforward to verify by induction on $m \leq n$ that if φ is Σ_0 and x, $y \in M^{m,p \restriction m}$, then

$$\varphi(x, h_M^m(y)) \text{ is uniformly } \Delta_1^{M^{m-1,p \restriction m-1}}(x,y). \tag{I.19}$$

Now let $A \in \boldsymbol{\Sigma}_\omega^M \cap \mathcal{P}(M^{n,p})$, say

$$x \in A \iff M \models \exists x_1 \forall x_2 \cdots \exists/\forall x_k \ \varphi(x, y, x_1, x_2, \cdots, x_k),$$

where φ is Σ_0 and $y \in M$. By Lemma 5.4, we may write

$$x \in A \iff \exists x_1' \in M^{n,p} \ \forall x_2' \in M^{n,p} \ \cdots \exists/\forall x_k \in M^{n,p}$$

$$\varphi(x, h_M^n(y'), h_M^n(x_1'), h_M^n(x_2'), \cdots h_M^n(x_k')),$$

where $y' \in M^{n,p}$. But then $A \in \boldsymbol{\Sigma}_\omega^{M^{n,p}}$, with the help of (I.19) for $m = n$. $\dashv$

A more careful look at the proofs of Lemmata 5.4 and 5.5 shows the following.

5.6 Lemma. *Let $n < \omega$. Let M be acceptable and $p \in R_M^n$. Let $A \subseteq M^{n,p}$ be $\boldsymbol{\Sigma}_{n+1}^M$. Then A is $\boldsymbol{\Sigma}_1^{M^{n,p}}$.*

5.7 Definition. M is *n-sound* iff $R_M^n = P_M^n$. M is *sound* iff M is n-sound for all $n < \omega$.

We shall prove later (cf. Lemma 9.2) that every J_α is sound. In fact, a key requirement on initial segments of a core model is that they be sound.

We can now formulate a general downward extension of embeddings lemma as the conjunction of the following three lemmata which, in turn, are immediate consequences of the corresponding lemmata for the first projectum.

5.8 Lemma. *Let $\bar{M}$, M be acceptable and $\pi : \bar{M}^{n,\bar{p}} \underset{\Sigma_0}{\longrightarrow} M^{n,p}$, where $\bar{p} \in R^n_{\bar{M}}$. Then there is a unique map $\tilde{\pi} \supseteq \pi$ such that* $\mathrm{dom}(\tilde{\pi}) = \bar{M}$, $\tilde{\pi}(\bar{p}) = p$ *and, setting $\tilde{\pi}_i = \tilde{\pi} \upharpoonright H^i_{\bar{M}}$,*

$$\tilde{\pi}_i : \bar{M}^{i,\bar{p}\upharpoonright i} \underset{\Sigma_0}{\longrightarrow} M^{i,p\upharpoonright i} \;\; \textit{for} \;\; i \leq n.$$

The map $\tilde{\pi}_i$ is in fact Σ_1-preserving for $i \in n$.

5.9 Lemma. *Suppose that $\bar{M}$, M, $\bar{p}$, p, π, $\tilde{\pi}$, $\tilde{\pi}_i$, $i \leq n$, are as above and $p \in R^n_M$. Let $\pi : \bar{M}^{n,\bar{p}} \underset{\Sigma_\ell}{\longrightarrow} M^{n,p}$ where $\ell \in \omega$. Then*

$$\tilde{\pi}_i : \bar{M}^{i,\bar{p}\upharpoonright i} \underset{\Sigma_{\ell+n-i}}{\longrightarrow} M^{i,p\upharpoonright i} \;\; \textit{for} \;\; i \leq n.$$

Hence, $\tilde{\pi}_0 : \bar{M} \underset{\Sigma_{\ell+n}}{\longrightarrow} M$.

5.10 Lemma. *Let $\pi : N \underset{\Sigma_0}{\longrightarrow} M^{n,p}$, where M is as above. Then there are unique $\bar{M}$, $\bar{p}$ such that $\bar{p} \in R^n_{\bar{M}}$ and $N = \bar{M}^{n,\bar{p}}$.*

The general upward extension of embeddings lemma is the conjunction of the following lemma together with Lemmata 5.8 and 5.9.

5.11 Lemma. *Let $\pi : \bar{M}^{n,\bar{p}} \to N$ be strong, where $\bar{M}$ is an acceptable J-structure and $\bar{p} \in R^n_{\bar{M}}$. Then there are unique M, p such that M is acceptable, $p \in R^n_M$ and $M^{n,p} = N$. Moreover, if $\tilde{\pi}$ is as in Lemma* 5.8, *then $\tilde{\pi}$ is strong.*

If π and $\tilde{\pi}$ are as in Lemma 5.8 then $\tilde{\pi}$ is often called the *n-completion* of π.

If we take $\pi : \bar{M}^{n,\bar{p}} \to M^{n,p}$ as in Lemma 5.8 and form the corresponding extension $\tilde{\pi}$, we can in fact do better than stated there. It is easy to see that for every appropriate $\bar{q}$ and $q = \tilde{\pi}(\bar{q})$,

$$\tilde{\pi}_i : \bar{M}^{i,\bar{q}\upharpoonright i} \underset{\Sigma_1}{\longrightarrow} M^{i,q\upharpoonright i} \;\; \text{for} \;\; i \in n. \tag{I.20}$$

This suggests the general notion of a $\Sigma^{(n)}_\ell$-preserving embedding, where n indicates preservation at the n-th level, i.e. if $\tilde{\pi}(\bar{q}) = q$ then (I.20) and

$$\tilde{\pi}_n : \bar{M}^{n,\bar{q}} \underset{\Sigma_\ell}{\longrightarrow} M^{n,q}. \tag{I.21}$$

It turns out that there is a canonical class of formulae, the so called $\Sigma_\ell^{(n)}$-formulae, such that the above embeddings are exactly those which are elementary with respect to this class. This idea leads towards Jensen's elegant Σ^* theory which is dealt with in [15, Sections 1.6 ff.].

Following [8, §2], though, we shall call $\Sigma_1^{(n)}$ elementary maps $r\Sigma_{n+1}$ elementary. Here is our official definition, which presupposes that the structures involved possess very good parameters; it will play a rôle in the last two sections.

5.12 Definition. Let M, N be acceptable, let $\pi\colon M \to N$, and let $n < \omega$. Then π is called $r\Sigma_{n+1}$ *elementary* provided that there is $p \in R_M^n$ with $\pi(p) \in R_N^n$, and for all $i \leq n$,

$$\pi \restriction H^M_{\rho_i(M)} \colon M^{i,p\restriction i} \underset{\Sigma_1}{\longrightarrow} N^{i,\pi(p)\restriction i}. \tag{I.22}$$

The map π is called *weakly* $r\Sigma_{n+1}$ *elementary* provided that there is $p \in R_M^n$ with $\pi(p) \in R_N^n$, and for all $i < n$, (I.22) holds, and

$$\pi \restriction H^M_{\rho_n(M)} \colon M^{n,p} \underset{\Sigma_0}{\longrightarrow} N^{n,\pi(p)}.$$

If $\pi\colon M \to N$ is (weakly) $r\Sigma_{n+1}$ elementary then typically both M and N will be n-sound; however, neither M nor N has to be $(n+1)$-sound. It is possible to generalize this definition so as to not assume that very good parameters exist (cf. [8, §2]).

Lemma 5.8 therefore says that the map π can be extended to its n-completion $\tilde{\pi}$ which is weakly $r\Sigma_{n+1}$ elementary, and Lemma 5.9 says that if π is Σ_1 elementary to begin with then the n-completion $\tilde{\pi}$ will end up being $r\Sigma_{n+1}$ elementary.

Moreover, if a map $\pi\colon M \to N$ is $r\Sigma_{n+1}$ elementary then it respects h^{n+1} by Theorem 1.15, i.e.:

5.13 Lemma. *Let $n < \omega$, and let M, N be acceptable. Let $\pi\colon M \to N$ be $r\Sigma_{n+1}$ elementary. Then for all appropriate x,*

$$\pi(h_M^{n+1}(x)) = h_N^{n+1}(\pi(x)).$$

6. Standard parameters

Finite sets of ordinals are well-ordered in a simple canonical way.

6.1 Definition. Let $a, b \in [\mathbf{On}]^{<\omega}$. Set

$$a <^* b \Longleftrightarrow \exists\, \alpha \in b\ (a \setminus (\alpha+1) = b \setminus (\alpha+1)\ \wedge\ \alpha \notin a).$$[8]

[8]I.e., $\max(a \triangle b) \in b$.

The ordering $<^*$ has a rudimentary definition, therefore it is absolute for transitive rudimentarily closed structures and is also preserved under embeddings which are Σ_0 elementary. If we view finite sets of ordinals as finite decreasing sequences, $a <^* b$ precisely when a precedes b lexicographically. Moreover, we easily get the following.

6.2 Lemma. $[\mathbf{On}]^{<\omega}$ *is well-ordered by* $<^*$.

Let M be acceptable, and $n < \omega$. The well-ordering $<^*$ induces a well-ordering of Γ^n_M by confusing $p \in \Gamma^n_M$ with $\bigcup \mathrm{ran}(p)$ (i.e., by identifying p with the obvious set of ordinals; cf. above). We shall denote this latter well-ordering also by $<^*$.

6.3 Definition. Let M be acceptable. The $<^*$-least $p \in P^n_M$ is called the n^{th} *standard parameter of* M and is denoted by $p_n(M)$. We shall write M^n for $M^{n,p_n(M)}$; M^n is called the n^{th} *standard reduct of* M.

6.4 Lemma. *Let* $p \in R^n_M$. *Then* p *can be lengthened to some* $p' \in P^{n+1}_M$, *i.e., there is some* $p' \in P^{n+1}_M$ *with* $p' \restriction n = p$.

