Proper forcing and remarkable cardinals II

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Abstract. The current paper proves the results announced in [6].

We isolate a new large cardinal concept, "remarkability." Consistencywise, remarkable cardinals are between ineffable and ω-Erdös cardinals. They are characterized by the existence of "$0^\sharp$-like" embeddings; however, they relativize down to L. It turns out that the existence of a remarkable cardinal is equiconsistent with $L(\mathbb{R})$ absoluteness for proper forcings. In particular, said absoluteness does not imply Π^1_1 determinacy.

Large cardinals are widely used for measuring the consistency strength of set theoretic principles. The current paper isolates a new large cardinal concept, "remarkability," which measures the power of proper forcing to change (in a certain respect) the shape of the universe. In fact, in this paper we shall give proofs of the main result announced in [6].

Let $\mathcal{F}$ be a class of set-sized posets. We say that $L(\mathbb{R})$ is absolute for forcings of type $\mathcal{F}$ if for all $P \in \mathcal{F}$, for all G being P-generic over V, for all formulae $\Phi(\vec{v})$, and for all $\vec{x} \in \mathbb{R}^V$ do we have that

$$L(\mathbb{R}^V) \models \Phi(\vec{x}) \Leftrightarrow L(\mathbb{R}^{V[G]}) \models \Phi(\vec{x}).$$

We say that $L(\mathbb{R})$ is absolute for c.c.c. (or, proper, ..., set) forcing if $L(\mathbb{R})$ is absolute for forcings of type $\mathcal{F}$ where $\mathcal{F} = \{P\colon P \text{ has the c.c.c.}\}$ (or, $\mathcal{F} = \{P\colon P \text{ is proper}\}$, ..., $\mathcal{F} = \{P\colon P \text{ is any poset (in } V)\}$).

The existence of large cardinals (for example, of a proper class of Woodin cardinals) implies that $L(\mathbb{R})$ is absolute for set forcing. (This is due Woodin.) The upshot is that an $L(\mathbb{R})$-stability of this sort even proves that AD, the Axiom of Determinacy, holds in $L(\mathbb{R})$. (This is due to Woodin; a slightly weaker version of it was shown later and independently by Steel; cf. [9].) However, $L(\mathbb{R})$ absoluteness for forcings of type $\mathcal{F}$ can be considerably weaker than $AD^{L(\mathbb{R})}$ if $\mathcal{F}$ is sufficiently

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"small." For example, a theorem of Kunen says that $L(\mathbb{R})$ absoluteness for c.c.c. forcing is equiconsistent with a weakly compact cardinal.

Results of Foreman, Magidor, Shelah, and Woodin can be used to see that semi-proper forcing can change the size of $\theta^{L(\mathbb{R})}$ – even in the presence of supercompact cardinals. (Recall that θ is defined to be the supremum of the order types of all pre-wellorderings of $\mathbb{R}$.) Hence the existence of large cardinals cannot imply that the boldface theory of $L(\mathbb{R})$ is absolute for set forcing, where "boldface" means that reals from the ground model as well as ordinals are allowed as parameters. This is tight in the sense that the main theorems of [3] and [4] say that – under appropriate assumptions – the boldface theory of $L(\mathbb{R})$ cannot be changed by set-sized *proper forcing*.

Proper forcing was discovered by Shelah (cf. [7]). Recall that a poset P is called proper if for all $\alpha \geq \omega_1$ and for all G being P-generic over V we have that every $S \subset [\alpha]^\omega$ from V which is stationary in V remains stationary in $V[G]$. How strong is $L(\mathbb{R})$ absoluteness for proper forcing? In particular, which amount of determinacy does it imply?

This question is particularily interesting, as the forcing which is used for proving that $L(\mathbb{R})$ absoluteness for set forcing gives $\mathbf{\Pi}^1_1$ determinacy, say, will collapse ω_1 (it is $Col(\omega, \lambda)$ for some λ). The question thus really is whether one can use more "coding like" forcings instead, to get the same conclusion, and whether the coding is proper. Our paper [5] gave some partial answers; it is shown there that "coding is reasonable and stationary preserving" (cf. [5] for details). Here we prove that "coding is not proper, in general."

Our main theorem, 3.6, will say that $L(\mathbb{R})$ absoluteness for proper forcing is equiconsistent with the existence of what we shall call a *remarkable cardinal*, and the same holds for boldface $L(\mathbb{R})$ absoluteness for proper forcing as well as the $L(\mathbb{R})$ anti coding theorem for proper forcings (cf. 2.5). As remarkable cardinals turn out to be compatible with $V = L$, this means that even the conclusions of the main theorems of [3] and [4] do not imply Π^1_1 determinacy.