Proof. This follows immediately from Lemma 5.3 (c). ⊣

6.5 Corollary. *Let* $n > 0$. *Let* M *be* n*-sound. Then* $p_{n-1}(M) = p_n(M) \restriction (n-1)$.

Proof. By Lemma 6.4. ⊣

6.6 Definition. Let M be acceptable. Suppose that for all $n < \omega$, $p_n(M) = p_{n+1}(M) \restriction n$. Then we set $p(M) = \bigcup_{n<\omega} p_n(M)$. $p(M)$ is called the *standard parameter of* M.

We shall often confuse $p(M)$ with $\bigcup \mathrm{ran}(p(M))$.

6.7 Corollary. *Let* M *be sound. Then* $p(M)$ *exists, i.e., for all* $n < \omega$, $p_n(M) = p_{n+1}(M) \restriction n$.

Proof. By Corollary 6.5. ⊣

6.8 Lemma. M *is sound iff* $p_n(M) \in R^n_M$ *for all* $n \in \omega$.

Proof. We shall prove the non-trivial direction ($\Leftarrow$). For each $n > 0$ we shall prove

$$p_n(M) \in R^n_M \Longrightarrow R^n_M = P^n_M. \tag{I.23}$$

This holds trivially for $n = 0$. Now suppose $n > 0$ is least such that the statement (I.23) fails. Hence $P^n_M \setminus R^n_M \neq \emptyset$ (cf. Lemma 5.3 (a)). Let q be the $<^*$-least element of $P^n_M \setminus R^n_M$. This means that $p <^* q$, where $p = p_n(M)$. By Lemma 5.3, we may let $i < n$ be least such that $q(i) \notin R_{M^{i,q\restriction i}}$.

Let us first consider the case $n = 1$.

Then, of course, $p(0) <^* q(0)$. Using the downward extension of embeddings lemma, we may let $\bar{M}$, $\bar{q}$, π be unique such that

$$\begin{cases} \bar{q} \in R_{\bar{M}} \\ \pi : \bar{M} \to M \text{ is } \Sigma_1 \text{ elementary} \\ \pi(\bar{q}) = q(0) \\ \pi \restriction H^M_{\rho_1(M)} = \text{id}. \end{cases} \tag{I.24}$$

As $q(0) \notin R_M$, $\pi \neq \text{id}$. Because $p(0) \in R_M$, there are e and $z \in [\rho_1(M), M \cap \text{On})^{<\omega}$ such that

$$q(0) = h_M(e, \langle z, p(0)\rangle).$$

As $p(0) <^* q(0)$, by elementarity we get that

$$\bar{M} \models \exists p' <^* \bar{q}\ (\bar{q} = h(e, \langle z, p'\rangle)).$$

Letting p^* be a witness, we may conclude that by elementarity again

$$\pi(p^*) <^* q(0) \wedge q(0) = h_M(e, \langle z, \pi(p^*)\rangle, \tag{I.25}$$

where we may and shall assume that $\pi(p^*) \in [\rho_1(M), M \cap \text{On})^{<\omega}$. We have that $\pi(p^*) \in P_M$ by I.25. But $\pi(p^*) \in \text{ran}(\pi)$ and $\pi \neq \text{id}$, so that we must also have that $\pi(p^*) \notin R_M$. Because $\pi(p^*) <^* q(0)$, we have a contradiction to the choice of q.

Now let us consider the case $n > 1$.

If $p \restriction n-1 = q \restriction n-1$ then we may apply the above argument to the reduct $M^{n-1, p\restriction n-1}$. We may thus assume that $p \restriction n-1 <^* q \restriction n-1$. Let $i < n$ be least such that $p \restriction i <^* q \restriction i$. We shall assume that $i = 1$ and $n = 2$ for notational convenience. The general case is similar to this special case and is left to the reader.

As $p_2(M) \in R^2_M$, Lemma 5.3 (d) yields that $P^1_M = R^1_M$. As $p(0) <^* q(0)$, there is some $j \in \omega$ and $z \in [\rho_1(M), M \cap \text{On})^{<\omega}$ such that

$$\exists p'(p' <^* q(0) \wedge q(0) = h_M(j, \langle z, p'\rangle)).$$

If i is the Gödel number of this Σ_1 formula, we thus have that $A^{q(0)}_M(i, z)$ holds true. Let X be the smallest Σ_1 submodel of the (first) reduct $M^{q(0)}$. There is then some $z_0 \in X$ such that $A^{q(0)}_M(i, z_0)$ holds true. Therefore,

$$\exists p'(p' <^* q(0) \wedge q(0) = h_M(j, \langle z_0, p'\rangle)). \tag{I.26}$$

So there is some p', call it $\bar{q}(0)$, witnessing I.26 which is an element of the smallest Σ_1 substructure of M which contains both $q(0)$ and z_0. There is then also some k with $\bar{q}(0) = h_M(k, \langle z_0, q(0)\rangle)$.

Now set $\bar{q} = \bar{q}(0) \cup q(1)$. Then $\bar{q} \in P^2_M$, because $q(0) = h_M(j, \langle z_0, \bar{q}(0)\rangle)$ and $q \in P^2_M$. We'll now show that $\bar{q} \notin R^2_M$. As $\bar{q}(0) <^* q(0)$ (and hence $\bar{q} <^* q$) this will contradict the choice of q and finish the proof.

Well, to see that $\bar{q} \notin R^2_M$ it suffices to verify that if

$$Y = h_{M^{\bar{q}(0)}}(\rho_2(M) \cup \{q(1)\})$$

then $Y \neq M^{\bar{q}(0)}$. As $\bar{q}(0) = h_M(k, \langle z_0, q(0)\rangle)$, we can find a recursive $f : \omega \to \omega$ such that for all $\ell \in \omega$ and for all x,

$$A_M^{\bar{q}(0)}(\ell, x) \iff A_M^{q(0)}(f(\ell), \langle x, z_0\rangle).$$

Therefore, as $z_0 \in X$,

$$Y \subseteq h_{M^{q(0)}}(\rho_2(M) \cup \{q(1)\}).$$

But $q(1) \notin R^{M^{q(0)}}$, and so $Y \neq M^{\bar{q}(0)}$. ⊣

There is a class of structures, for which the above characterization of soundness has a particularly nice form. This class comprises all of the structures J_α where α is a limit ordinal. Moreover, the same applies to sufficiently iterable premice, which are the building blocks of core models.

7. Solidity witnesses

Solidity witnesses are witnesses to the fact that a given ordinal is a member of the standard parameter. The key fact will be that being a witness is preserved under Σ_1 elementary maps, so that witnesses can be used for showing that standard parameters are mapped to standard parameters.

Whereas the pure theory of witnesses is easy to grasp, it is one of the deepest results of inner model theory that the structures considered there (viz., iterable premice) do contain witnesses.

7.1 Definition. Let M be an acceptable structure, let $p \in [\mathrm{On} \cap M]^{<\omega}$, and let $\nu \in p$. Let W be another acceptable structure with $\nu \subseteq W$, and let $r \in [\mathrm{On} \cap W]^{<\omega}$. We say that (W, r) is a *witness for* $\nu \in p$ w.r.t. M, p iff for every Σ_1 formula $\varphi(v_0, ..., v_{l+1})$ and for all ξ_0, ..., $\xi_l < \nu$

$$M \models \varphi(\xi_0, ..., \xi_l, p \setminus (\nu + 1)) \implies W \models \varphi(\xi_0, ..., \xi_l, r). \tag{I.27}$$

In this situation, we shall often suppress r and call W a witness. The proof of Lemma 7.2 will show that if a witness exists then there is also one where $\implies$ may be replaced by $\iff$ in I.27.

7.2 Lemma. *Let M be an acceptable structure, and let $p \in P_M$. Suppose that for each $\nu \in p$ there is a witness W for $\nu \in p$ w.r.t. M, p such that $W \in M$. Then $p = p_1(M)$.*

Proof. Suppose not. Then $p_1(M) <^* p$, and we may let $\nu \in p \setminus p_1(M)$ be such that $p \setminus (\nu+1) = p_1(M) \setminus (\nu+1)$. Let us write q for $p \setminus (\nu+1) = p_1(M) \setminus (\nu+1)$. Let $(W,r) \in M$ be a witness for $\nu \in p$ w.r.t. M, p. Let $A \in \Sigma_1^M(\{p_1(M)\})$ be such that $A \cap \rho_1(M) \notin M$.

Let k be the number of elements of $p \cap \nu$, and if $k > 0$ then let $\xi_1 < ... < \xi_k$ be such that $p_1(M) \cap \nu = \{\xi_1, ..., \xi_k\}$. There is a Σ_1 formula $\varphi(v_0, ..., v_{k+1})$ such that

$$\xi \in A \iff M \models \varphi(\xi, \xi_1, ..., \xi_k, q).$$

Because $(W,r) \in M$ is a witness for $\nu \in p$ w.r.t. M, p, we have that

$$M \models \bar{\varphi}(\xi, \xi_1, ..., \xi_k, q) \implies W \models \bar{\varphi}(\xi, \xi_1, ..., \xi_k, r)$$

for every $\xi < \rho_1(M) \leq \nu$ and every $\bar{\varphi}$ which is Σ_1.

Let $\alpha = \sup(h_W(\nu \cup \{r\}) \cap \mathrm{On})$, and let $\bar{W} = J_\alpha^B$ (where $W = J_\beta^B$, some $\beta \geq \alpha$). By looking at the canonical elementary embedding from $h_M(\nu \cup \{q\})$ into $\bar{W}$, which is Σ_0 elementary and cofinal (and hence Σ_1 elementary) we get that

$$M \models \bar{\varphi}(\xi, \xi_1, ..., \xi_k, q) \iff \bar{W} \models \bar{\varphi}(\xi, \xi_1, ..., \xi_k, r) \tag{I.28}$$

for every $\xi < \rho_1(M) \leq \nu$ and every $\bar{\varphi}$ which is Σ_1. In particular, I.28 holds with $\bar{\varphi}$ replaced by φ and every $\xi < \rho_1(M) \leq \nu$. As $\bar{W} \in M$, this shows that in fact $A \cap \rho_1(M) \in M$. Contradiction! $\dashv$

7.3 Definition. Let M be acceptable, let $p \in [On \cap M]^{<\omega}$, and let $\nu \in p$. We denote by $W_M^{\nu,p}$ the transitive collapse of $h_M(\nu \cup (p \setminus (\nu+1)))$. We call $W_M^{\nu,p}$ the *standard witness for* $\nu \in p$ w.r.t. M, p.

7.4 Lemma. *Let M be acceptable, and let $\nu \in p \in P_M$. The following are equivalent.*

(1) $W_M^{\nu,p} \in M$.

(2) There is a witness W for $\nu \in p$ w.r.t. M, p such that $W \in M$.

Proof. We have to show (2) $\implies$ (1). Let $\sigma \colon W_M^{\nu,p} \to M$ be the inverse of the transitive collapse. We may also let $\sigma^* \colon W_M^{\nu,p} \to W$ be defined by $h_{W_M^{\nu,p}}(\xi, \sigma^{-1}(p \setminus \nu+1)) \mapsto h_W(\xi, r)$, $\xi < \nu$. Set $\alpha = \sup(h_W(\nu \cup \{r\}) \cap \mathrm{On})$, and let $\bar{W} = J_\alpha^A$ (where $W = J_\beta^A$, some $\beta \geq \alpha$). Again, $\sigma^* \colon W_M^{\nu,p} \to \bar{W}$ is Σ_1 elementary.

Let us assume w.l.o.g. that $W_M^{\nu,p}$ is not an initial segment of M.

Now if $\sigma(\nu) = \nu$ then a witness to $\rho_1(M)$ is definable over $W_M^{\nu,p}$, and hence over $\bar{W}$. But as $\bar{W} \in M$, this witness to $\rho_1(M)$ would then be in M. Contradiction!

We thus have that ν is the critical point of σ. Thus, if $M = J_\gamma^B$, we know that $\sigma(\nu)$ is regular in M and so $J_{\sigma(\nu)}^B \models \mathsf{ZFC}^-$. We may code $W_M^{\nu,p}$ by some $a \subseteq \nu$, Σ_1-definably over $W_M^{\nu,p}$. Using σ^*, a is definable over $\bar{W}$, so that $a \in M$. In fact, $a \in J^B_{\sigma(\nu)}$ by acceptability. We can thus decode a in $J^B_{\sigma(\nu)}$, which gives $W^{\nu,p}_M \in J^B_{\sigma(\nu)} \subseteq M$. ⊣

7.5 Definition. Let M be an acceptable structure. We say that M is 1-*solid* iff

$$W_M^{\nu,p_1(M)} \in M$$

for every $\nu \in p_1(M)$.

7.6 Lemma. *Let $\bar{M}$, M be acceptable structures, and let $\pi\colon \bar{M} \xrightarrow[\Sigma_1]{} M$. Let $\bar{\nu} \in \bar{p} \in [On \cap \bar{M}]^{<\omega}$, and set $\nu = \pi(\bar{\nu})$ and $p = \pi(\bar{p})$. Let $(\bar{W}, \bar{r})$ be a witness for $\bar{\nu}$ w.r.t. $\bar{M}$, $\bar{p}$ such that $\bar{W} \in \bar{M}$, and set $W = \pi(\bar{W})$ and $r = \pi(\bar{r})$. Then (W, r) is a witness for ν w.r.t. M, p.*

Proof. Let $\varphi(v_0, ..., v_{l+1})$ be an arbitrary Σ_1 formula. We know that

$$\bar{M} \models \forall \xi_0 ... \xi_l < \bar{\nu}[\varphi(\xi_0, ..., \xi_l, \bar{p} \setminus (\bar{\nu}+1)) \to \bar{W} \models \varphi(\xi_0, ..., \xi_l, \bar{r})].$$

As π is Π_1 elementary, this yields that

$$M \models \forall \xi_0 ... \xi_l < \nu[\varphi(\xi_0, ..., \xi_l, p \setminus (\nu+1)) \to W \models \varphi(\xi_0, ..., \xi_l, r)].$$

We thus conclude that (W, r) is a witness for ν w.r.t. M, p. ⊣

7.7 Corollary. *Let $\bar{M}$, M be acceptable structures, and let $\pi\colon \bar{M} \xrightarrow[\Sigma_1]{} M$. Suppose that $\bar{M}$ is 1-solid and $\pi(p_1(\bar{M})) \in P_M$. Then $p_1(M) = \pi(p_1(\bar{M}))$, and M is 1-solid.*

The proof of the following lemma is virtually the same as the proof of Lemma 7.6.

7.8 Lemma. *Let $\bar{M}$, M be acceptable structures, and let $\pi\colon \bar{M} \xrightarrow[\Sigma_1]{} M$. Let $\bar{\nu} \in \bar{p} \in [On \cap \bar{M}]^{<\omega}$, and set $\nu = \pi(\bar{\nu})$ and $p = \pi(\bar{p})$. Let $(\bar{W}, \bar{r}) \in \bar{M}$ be such that, setting $W = \pi(\bar{W})$ and $r = \pi(\bar{r})$, (W, r) is a witness for ν w.r.t. M, p. Then $(\bar{W}, \bar{r})$ is a witness for $\bar{\nu} \in \bar{p}$ w.r.t. $\bar{M}$, $\bar{p}$.*

7.9 Corollary. *Let $\bar{M}$, M be acceptable structures, and let $\pi\colon \bar{M} \xrightarrow[\Sigma_1]{} M$. Suppose that M is 1-solid, and that in fact $W_M^{\nu,p_1(M)} \in \mathrm{ran}(\pi)$ for every $\nu \in p_1(M)$. Then $p_1(\bar{M}) = \pi^{-1}(p_1(M))$, and $\bar{M}$ is 1-solid.*

The following definition just extends Definition 7.5.

7.10 Definition. Let M be an acceptable structure. If $0 < n < \omega$ then we say that M is n-solid if for every $k < n$, $p_1(M^k) = p_{k+1}(M)(k) = p_n(M)(k)$ and M^k is 1-solid, i.e., if

$$W_{M^k}^{\nu,p_1(M^k)} \in M^k$$

for every $\nu \in p_1(M^k)$.

7.11 Lemma. *Let $\bar{M}$, M be acceptable, let $n > 0$, and let $\pi \colon \bar{M} \to M$ be $r\Sigma_n$ elementary as being witnessed by $p_{n-1}(M)$. If $\bar{M}$ is n-solid and $\pi(p_1(\bar{M}^{n-1})) \in P_{M^{n-1}}$ then $p_n(M) = \pi(p_n(\bar{M}))$ and M is n-solid.*

7.12 Lemma. *Let $\bar{M}$, M be acceptable, let $n > 0$, and let $\pi \colon \bar{M} \to M$ be $r\Sigma_n$ elementary as being witnessed by $\pi^{-1}(p_{n-1}(M))$. Suppose that M is n-solid, and in fact $W^{\nu, p_1(M^k)}_{M^k} \in \mathrm{ran}(\pi)$ for every $k < n$. If $\pi^{-1}(p_{n-1}(M)) \in P^{n-1}_{\bar{M}}$ then $p_n(\bar{M}) = \pi^{-1}(p_n(\bar{M}))$ and $\bar{M}$ is n-solid.*

The ultrapower maps we shall construct in the next section shall be elementary in the sense of the following definition. (Cf. [8, Definition 2.8.4].)

7.13 Definition. Let both M and N be acceptable, let $\pi \colon M \to N$, and let $n < \omega$. Then π is called an *n-embedding* if the following hold true.

(1) Both M and N are n-sound,
(2) π is $r\Sigma_{n+1}$ elementary,
(3) $\pi(p_k(M)) = p_k(N)$ for every $k \leq n$, and
(4) $\pi(\rho_k(M)) = \rho_k(N)$ for every $k < n$ and $\rho_n(N) = \sup(\pi\text{"}\rho_n(M))$.

Other examples for n-embeddings are typically obtained as follows. Let M be acceptable, and let, for $n \in \omega$, $\mathfrak{C}_n(M)$ denote the transitive collapse of $h^n_M\text{"}(\rho_n(M) \cup \{p_n(M)\})$. $\mathfrak{C}_n(M)$ is called the n^{th} *core of* M. The natural map from $\mathfrak{C}_{n+1}(M)$ to $\mathfrak{C}_n(M)$ will be an n-embedding under favourable circumstances.

8. Fine ultrapowers

This section deals with the construction of "fine structure preserving" embeddings. Inner model theory is in need of such maps in two main contexts: first, in "lift up arguments" which are crucial for instance in the proof of the Covering Lemma for L or higher core models and in the proof of $\square_\kappa$ in such models (cf. [5] and the next section), and second, in performing iterations of premice (cf. [5], [9], and [12]). This section will deal with the construction of such embeddings from an abstract point of view. The combinatorial objects which are used for defining such maps are called "extenders."

The following definition makes use of notational conventions which are stated right after it.

8.1 Definition. Let M be acceptable. Then $E = \langle E_a ; a \in [\nu]^{<\omega} \rangle$ is called a *(κ, ν)-extender over M with critical points $\langle \mu_a ; a \in [\nu]^{<\omega} \rangle$* provided the following hold true.