A technical lemma. Our proofs will use the following simple and well-known lemma. For completeness, we have indicated its proof.

Lemma 0.1 *Let $\mathcal{M} = (M; (R_i : i < n))$ and $\mathcal{N} = (N; (S_i : i < m))$ be models such that $n \leq m$, R_i has the same arity as S_i for $i < n$, and M is countable. Then there is a tree T of height $\leq \omega$ searching for $(R_i : n \leq i < m)$ together with an elementary embedding*

$$\pi : (M; (R_i : i < m)) \to (N; (S_i : i < m)).$$

Proof. Let $(e_i : i < \omega)$ be an enumeration of M, and let $(\Phi_i(\vec{v}) : i < \omega)$ be an enumeration of all formulae of the language associated with $\mathcal{N}$. Let $\sharp(i)$ denote the arity of R_i (= of S_i) for $i < n$. Let $\gamma\colon \omega \to \omega \times {}^{<\omega}\omega$ be such that $\Phi_{\gamma(i)_0}$ has the variables with indices $< dom(\gamma(i)_1)$ as its free variables and $ran(\gamma(i)_1) \subset i-1$, and such that γ is "onto" in the obvious sense. Let $\mathcal{F}$ be a Skolem function for $\mathcal{N}$; more precisely, let $\mathcal{F}(i, \vec{x})$ be such that

$$\mathcal{N} \models \exists y\ \Phi_i(y, \vec{x}) \Rightarrow \Phi_i(\mathcal{F}(i, \vec{x}), \vec{x})$$

(if there is no such y then we let $\mathcal{F}(i, \vec{x})$ undefined). Let the k^{th} level of T consist of sequences $f\colon k \to N$ such that $f \restriction k-1 \in (k-1)^{st}$ level of T,

$$\forall i < n \forall \{l_1, ..., l_{\sharp(i)}\} \subset k\ (\ R_i(e_{l_1}, ..., e_{l_{\sharp(i)}}) \Leftrightarrow S_i(f(l_1), ..., f(l_{\sharp(i)}))\ ),\ \text{and}$$

$$f(k-1) = \mathcal{F}(\gamma(k)_0, f \circ \gamma(k)_1(1), ..., f \circ \gamma(k)_1(dom(\gamma(k)_1) - 1))$$

(if this is defined, otherwise we let $f(k-1) =$ an arbitrary element of N).

Now if $f\colon \omega \to N$ is given by an infinite branch through T then it is easy to see that by setting $R_i(e_{l_1}, ..., e_{l_p}) \Leftrightarrow S_i(f(l_1), ..., f(l_p))$ for $n \leq i < m$ and $\pi(e_i) = f(i)$ we get relations and an embedding as desired. On the other hand, any such relations together with some such embedding define an infinite branch through T.

$\square$ (0.1)

As an immediate corollary to this proof we get the following.

Lemma 0.2 *Let $\mathcal{M} = (M; (R_i\colon i < n))$ and $\mathcal{N} = (N; (S_i\colon i < m))$ be models such that $n \leq m$, R_i has the same arity as S_i for $i < n$, and M is countable. Let Q be an admissible set such that $\mathcal{M}, \mathcal{N} \in Q$, and M is countable in Q. If in V there are R_i, $n \leq i < m$, together with an elementary embedding*

$$\pi\colon (M; (R_i\colon i < m)) \to (N; (S_i\colon i < m))$$

then such R_i, π also exist in Q.

1 Remarkable cardinals.

We commence with an official definition.

Definition 1.1 *A cardinal κ is called* remarkable *iff for all regular cardinals $\theta > \kappa$ there are π, M, $\bar{\kappa}$, σ, N, and $\bar{\theta}$ such that the following hold:*

- *$\pi\colon M \to H_\theta$ is an elementary embedding,*
- *M is countable and transitive,*

- $\pi(\bar\kappa) = \kappa$,
- $\sigma\colon M \to N$ *is an elementary embedding with critical point* $\bar\kappa$*,*
- N *is countable and transitive,*
- $\bar\theta = M \cap OR$ *is a regular cardinal in* N*,* $\sigma(\bar\kappa) > \bar\theta$*, and*
- $M = H^N_{\bar\theta}$*, i.e.,* $M \in N$ *and* $N \models$ *"*M *is the set of all sets which are hereditarily smaller than* $\bar\theta$*."*

It is the last clause of 1.1 which gives remarkable cardinals their strength.

As a matter of fact, "remarkability" relativizes down to L, i.e., any remarkable cardinal is also remarkable in L (cf. 1.7 below). Hence the existence of remarkable cardinals is consistent with $V = L$. It is an easy exercise to verify that every remarkable cardinal is totally indescribable. In particular, the least measurable cardinal is not remarkable. However, every strong cardinal *is* remarkable, and we shall see below (cf. 1.3) that every Silver indiscernible is remarkable in L.