(1) (Ultrafilter property) For $a \in [\nu]^{<\omega}$ we have that E_a is an ultrafilter on the set $\mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ which is κ-complete w.r.t. sequences in M; moreover, μ_a is the least μ such that $[\mu]^{\mathrm{Card}(a)} \in E_a$.

(2) (Coherency) For $a, b \in [\nu]^{<\omega}$ with $a \subseteq b$ and for $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ we have that $X \in E_a \Leftrightarrow X^{ab} \in E_b$.

(3) (Uniformity) $\mu_{\{\kappa\}} = \kappa$.

(4) (Normality) Let $a \in [\nu]^{<\omega}$ and $f\colon [\mu_a]^{\mathrm{Card}(a)} \to \mu_a$ with $f \in M$. If

$$\{u \in [\mu_a]^{\mathrm{Card}(a)}; f(u) < \max(u)\} \in E_a$$

then there is some $\beta < \max(a)$ such that

$$\{u \in [\mu_a]^{\mathrm{Card}(a \cup \{\beta\})}; f^{a, a\cup\{\beta\}}(u) = u_\beta^{a \cup \{\beta\}}\} \in E_{a \cup \{\beta\}}.$$

We write $\sigma(E) = \sup\{\mu_a + 1; a \in [\nu]^{<\omega}\}$. The extender E is called *short* if $\sigma(E) = \kappa + 1$; otherwise E is called *long*.

Let $b = \{\beta_1 < \ldots < \beta_n\}$, and let $a = \{\beta_{j_1} < \ldots < \beta_{j_m}\} \subseteq b$. If $u = \{\xi_1 < \ldots < \xi_n\}$ then we write u_a^b for $\{\xi_{j_1} < \ldots < \xi_{j_m}\}$; we also write $u_{\beta_i}^b$ for ξ_i. If $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)})$, then we write X^{ab} for $\{u \in [\mu_b]^{\mathrm{Card}(b)}; u_a^b \in X\}$. Finally, if f has domain $[\mu_a]^{\mathrm{Card}(a)}$ then we write $f^{a,b}$ for that g with domain $[\mu_b]^{\mathrm{Card}(b)}$ such that $g(u) = f(u_a^b)$. Finally, we write pr for the function which maps $\{\beta\}$ to β (i.e., $\mathrm{pr} = \bigcup$).

Notice that if E is a (κ, ν)-extender over the acceptable J-model M with critical points μ_a, and if N is another acceptable J-model with $\mathcal{P}(\mu_a) \cap N = \mathcal{P}(\mu_a) \cap M$ for all $a \in [\nu]^{<\omega}$, then E is also an extender over N.

The currently known core models are built with just short extenders on their sequence (cf. [5], [9], [12]). On the other hand, already the proof of the Covering Lemma for L has to make use of long extenders.

The following is easy to verify.

8.2 Theorem. *Let M and N be acceptable, and let $\pi\colon M \underset{\Sigma_0}{\longrightarrow} N$ cofinally with critical point κ. Let $\nu \leq N \cap \mathrm{On}$. For each $a \in [\nu]^{<\omega}$ let μ_a be the least $\mu \leq M \cap \mathrm{On}$ such that $a \subseteq \pi(\mu)$, and set*

$$E_a = \{X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M; a \in \pi(X)\}.$$

Then $E = \langle E_a; a \in [\nu]^{<\omega}\rangle$ is a (κ, ν)-extender over M.

8.3 Definition. If $\pi\colon M \to N$, E, κ , and ν are as in the statement of Theorem 8.2 then E is called the (κ, ν)-*extender derived from* π.

8.4 Theorem. *Let $M = \langle J_\alpha^A, B\rangle$ be acceptable, and let $E = \langle E_a; a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. There are then N and π such that the following hold true.*

(a) $\pi\colon M \underset{\Sigma_0}{\longrightarrow} N$ cofinally with critical point κ,

(b) the well-founded part $\mathrm{wfp}(N)$ of N is transitive and $\nu \subseteq \mathrm{wfp}(N)$,

(c) $N = \{\pi(f)(a); a \in [\nu]^{<\omega}, f\colon [\mu_a]^{\mathrm{Card}(a)} \to M, f \in M\}$, and

(d) for $a \in [\nu]^{<\omega}$ we have that $X \in E_a$ if and only if $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ and $a \in \pi(X)$.

Moreover, N and π are unique up to isomorphism.

Proof. We do not construe (c) in the statement of this Theorem to presuppose that N be well-founded; in fact, this statement makes perfect sense even if N is *not* well-founded.

Let us first argue that N and π are unique up to isomorphism. Suppose that N, π and N', π' are both as in the statement of the Theorem. We claim that

$$\pi(f)(a) \mapsto \pi'(f)(a)$$

defines an isomorphism between N and N'. Notice for example that $\pi(f)(a) \in \pi(g)(b)$ if and only if, setting $c = a \cup b$,

$$c \in \pi(\{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{a,c}(u) \in g^{b,c}(u)\}),$$

which by (d) yields that

$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{a,c}(u) \in g^{b,c}(u)\} \in E_c,$$

and hence by (d) once more that

$$c \in \pi'(\{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{a,c}(u) \in g^{b,c}(u)\},$$

i.e., $\pi'(f)(a) \in \pi'(g)(b)$.

The existence is shown by an ultrapower construction. Let us set

$$D = \{\langle a, f\rangle; a \in [\nu]^{<\omega}, f \colon [\mu_a]^{\mathrm{Card}(a)} \to M, f \in M\}.$$

For $\langle a, f\rangle$, $\langle b, g\rangle \in D$ let us write

$$\langle a, f\rangle \sim \langle b, g\rangle \iff \{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{a,c}(u) = g^{b,c}(u)\} \in E_c, \text{ for } c = a \cup b.$$

We may easily use (1) and (2) of Definition 8.1 to see that $\sim$ is an equivalence relation on D. If $\langle a, f\rangle \in D$ then let us write $[a, f] = [a, f]^M_E$ for the equivalence class $\{\langle b, g\rangle \in D; \langle a, f\rangle \sim \langle b, g\rangle\}$, and let us set

$$\tilde{D} = \{[a, f]; \langle a, f\rangle \in D\}.$$

Let us also define, for $[a, f]$, $[b, g] \in \tilde{D}$,

$$\begin{aligned}
[a, f] \mathrel{\tilde{\in}} [b, g] &\iff \{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{a,c}(u) \in g^{b,c}(u)\} \in E_c, \text{ for } c = a \cup b \\
\tilde{A}([a, f]) &\iff \{u \in [\mu_a]^{\mathrm{Card}(a)}; f(u) \in A\} \in E_a \\
\tilde{B}([a, f]) &\iff \{u \in [\mu_a]^{\mathrm{Card}(a)}; f(u) \in B\} \in E_a
\end{aligned}$$

Notice that the relevant sets are members of M, as M is rud_A-closed and amenable. Moreover, by (1) and (2) of Definition 8.1, $\tilde{\in}$, $\tilde{A}$, and $\tilde{B}$ are well-defined. Let us set

$$N = \langle \tilde{D}, \tilde{\in}, \tilde{A}, \tilde{B}\rangle.$$

Claim 1. (Łoś Theorem) Let $\varphi(v_1, ..., v_k)$ be a Σ_0 formula, and let $\langle a_1, f_1\rangle$, ..., $\langle a_k, f_k\rangle \in D$. Then

$$N \models \varphi([a_1, f_1], ..., [a_k, f_k]) \iff$$
$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; M \models \varphi(f_1^{a_1,c}(u), ..., f_k^{a_k,c}(u))\} \in E_c \text{ for } c = a_1 \cup ... \cup a_k.$$

Notice again that the relevant sets are members of M. Claim 1 is shown by induction on the complexity of φ, by exploiting (1) and (2) of Definition 8.1. Let us illustrate this by verifying the direction from right to left in the case that, say, $\varphi \equiv \exists v_0 \in v_1\ \psi$ for some Σ_0 formula ψ.

We assume that, setting $c = a_1 \cup ... \cup a_k$,

$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; M \models \exists v_0 \in v_1\ \psi(f_1^{a_1,c}(u), ..., f_k^{a_k,c}(u))\} \in E_c.$$

Let us define $f_0 \colon [\mu_c]^{\mathrm{Card}(c)} \to \mathrm{ran}(f_1)$ as follows.

$$f_0(u) = \begin{cases} \text{the } <_M -\text{smallest } x \in \mathrm{ran}(f_1) \text{ with} & \\ \qquad M \models \psi(x, f_1^{a_1,c}, ..., f_k^{a_k,c}(u)) & \text{if some such } x \text{ exists,} \\ \emptyset & \text{otherwise.} \end{cases}$$

The point is that $f_0 \in M$, because M is rud_A-closed and amenable. But we then have that

$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; M \models f_0(u) \in f_1^{a_1,c}(u) \wedge \psi(f_1^{a_1,c}(u), ..., f_k^{a_k,c}(u))\} \in E_c,$$

which inductively implies that

$$N \models [c, f_0] \in [a_1, f_1] \wedge \psi([a_1, f_1], ..., [a_k, f_k]),$$

and hence that

$$N \models \exists v_0 \in v_1\ \psi([a_1, f_1], ..., [a_k, f_k]).$$

Given Claim 1, we may and shall from now on identify, via the Mostowski collapse, the well-founded part $\mathrm{wfp}(N)$ of N with a transitive structure. In particular, if $[a, f] \in \mathrm{wfp}(N)$ then we identify the equivalence class $[a, f]$ with its image under the Mostowski collapse.

Let us now define $\pi \colon M \to N$ by

$$\pi(x) = [0, c_x], \text{ where } c_x \colon [\mu_0]^0 \to \{x\}.$$

We aim to verify that N, π satisfy (a), (b), (c), and (d) from the statement of Theorem 8.4.

Claim 2. If $\alpha < \nu$ and $[a, f] \tilde{\in} [\{\alpha\}, \mathrm{pr}]$ then $[a, f] = [\{\beta\}, \mathrm{pr}]$ for some $\beta < \alpha$.

In order to prove Claim 2, let $[a, f] \mathrel{\tilde{\in}} [\{\alpha\}, \mathrm{pr}]$. Set $b = a \cup \{\alpha\}$. By the Łoś Theorem,

$$\{u \in [\mu_b]^{\mathrm{Card}(b)}; f^{a,b}(u) \in \mathrm{pr}^{\{\alpha\},b}(u)\} \in E_b.$$

By (4) of Definition 8.1, there is some $\beta < \alpha$ such that, setting $c = b \cup \{\beta\}$,

$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{a,c}(u) = \mathrm{pr}^{\{\beta\},c}(u)\} \in E_c,$$

and hence, by the Łoś Theorem again,

$$[a, f] = [\{\beta\}, \mathrm{pr}].$$

Claim 2 implies, via a straightforward induction, that

$$[\{\alpha\}, \mathrm{pr}] = \alpha \text{ for } \alpha < \nu. \tag{I.29}$$

In particular, (b) from the statement of Theorem 8.4 holds.

Claim 3. If $a \in [\nu]^{<\omega}$ then $[a, \mathrm{id}] = a$.

If $[b, f] \mathrel{\tilde{\in}} [a, \mathrm{id}]$ then by the Łoś Theorem, setting $c = a \cup b$,

$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{b,c}(u) \in u_a^c\} \in E_c.$$

However, as E_c is an ultrafilter, there must then be some $\alpha \in a$ such that

$$\{u \in [\mu_c]^{\mathrm{Card}(c)}; f^{b,c}(u) = u_\alpha^c\} \in E_c,$$

and hence by the Łoś Theorem and (I.29)

$$[b, f] = [\{\alpha\}, \mathrm{pr}] = \alpha.$$

On the other hand, if $\alpha \in a$ then it is easy to see that $\alpha \in [a, \mathrm{id}]$. This shows Claim 3.

Claim 4. $[a, f] = \pi(f)(a)$.

Notice that this statement makes sense even if $[a, f] \notin \mathrm{wfp}(N)$.
Let $b = a \cup \{0\}$. We have that

$$\{u \in [\mu_b]^{\mathrm{Card}(b)}; f^{a,b}(u) = ((c_f)^{\{0\},b}(u))(\mathrm{id}^{a,b}(u))\} = [\mu_b]^{\mathrm{Card}(b)} \in E_b,$$

by (1) of Definition 8.1, and therefore by the Łoś Theorem and Claim 3,

$$[a, f] = [0, c_f]([a, \mathrm{id}]) = \pi(f)(a).$$

Claim 4 readily implies (c) from the statement of Theorem 8.4.

Claim 5. $\kappa = \mathrm{cr}(\pi)$.

Let us first show that $\pi \upharpoonright \kappa = \mathrm{id}$. We prove that $\pi(\xi) = \xi$ for all $\xi < \kappa$ by induction on ξ.

Let $\xi < \kappa$. Suppose that $[a, f] \mathrel{\tilde{\in}} \pi(\xi) = [0, c_\xi]$. Set $b = a \cup \{\xi\}$. Then

$$\{u \in [\mu_b]^{\mathrm{Card}(b)}; f^{a,b}(u) < \xi\} \in E_b.$$

As E_b is κ-complete w.r.t. sequences in M (cf. (1) of Definition 8.1), there is hence some $\bar{\xi} < \xi$ such that

$$\{u \in [\mu_b]^{\mathrm{Card}(b)}; f^{a,b}(u) = \bar{\xi}\} \in E_b,$$

and therefore $[a, f] = \pi(\bar{\xi})$ which is $\bar{\xi}$ by the inductive hypothesis. Hence $\pi(\xi) \subseteq \xi$. It is clear that $\xi \subseteq \pi(\xi)$.

We now prove that $\pi(\kappa) > \kappa$ (if $\pi(\kappa) \notin \mathrm{wfp}(N)$ we mean that $\kappa \tilde{\in} \pi(\kappa)$) which will establish Claim 5. Well, $\mu_{\{\kappa\}} = \kappa$, and

$$\{u \in [\kappa]^1; \mathrm{pr}(u) < \kappa\} = [\kappa]^1 \in E_{\{\kappa\}},$$

from which it follows, using the Łoś Theorem, that $\kappa = [\{\kappa\}, \mathrm{pr}] < [0, c_\kappa] = \pi(\kappa)$.

The following, together with Claim 1 and Claim 5, will establish (a) from the statement of Theorem 8.4.

Claim 6. For all $[a, f] \in N$ there is some $y \in M$ with $[a, f] \mathrel{\tilde{\in}} \pi(y)$.

To verify Claim 6, it is easy to see that we can just take $y = \mathrm{ran}(f)$.

It remains to prove (d) from the statement of Theorem 8.4.

Let $X \in E_a$. By (1) of Definition 8.1,

$$X = \{u \in [\mu_a]^{\mathrm{Card}(a)}; u \in X\} \in E_a,$$

which, by the Łoś Theorem and Claim 3, gives that $a = [a, \mathrm{id}] \mathrel{\tilde{\in}} [0, c_X] = \pi(X)$.

On the other hand, suppose that $X \in \mathcal{P}([\mu_a]^{\mathrm{Card}(a)}) \cap M$ and $a \in \pi(X)$. Then by Claim 3, $[a, \mathrm{id}] = a \in \pi(X) = [0, c_X]$, and thus by the Łoś Theorem

$$X = \{u \in [\mu_a]^{\mathrm{Card}(a)}; u \in X\} \in E_a.$$

We have shown Theorem 8.4. $\dashv$

8.5 Definition. Let M, E, N, and π be as in the statement of Theorem 8.4. We shall denote N by $\mathrm{Ult}_0(M; E)$ and call it the Σ_0 ultrapower of M by E, and we call $\pi \colon M \to N$ the Σ_0 *ultrapower map* (*given by* E). We shall also write π_E for π.

8.6 Definition. Let M be acceptable, and let E be a (κ, ν)-extender over M. Let $n < \omega$ be such that $\rho_n(M) > \sigma(E)$. Suppose that M is n-sound, and set $p = p_n(M)$. Let

$$\pi \colon M^{n,p} \to \bar{N}$$

be the Σ_0 ultrapower map given by E. Suppose that

$$\tilde{\pi} \colon M \to N$$

is as given by the proof of Lemmata 4.2 and 5.11. Then we write $\mathrm{Ult}_n(M; E)$ for N and call it the $r\Sigma_{n+1}$ *ultrapower of* M *by* E, and we call $\tilde{\pi}$ the $r\Sigma_{n+1}$ *ultrapower map* (*given by* E).

A comment is in order here. Lemmata 4.2 and 5.11 presupposes that π is strong (cf. Definition 4.1). However, the construction of the term model in section 4 does not require π to be strong, nor does it even require the target model $\bar{N}$ to be well-founded. Consequently, we can make sense of $\mathrm{Ult}_n(M; E)$ even if π is not strong or $\bar{N}$ is not well-founded. This is why we have "the proof of Lemmata 4.2 and 5.11" in the statement of Definition 8.6. We shall of course primarily be interested in situations where $\mathrm{Ult}_n(M; E)$ is well-founded after all. In any event, we shall identify the well-founded part of $\mathrm{Ult}_n(M; E)$ with its transitive collapse.

One can also construct $r\Sigma_{n+1}$ ultrapower maps without assuming that the model one takes the ultrapower of is n-sound; this is done by pointwise lifting up a directed system converging to the model in question. However, the construction of Definition 8.6 seems to be broad enough for most applications.

Recall Definition 5.12. It is clear in the light of the upward extension of embeddings lemma that any $r\Sigma_{n+1}$ ultrapower map is $r\Sigma_{n+1}$ elementary (and hence the name). The following will give more information.

8.7 Theorem. *Let M be acceptable, and let E be a (κ, ν)-extender over M. Let $n < \omega$ be such that $\rho_n(M) > \sigma(E)$. Suppose that M is n-sound and $(n+1)$-solid. Let*

$$\pi \colon M \to \mathrm{Ult}_n(M; E)$$

be the $r\Sigma_{n+1}$ ultrapower map given by E. Assume that $\mathrm{Ult}_n(M; E)$ *is transitive, and that $\pi(p_{n+1}(M)) \in P_N^{n+1}$.*

Then π is an n-embedding, $\mathrm{Ult}_n(M; E)$ *is $(n{+}1)$-solid, and* $\pi(p_{n+1}(M)) = p_{n+1}(N)$.

Proof. Set $N = \mathrm{Ult}_n(M; E)$. That N is n-sound follows from the upward extension of embeddings lemma. N is $(n+1)$-solid and $\pi(p_{n+1}(M)) = p_{n+1}(N)$ by Lemma 7.11. By construction we have that π is the upward extension of

$$\pi \restriction M^n \colon M^n \to \mathrm{Ult}_0(M^n; E),$$

so that by the upward extension of embeddings lemma we shall now have that $N^n = \mathrm{Ult}_0(M^n; E)$, and therefore $\rho_n(N) = N^n \cap \mathrm{On} = \mathrm{Ult}_0(M^n; E) \cap \mathrm{On}$; however, $\pi \restriction M^n$ is cofinal in $\mathrm{Ult}_0(M^n; E)$ by Theorem 8.4, and thus $\rho_n(N) = \sup"\rho_n(M)$. The upward extension of embeddings lemma also implies that $\pi(\rho_k(M)) = \rho_k(N)$ for all $k < n$. ⊣

The following is sometimes called the "Interpolation Lemma." We leave the (easy) proof to the reader.