The following two lemmata 1.2 and 1.4 will give information as to where remarkable cardinals sit in the large cardinal hierarchy. Cf. [2] for definitions of the large cardinal concepts mentioned.

Lemma 1.2 *Let* $\kappa \to (\omega)^{<\omega}$*. Then there are* $\alpha < \beta < \omega_1$ *such that* $L_\beta \models$ *"*$ZFC + \alpha$ *is a remarkable cardinal."*

Proof. We may assume that $V = L$, as $\kappa \to (\omega)^{<\omega}$ relativizes down to L. Let $\pi\colon L_\gamma \to L_\kappa$ be an elementary embedding such that $ran(\pi)$ is the Skolem hull in L_κ of ω many indiscernibles for L_κ. Let α, β (with $\alpha < \beta$) be the images of the first two indiscernibles under π^{-1}. Of course, $L_\beta \models ZFC$, as any of the indiscernibles is inaccessible in L. We claim that α is remarkable in L_β.

Let $\theta < \beta$ be regular in L_β with $\theta > \alpha$. There is $\sigma\colon L_\gamma \to L_\gamma$ with $\sigma(\alpha) = \beta$, obtained from shifting the indiscernibles. I.e., there is some countable $L_{\bar\theta}$ (namely, L_θ) together with some $\bar\pi\colon L_{\bar\theta} \to L_{\pi(\theta)}$ (namely, $\pi \restriction L_\theta$) such that $\pi(\alpha)$ is in the range of $\bar\pi$, and there is some $\bar\sigma\colon L_{\bar\theta} \to L_{\tilde\theta}$ (namely, $\sigma \restriction L_\theta$) with critical point $\bar\pi^{-1}(\pi(\alpha))$ such that $\tilde\theta$ is countable, $\bar\theta$ is a regular cardinal in $L_{\tilde\theta}$, and $\bar\sigma(\bar\pi^{-1}(\pi(\alpha))) > \bar\theta$. As $\pi(\beta)$ is inaccessible in L, the same holds in $L_{\pi(\beta)}$. Pulling it back via π^{-1} we get that in L_β do we have that there is some countable $L_{\bar\theta}$ together with some $\bar\pi\colon L_{\bar\theta} \to L_\theta$ such that α is in the range of $\bar\pi$, and there is some $\bar\sigma\colon L_{\bar\theta} \to L_{\tilde\theta}$ with critical point $\bar\pi^{-1}(\alpha)$ such that $\tilde\theta$ is countable, $\bar\theta$ is a regular cardinal in $L_{\tilde\theta}$, and $\bar\sigma(\bar\pi^{-1}(\alpha)) > \bar\theta$. As $\theta > \alpha$ was an arbitrary regular cardinal in L_β, we have shown that α is remarkable in L_β.

□ (1.2)

As an immediate corollary to this proof we get:

Lemma 1.3 *Suppose that $0^\sharp$ exists. Then every Silver indiscernible is remarkable in L.*

Proof. A slight variation of the previous proof gives that $L_\beta \models$ "α is remarkable" whenever $\alpha < \beta$ are both indiscernibles for L. But then every Silver indiscernible is remarkable in L.

$\square$ (1.3)

Lemma 1.4 *Let κ be remarkable. Then there are $\alpha < \beta < \omega_1$ such that $L_\beta \models$ "ZFC $+$ α is a ineffable cardinal."*

Proof. Let $\theta = \kappa^+$, and let π, M, σ, and N be as in 1.1. Let $\alpha = \pi^{-1}(\kappa)$ and let $\beta = \sigma(\alpha)$. It is easy to see that $L_\beta \models ZFC$. We claim that α is ineffable in L_β.

Let $(A_i : i < \alpha) \in L_\beta$ be such that $A_i \subset i$ for all $i < \alpha$, and let $C \in L_\beta$ be club in α. There is $(A_i : \alpha \leq i < \beta)$ such that $\sigma((A_i : i < \alpha)) = (A_i : i < \beta)$. Notice that $A_\alpha \in M$, as $\mathcal{P}(\alpha) \cap N = \mathcal{P}(\alpha) \cap M$ by the properties of M, σ, and N. Now of course $A_\alpha = \sigma(A_\alpha) \cap \alpha$, and also $\alpha \in \sigma(C)$. This gives that $\alpha \in \sigma(\{i < \alpha : A_i = A_\alpha \cap i\}) \cap \sigma(C)$, and thus via σ we have that $\{i < \alpha : A_i = A_\alpha \cap i\} \cap C \neq \emptyset$. As C was arbitrary, we have shown that α is ineffable in L_β.