8.8 Lemma. *Let $n < \omega$. Let $\bar{M}$, M be acceptable, and let*

$$\pi\colon \bar{M} \longrightarrow M$$

be $r\Sigma_{n+1}$ elementary. Let $\nu \leq M \cap \mathrm{On}$, and let E be the (κ, ν)-extender derived from π.

There is then a weakly $r\Sigma_{n+1}$ elementary embedding

$$\sigma\colon \mathrm{Ult}_n(\bar{M}; E) \to M$$

such that $\sigma \restriction \nu = \mathrm{id}$ and $\sigma \circ \pi_E = \pi$.

If π is as in Theorem 8.7 then it is often crucial to know that $\rho_{n+1}(M) = \rho_{n+1}(\mathrm{Ult}_n(M; E))$. In order to be able to prove this we need that $\langle M, E\rangle$ satisfies additional hypotheses.

8.9 Definition. Let M be acceptable, and let $E = \langle E_a; a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. Then E is *close to* M if for every $a \in [\nu]^{<\omega}$,

(1) E_a is $\Sigma_1^M(\{q\})$ for some $q \in M$, and

(2) if $Y \in M$, $M \models \mathrm{Card}(Y) \leq \kappa$, then $E_a \cap Y \in M$.

The following theorem is the key tool for proving the preservation of the standard parameter in iterations of mice.

8.10 Theorem. *Let M be acceptable, and let $E = \langle E_a; a \in [\nu]^{<\omega}\rangle$ be a short (κ, ν)-extender over M which is close to M. Suppose that $n < \omega$ is such that $\rho_{n+1}(M) \leq \kappa < \rho_n(M)$, M is n-sound, and $\mathrm{Ult}_n(M; E)$ is transitive. Then*

$$\begin{aligned} \mathcal{P}(\kappa) \cap M &= \mathcal{P}(\kappa) \cap \mathrm{Ult}_n(M; E), \text{ and} \\ \rho_{n+1}(M) &= \rho_{n+1}(\mathrm{Ult}_n(M; E)). \end{aligned}$$

In particular, if M is $(n+1)$-solid then $\mathrm{Ult}_n(M; E)$ is $(n+1)$-solid and $\pi(p_{n+1}(M)) = p_{n+1}(\mathrm{Ult}_n(M; E))$.

Proof. Set $N = \mathrm{Ult}_n(M; E)$. Trivially, $\mathcal{P}(\kappa) \cap M \subseteq \mathcal{P}(\kappa) \cap N$. To show that $\mathcal{P}(\kappa) \cap N \subseteq \mathcal{P}(\kappa) \cap M$, let $X \in \mathcal{P}(\kappa) \cap N$. Let $X = [a, f]_E^{M^n}$. As E is short, $\mu_a \leq \kappa$; in fact, w.l.o.g., $\mu_a = \kappa$. Hence for $\xi < \kappa$, if we set

$$X_\xi = \{u \in [\kappa]^{\mathrm{Card}(a)}; \xi \in f(u)\}$$

then $\{X_\xi; \xi < \kappa\} \in M^n \subseteq M$. But $\xi \in X \Leftrightarrow X_\xi \in E_a$ by the Łoś Theorem, and since E is close to M we get that $X \in M$.

If $A \in \Sigma_1^{M^n}(\{q\})$, where $q \in M^n$, then $A \cap \rho_{n+1}(M) \in \Sigma_1^{N^n}(\{\pi(q), \rho_{n+1}(M)\})$. Because $\mathcal{P}(\kappa) \cap N \subseteq \mathcal{P}(\kappa) \cap M$, we thus have that $\rho_{n+1}(N) \leq \rho_{n+1}(M)$.

To show that $\rho_{n+1}(M) \leq \rho_{n+1}(N)$, let $A \in \Sigma_1^{N^n}(\{q\})$ for some $q \in N^n$. Let $q = [a, f]_E^{M^n}$, and let

$$x \in A \Longleftrightarrow N^n \models \varphi(x, [a, f]),$$

where φ is Σ_1. If $M^n = \langle J_\alpha^B, A\rangle$ and $\delta < \alpha$ then we shall write $M^{n,\delta}$ for $\langle J_\delta^B, A \cap J_\delta^B\rangle$. For $\delta < N^n \cap \mathrm{On}$, $N^{n,\delta}$ is defined similarly. Because $\pi_E \colon M^n \to N^n$ is cofinal, we have that

$$x \in A \Longleftrightarrow \exists \delta < M^n \cap \mathrm{On}\ N^{n,\pi_E(\delta)} \models \varphi(x, [a, f]).$$

By the Łoś Theorem, we may deduce that for $\xi < \kappa$,

$$\xi \in A \Longleftrightarrow \exists \delta < M^n \cap \mathrm{On}\ \{u \in [\kappa]^{\mathrm{Card}(a)}; M^{n,\delta} \models \varphi(\xi, f(u))\} \in E_a.$$

But now E is close to M, so that E_a is $\Sigma_1^{M^n}(\{q'\})$ for some $q' \in M^n$, which implies that $A \cap \kappa$ is $\Sigma_1^{M^n}(\{q', \kappa, f\})$.

We now finally have that $\pi(p_{n+1}(M)) \in P_M^{n+1}$. Therefore, if M is $(n+1)$-solid then $\mathrm{Ult}_n(M; E)$ is $(n+1)$-solid and $\pi(p_{n+1}(M)) = p_{n+1}(\mathrm{Ult}_n(M; E))$ by Lemma 7.12. $\dashv$

We now turn towards criteria for $\mathrm{Ult}_n(M; E)$ being well-founded.

8.11 Definition. Let M be acceptable, and let $E = \langle E_a; a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. Let $\lambda < \mathrm{Card}(\kappa)$ be an infinite cardinal (in V). Then E is called λ-*complete* provided the following holds true. Suppose that $\langle\langle a_i, X_i\rangle; i < \lambda\rangle$ is such that $X_i \in E_{a_i}$ for all $i < \lambda$. Then there is some order-preserving map $\tau \colon \bigcup_{i<\lambda} a_i \to \sigma(E)$ such that τ"$a_i \in X_i$ for every $i < \lambda$.

8.12 Lemma. *Let M be acceptable, and let $E = \langle E_a; a \in [\nu]^{<\omega}\rangle$ be a (κ, ν)-extender over M. Let $\lambda < \mathrm{Card}(\kappa)$ be an infinite cardinal. Then E is λ-complete if and only if for every $U \underset{\Sigma_0}{\prec} \mathrm{Ult}_0(M; E)$ of size λ there is some $\varphi \colon U \underset{\Sigma_0}{\longrightarrow} M$ such that $\varphi \circ \pi_E(x) = x$ whenever $\pi_E(x) \in U$.*

Proof. "$\Rightarrow$": Let $U \underset{\Sigma_0}{\prec} \mathrm{Ult}_0(M; E)$ be of size λ. Write $U = \{[a, f]; \langle a, f\rangle \in \bar{U}\}$ for some $\bar{U}$ of size λ. Let $\langle\langle a_i, X_i\rangle; i < \lambda\rangle$ be an enumeration of all pairs $\langle c, X\rangle$ such that there is a Σ_0 formula ψ and there are $\langle a^1, f_1\rangle$, ..., $\langle a^k, f_k\rangle \in \bar{U}$ with $c = a^1 \cup ... \cup a^k$ and

$$X = \{u \in [\mu_c]^{\mathrm{Card}(c)}; M \models \psi(f_1^{a^1,c}(u), ..., f_k^{a^k,c})\} \in E_c.$$

Let $\tau\colon \bigcup_{i<\lambda} a_i \to \sigma(E)$ be order-preserving such that $\tau"a_i \in X_i$ for every $i < \lambda$. Let us define $\varphi\colon U \to M$ by setting $\varphi([a,f]) = f(\tau"(a))$ for $\langle a, f\rangle \in \bar{U}$.

We get that φ is well-defined and Σ_0 elementary by the following reasoning. Let $\psi(v_1,...,v_k)$ be Σ_0, and let $\langle a^j, f_j\rangle \in U$, $1 \leq j \leq k$. Set $c = a^1 \cup ... \cup a^k$. We then get that

$$\begin{aligned} U \models & \psi([a^1,f_1],...,[a^k,f_k]) && \iff \\ \mathrm{Ult}_0(M;E) \models & \psi([a^1,f_1],...,[a^k,f_k]) && \iff \\ \{u \in [\mu_c]^{\mathrm{Card}(c)}; M \models & \psi(f_1^{a^1,c}(u),...,f_k^{a^k,c}(u))\} \in E_c && \iff \\ \tau"c \in \{u \in [\mu_c]^{\mathrm{Card}(c)}; & M \models \psi(f_1^{a^1,c}(u),...,f_k^{a^k,c}(u))\} && \iff \\ M \models & \psi(f_1(\tau"a^1),...,f_k(\tau"a^k)). \end{aligned}$$

We also get that $\varphi \circ \pi_E(x) = \varphi([\emptyset, c_x]) = c_x(\emptyset) = x$.

"$\Leftarrow$": Let $\langle\langle a_i, X_i\rangle; i < \lambda\rangle$ be such that $X_i \in E_{a_i}$ for all $i < \lambda$. Pick $U \underset{\Sigma_0}{\prec} \mathrm{Ult}_0(M;E)$ with $\{a_i, X_i; i < \lambda\} \subseteq U$, $\mathrm{Card}(U) = \lambda$, and let $\varphi\colon U \underset{\Sigma_0}{\longrightarrow} M$ be such that $\varphi\circ\pi_E(x) = x$ whenever $\pi_E(x) \in U$. Set $\tau = \varphi \upharpoonright \bigcup_{i<\lambda} a_i$. Then $\tau"a_i = \varphi(a_i) \in \varphi \circ \pi_E(X_i) = X_i$ for all $i < \lambda$. Clearly, $\mathrm{ran}(\tau) \subseteq \sigma(E)$. $\dashv$

8.13 Corollary. *Let M be acceptable, and let E be an $\aleph_0$-complete (κ,ν)-extender over M. Then $\mathrm{Ult}_0(M;E)$ is well-founded. In fact, if $n < \omega$ is such that $\rho_n(M) \geq \sigma(E)$ then $\mathrm{Ult}_n(M;E)$ is well-founded.*

The concept of $\aleph_0$-completeness is relevant for constructing inner models below the "sharp" for an inner model with a proper class of strong cardinals (cf. [11]). There are strengthenings of the concept of $\aleph_0$-completeness which are needed for the construction of inner models beyond the "sharp" for an inner model with a proper class of strong cardinals (cf. for instance [13, Definition 1.2], [7, Definition 1.6]).

8.14 Lemma. *Let λ be an infinite cardinal, and let θ be regular. Let $\pi\colon \bar{H} \to H_\theta$, where $\bar{H}$ is transitive and ${}^\lambda\bar{H} \subseteq \bar{H}$. Suppose that $\pi \neq \mathrm{id}$, and set $\kappa = \mathrm{cr}(\pi)$. Let M be acceptable, let ρ be regular in M, and suppose that $H_\rho^M \subseteq \bar{H}$. Set $\nu = \sup\pi"\rho$, and let E be the (κ,ν)-extender derived from $\pi \upharpoonright H_\rho^M$. Then E is λ-complete.*

Proof. Let $\langle\langle a_i, X_i\rangle; i < \lambda\rangle$ be such that $X_i \in E_{a_i}$, and hence $a_i \in \pi(X_i)$, for all $i < \lambda$. As ${}^\lambda\bar{H} \subseteq \bar{H}$, $\langle X_i; i < \lambda\rangle \in \bar{H}$. Let $\sigma\colon \mathrm{otp}(\bigcup_{i<\lambda} a_i) \cong \gamma$ be the transitive collapse; notice that $\gamma < \lambda^+ < \kappa$. For each $i < \lambda$ let $\bar{a}_i = \sigma"a_i$. We have that $\langle \bar{a}_i; i < \lambda\rangle \in \bar{H}$. But now

$$H_\theta \models \exists\tau\colon \gamma \tilde{\to} \mathrm{On}\ \forall i < \lambda\ \tau"\bar{a}_i \in \pi(\langle X_j; j < \lambda\rangle)(i),$$

as witnessed by σ. Therefore,

$$\bar{H} \models \exists\tau\colon \gamma \tilde{\rightarrow} \mathrm{On}\ \forall i < \lambda\ \tau"\bar{a}_i \in X_i.$$

Hence, if $\tau \in \bar{H}$ is a witness to this fact then $\tau \circ \sigma\colon \bigcup_{i<\lambda} a_i \to \mathrm{On}$ is such that $\tau \circ \sigma" a_i \in X_i$ for every $i < \lambda$. $\dashv$

We leave it to the reader to find variants of this result. For instance, extenders derived from canonical ultrapower maps witnessing that a given cardinal κ is measurable are λ-complete for every $\lambda < \kappa$.

9. Applications to L

In this final section we shall illustrate how to use the above machinery in the simplest case – in the constructible universe L. The theory developed above is, however, general enough so that it can be used for all the currently known core models.

We shall first prove two important lemmata. Recall that we index the J-hierarchy with limit ordinals.

9.1 Lemma. *For each limit ordinal α, J_α is acceptable.*

9.2 Lemma. *For each limit ordinal α, J_α is sound.*

We shall prove these two lemmata simultaneously. The proof goes by induction on α in a zig-zag way in the sense that we use soundness of J_α to prove the acceptability of $J_{\alpha+\omega}$ and then, knowing this, its soundness.

Proof. The case $\alpha = \omega$ is trivial. Now suppose both lemmata hold for all limit ordinals $\beta < \alpha$.

Claim 1. J_α is acceptable.

This is trivial for α being a limit of limit ordinals. For $\alpha = \beta + \omega$ it is clear that the only thing we have to prove is the following:

$$\begin{aligned}&\text{If there is } \tau < \beta \text{ and } a \subseteq \tau \text{ such that } a \in J_{\beta+\omega} \setminus J_\beta,\\ &\text{then there is an } f \in J_{\beta+\omega} \text{ such that } f : \tau \xrightarrow{\text{onto}} \beta.\end{aligned} \tag{I.30}$$

We prove (I.30). Suppose there are such τ, a and take τ to be the least one such that there is a as above. Then

$$\tau = \rho_\omega(J_\beta). \tag{I.31}$$

To see (I.31) note first that if n is such that $\rho_n(J_\beta) = \rho_\omega(J_\beta)$, then we have a new $\mathbf{\Sigma}_1^{J_\beta^n}$ subset of $\rho_\omega(J_\beta)$. Such a set is $\mathbf{\Sigma}_\omega^{J_\beta}$ by Lemma 5.5, and is hence in $J_\beta + \omega \setminus J_\beta$ by Lemma 1.7. Therefore, $\tau \leq \rho_\omega(J_\beta)$.

Let $a \subseteq \tau$ such that $a \in J_{\beta+\omega} \setminus J_\beta$. Then $a \in \mathbf{\Sigma}_n^{J_\beta}$ for some $n \in \omega$ by Lemma 1.7. By the above inequality, $a \subseteq \rho_n(J_\beta)$. Lemma 5.6 then yields that a is $\mathbf{\Sigma}_1^{J_\beta^{n-1}}$, since, by the induction hypothesis, J_β is sound. Consequently, $\rho_\omega(J_\beta) \leq \tau$. This proves (I.31).

Now we use the induction hypthesis once again to verify (I.30). By the soundness of J_β and by Lemmata 5.4 and 5.5, there is some $f \in \mathbf{\Sigma}_\omega^{J_\beta}$ such that $f\colon \rho_\omega(J_\beta) \to J_\beta$ is surjective. By Lemma 1.7, $f \in J_{\beta+\omega}$. This shows (I.30) and therefore also Claim 1.

Claim 2. J_α is sound.

We shall make use of Lemma 6.8 here. Hence, for $n < \omega$ we prove

$$p_n(J_\alpha) \in R^n_{J_\alpha}. \tag{I.32}$$

Suppose this is false. Pick the first n such that $p = p_n(J_\alpha) \notin R^n_{J_\alpha}$. Let a be $\Sigma_1^{J_\alpha^{n-1}}(\{p\})$ such that $a \cap \rho_n(J_\alpha) \notin J_\alpha$. Using the downward extension of embeddings lemma we construct unique $J_{\bar\alpha}$, $\bar p$, π such that

$$\begin{cases} \bar p \in R^n_{J_{\bar\alpha}} \\ \pi : J_{\bar\alpha}^{n-1,\bar p\restriction n-1} \to J_\alpha^{n-1,p\restriction n-1} \text{ is } \Sigma_1 \text{ elementary} \\ \pi(\bar p(n-1)) = p(n-1) \\ \pi \restriction J_{\rho_n(J_\alpha)} = \mathrm{id}. \end{cases} \tag{I.33}$$

Hence $a \cap \rho_n(J_\alpha) = \bar a \cap \rho_n(J_{\bar\alpha})$ where $\bar a$ is $\Sigma_1^{J_{\bar\alpha}^{n-1}}(\{\bar p(n-1)\})$ by the same definition. Hence $\bar\alpha$ cannot be less than α, since otherwise $a \cap \rho_n(J_\alpha) \in J_\alpha$. Consequently, $\bar\alpha = \alpha$. It is also clear by the construction that $\bar p \leq^* p$. But $p \leq^* \bar p$ since p is the standard parameter. Hence, $p = \bar p$. But this means $p \in R^n_{J_\alpha}$. Contradiction.

$\dashv$

Classical applications of the fine structure theory include Jensen's results that $\Diamond$ and $\Box$ hold in L and that L satisfies the Covering Lemma. The following is Jensen's Covering Lemma for L.

9.3 Theorem. *Suppose that $0^\#$ does not exist. Let X be a set of ordinals. Then there is $Y \in L$ with $Y \supseteq X$ and* $\mathrm{Card}(Y) \leq \mathrm{Card}(X) \cdot \aleph_1$.

This result is shown in [2] (cf. also [5]). In order to illustrate the fine structural techniques we have developed we shall now give a proof of a corollary to Theorem 9.3. Recall that a cardinal κ is called *countably closed* if $\lambda^{\aleph_0} < \kappa$ whenever $\lambda < \kappa$.

9.4 Corollary. *Let κ be a countably closed singular cardinal. If $0^\#$ does not exist then $\kappa^{+L} = \kappa^+$.*

Proof. We shall use the fact that the existence of $0^{\#}$ is equivalent with the existence of a non-trivial elementary embedding $\pi\colon L \to L$. Suppose that $0^{\#}$ does not exist and κ is a countably closed singular cardinal such that $\kappa^{+L} < \kappa^{+}$. We aim to derive a contradiction.

Let $X \subseteq \kappa^{+L}$ be cofinal with $\mathrm{otp}(X) < \kappa$. We may pick an elementary embedding

$$\pi\colon \bar{H} \to H_{\kappa^+}$$

such that $\bar{H}$ is transitive, ${}^{\omega}\bar{H} \subseteq \bar{H}$, $X \subseteq \mathrm{ran}(\pi)$, and $\mathrm{Card}(\bar{H}) = \mathrm{otp}(X)^{\aleph_0}$. As κ is countably closed, $\mathrm{Card}(\bar{H}) < \kappa$, which implies that $\pi \neq \mathrm{id}$. Set $\lambda = \pi^{-1}(\kappa^{+L})$, and let E be the $(\kappa, \pi(\lambda))$-extender over J_λ derived from $\pi \restriction J_\lambda$.

By Lemma 8.14, E is $\aleph_0$-complete. By Corollary 8.13, this implies the following.

Claim. Let $\alpha \geq \lambda$, $\alpha \in \mathrm{On} \cup \{\mathrm{On}\}$. Suppose that λ is a cardinal in J_α (which implies that E is an extender over J_α). Suppose that $\rho_n(J_\alpha) \geq \lambda$. Then $\mathrm{Ult}_n(J_\alpha; E)$ is transitive, and therefore $\mathrm{Ult}_n(J_\alpha; E) = J_\beta$ for some $\beta \in \mathrm{On} \cup \{\mathrm{On}\}$. (If $\alpha = \mathrm{On}$ then by J_α we mean L, and we want $n = 0$; we shall then have $J_\beta = L$ as well.)

Now because $0^{\#}$ does not exist, we cannot have that $\alpha = \mathrm{On}$ satisfies the hypothesis of the Claim. Let $\alpha \in \mathrm{On} \setminus \lambda$ be largest such that λ is a cardinal in J_α. Let $n < \omega$ be such that $\rho_{n+1}(J_\alpha) < \lambda \leq \rho_n(J_\alpha)$. By Lemma 9.2, we have that

$$J_\alpha = h^{n+1\,\text{''}}_{J_\alpha}(\rho_{n+1}(J_\alpha) \cup \{p\}),$$

where $p = p_{n+1}(J_\alpha)$. (Cf. Lemma 5.4.) Because π_E is $r\Sigma_{n+1}$ elementary by Theorem 8.7, Lemma 5.13 implies that

$$X \subseteq \pi\text{''} J_\alpha \subseteq h^{n+1\,\text{''}}_{J_\beta}(\pi(\rho_{n+1}(J_\alpha)) \cup \{\pi(p)\}).$$

But $\pi(\rho_{n+1}) \leq \kappa$, so that in particular

$$J_{\beta+\omega} \models \pi(\lambda) \text{ is not a cardinal.}$$

However, $\pi(\lambda) = \kappa^{+L}$. Contradiction! $\dashv$

We finally aim to prove $\Box_\kappa$ in L. This is the combinatorial principle the proof of which most heavily exploits the fine structure theory.

Let κ be an infinite cardinal. Recall that we say that $\Box_\kappa$ holds if and only if there is a sequence $\langle C_\nu; \nu < \kappa^+\rangle$ such that if ν is a limit ordinal, $\kappa < \nu < \kappa^+$, then C_ν is a club subset of ν with $\mathrm{otp}(C_\nu) \leq \kappa$ and whenever $\bar{\nu}$ is a limit point of C_ν then $C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$.

9.5 Theorem. *Suppose that $V = L$. Let $\kappa \geq \aleph_1$ be a cardinal. Then $\Box_\kappa$ holds.*

Proof. We shall verify that there is a club $C \subseteq \kappa^+$ and some $\langle C_\nu; \nu \in C \wedge \mathrm{cf}(\nu) > \omega\rangle$ such that if ν is a limit ordinal, $\kappa < \nu < \kappa^+$, then C_ν is a club subset of ν with $\mathrm{otp}(C_\nu) \leq \kappa$ and whenever $\bar{\nu}$ is a limit point of C_ν then $C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$. It is not hard to verify that this implies $\Box_\kappa$ (cf. [1, pp. 158ff.]).

Let $C = \{\nu < \kappa^+; J_\nu \prec_{\Sigma_\omega} J_{\kappa^+}\}$, a closed unbounded subset of κ^+.

Let $\nu \in C$. Obviously, κ is the largest cardinal of J_ν. We may let $\alpha(\nu)$ be the largest $\alpha \geq \nu$ such that either $\alpha = \nu$ or ν is a cardinal in J_α. By Lemma 1.7, $\rho_\omega(J_{\alpha(\nu)}) = \kappa$. Let $n(\nu)$ be that $n < \omega$ such that $\kappa = \rho_{n+1}(J_{\alpha(\nu)}) < \nu \leq \rho_n(J_{\alpha(\nu)})$.

If $\nu \in C$, then we define D_ν as follows. D_ν consists of all $\bar{\nu} \in C \cap \nu$ such that $n(\bar{\nu}) = n(\nu)$, and there is a weakly $r\Sigma_{n(\nu)+1}$ elementary embedding

$$\sigma\colon J_{\alpha(\bar{\nu})} \longrightarrow J_{\alpha(\nu)}.$$

such that $\sigma \upharpoonright \bar{\nu} = \mathrm{id}$, $\sigma(p_{n(\bar{\nu})+1}(J_{\alpha(\bar{\nu})})) = p_{n(\nu)+1}(J_{\alpha(\nu)})$, and if $\bar{\nu} \in J_{\alpha(\bar{\nu})}$ then $\nu \in J_{\alpha(\nu)}$ and $\sigma(\bar{\nu}) = \nu$. It is easy to see that if $\bar{\nu} \in D_\nu$ then there is exactly one map σ witnessing this, namely the one with

$$\sigma(h^{n(\bar{\nu})+1}_{J_{\alpha(\bar{\nu})}}(\xi, p_{n(\bar{\nu})+1}(J_{\alpha(\bar{\nu})}))) = h^{n(\nu)+1}_{J_{\alpha(\nu)}}(\xi, p_{n(\nu)+1}(J_{\alpha(\nu)}))$$

$\xi < \kappa$; we shall denote this map by $\sigma_{\bar{\nu},\nu}$.

Claim 1. Let $\nu \in C$. The following hold true.
(a) D_ν is closed.
(b) If $\mathrm{cf}(\nu) > \omega$ then D_ν is unbounded.
(c) If $\bar{\nu} \in D_\nu$ then $D_\nu \cap \bar{\nu} = D_{\bar{\nu}}$.

Proof of Claim 1. (a) and (c) are easy. Let us show (b). Suppose that $\mathrm{cf}(\nu) > \omega$. Set $\alpha = \alpha(\nu)$ and $n = n(\nu)$. Let $\beta < \nu$. We aim to show that $D_\nu \setminus \beta \neq \emptyset$.

Let $\pi\colon J_{\bar{\alpha}} \underset{\Sigma_{n+1}}{\longrightarrow} J_\alpha$ be such that $\bar{\alpha}$ is countable, $\beta \in \mathrm{ran}(\pi)$, and

$$\{W^{\nu, p_1(J^k_\alpha)}_{J^k_\alpha}; \nu \in p_1(J^k_\alpha), k \leq n\} \subseteq \mathrm{ran}(\pi).$$

Let $\bar{\nu} = \pi^{-1}(\nu)$ (if $\nu = \alpha$, we mean $\bar{\nu} = \bar{\alpha}$). Let

$$\pi' = \pi_{E_{\pi \upharpoonright J_{\bar{\nu}}}} : J_{\bar{\alpha}} \underset{r\Sigma_{n+1}}{\longrightarrow} \mathrm{Ult}_n(J_{\bar{\alpha}}; E_{\pi \upharpoonright J_{\bar{\nu}}}).$$

Write $J_{\alpha'} = \mathrm{Ult}_n(J_{\bar{\alpha}}; E_{\pi\upharpoonright J_{\bar{\nu}}})$. By Lemma 8.8, we may define a weakly $r\Sigma_{n+1}$ elementary embedding

$$k\colon J_{\alpha'} \longrightarrow J_\alpha$$

with $k \circ \pi' = \pi$. As $\beta \in \mathrm{ran}(\pi)$, $k^{-1}(\nu) > \beta$. Moreover, $k^{-1}(\nu) = \sup \pi"\bar{\nu} < \nu$, as $\mathrm{cf}(\nu) > \omega$. Therefore $\beta < k^{-1}(\nu) \in D_\nu$. $\dashv$

Now let $\nu \in C$. We aim to define C_ν. Set $\alpha = \alpha(\nu)$, and $n = n(\nu)$. Recursively, we define sequences $\langle \nu_i; i \leq \theta(\nu)\rangle$ and $\langle \xi_i; i < \theta(\nu)\rangle$ as follows. Set $\nu_0 = \min(D_\nu)$. Given ν_i with $\nu_i < \nu$, we let ξ_i be the least $\xi < \kappa$ such that

$$h^{n+1}_{J_\alpha}(\xi, p_{n+1}(J_\alpha)) \setminus \mathrm{ran}(\sigma_{\nu_i,\nu}) \neq \emptyset.$$

Given ξ_i, we let ν_{i+1} be the least $\bar{\nu} \in D_\nu$ such that

$$h^{n+1}_{J_\alpha}(\xi_i, p_{n+1}(J_\alpha)) \in \mathrm{ran}(\sigma_{\bar{\nu},\nu}).$$

Finally, given $\langle \nu_i; i < \lambda\rangle$, where λ is a limit ordinal, we set $\nu_\lambda = \sup(\{\nu_i; i < \lambda\}$. Naturally, $\theta(\nu)$ will be the least i such that $\nu_i = \nu$. We set $C_\nu = \{\nu_i; i < \theta(\nu)\}$.

The following is now easy to verify.

Claim 3. Let $\nu \in C$. The following hold true.
(a) $\langle \xi_i; i < \theta(\nu)\rangle$ is strictly increasing.
(b) $\mathrm{otp}(C_\nu) = \theta(\nu) \leq \kappa$.
(c) C_γ is closed.
(d) If $\bar{\nu} \in C_\nu$ then $C_\nu \cap \bar{\nu} = C_{\bar{\nu}}$.
(e) If D_ν is unbounded in ν then so is C_ν.

We have shown that $\Box_\kappa$ holds. ⊣

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