$\square$ (1.4)

The previous argument can easily be adopted to show that every remarkable cardinal is ineffable.

We now turn towards a useful characterization of remarkability.

Definition 1.5 *Let κ be a cardinal. Let G be $Col(\omega, <\kappa)$-generic over V, let $\theta > \kappa$ be a regular cardinal, and let $X \in [H_\theta^{V[G]}]^\omega$. We say that X* condenses remarkably *if $X = ran(\pi)$ for some elementary*

$$\pi : (H_\beta^{V[G \cap H_\alpha^V]}; \in, H_\beta^V, G \cap H_\alpha^V) \to (H_\theta^{V[G]}; \in, H_\theta^V, G)$$

where $\alpha = crit(\pi) < \beta < \kappa$ and β is a regular cardinal (in V).

Notice that in the situation of 1.5 we will have that α is inaccessible in V, $G \cap H_\alpha^V$ is $Col(\omega, <\alpha)$-generic over V, and hence β is a regular cardinal in $V[G \cap H_\alpha^V]$, too.

Lemma 1.6 *A cardinal κ is remarkable if and only if for all regular cardinals $\theta > \kappa$ do we have that*

$$\Vdash^V_{Col(\omega, <\kappa)} \text{ "}\{X \in [H_{\check\theta}^{V[\dot G]}]^\omega : X \text{ condenses remarkably}\} \text{ is stationary."}$$

PROOF. "$\Rightarrow$." Let κ be remarkable, and let $\theta > \kappa$ be a regular cardinal. We may pick $\pi\colon M \to H_{\theta^+}$ as in 1.1, but with θ^+ playing the role of θ. Let $\bar\kappa$, $\bar\theta = \pi^{-1}(\kappa, \theta)$, and let $\sigma\colon M \to N$ with critical point $\bar\kappa$ be such that N is countable and transitive, $\rho = M \cap OR$ is regular in N, $M = H_\rho^N$, and $\sigma(\bar\kappa) > \rho$. In V, we may pick G being $Col(\omega, < \bar\kappa)$-generic over M (and hence over N), and we may pick $G' \supset G$ being $Col(\omega, < \sigma(\bar\kappa))$-generic over N. We then have that σ naturally extends to $\tilde\sigma\colon M[G] \to N[G']$.

Let $\mathcal{M} = (H_{\bar\theta}^{M[G]}; \in, H_{\bar\theta}^M, G, (R_i\colon i < n)) \in M[G]$ be any model of finite type. Notice that $\mathcal{M} \in N[G']$ and is countable there. By the existence of $\tilde\sigma \restriction H_{\bar\theta}^{M[G]}\colon \mathcal{M} \to \tilde\sigma(\mathcal{M})$ together with 0.2, we get that in $N[G']$ there is an elementary embedding τ of $\mathcal{M}$ into $\tilde\sigma(\mathcal{M})$. This means that in $N[G']$ it is true that

$$\exists\alpha < \beta < \sigma(\bar\kappa)\exists\tau\ (\tau\colon (H_\beta^{V[G' \cap H_\alpha]}; \in, ...) \to \tilde\sigma(\mathcal{M}) \text{ with } \alpha = c.p.(\tau) \wedge \beta \text{ is regular }).$$

Pulling this back via $\tilde\sigma$ gives that in $M[G]$ it is true that

$$\exists\alpha < \beta < \bar\kappa\exists\tau\ (\tau\colon (H_\beta^{V[G \cap H_\alpha]}; \in, ...) \to \mathcal{M} \text{ with } \alpha = c.p.(\tau) \wedge \beta \text{ is regular }).$$

As $\mathcal{M}$ was arbitrary, we have shown that

$$\Vdash^M_{Col(\omega,<\bar\kappa)} \text{ "}\{X \in [H_{\check{\bar\theta}}^{M[\dot G]}]^\omega\colon X \text{ condenses remarkably }\} \text{ is stationary."}$$

Lifting this up via π gives

$$\Vdash^V_{Col(\omega,<\kappa)} \text{ "}\{X \in [H_{\check\theta}^{V[\dot G]}]^\omega\colon X \text{ condenses remarkably }\} \text{ is stationary."}$$

As θ was arbitrary, this proves "$\Rightarrow$."

"$\Leftarrow$." Let $\theta > \kappa$ be a regular cardinal, and suppose that

$$\Vdash^V_{Col(\omega,<\kappa)} \text{ "}\{X \in [H_{\check\theta}^{V[\dot G]}]^\omega\colon X \text{ condenses remarkably }\} \text{ is stationary."}$$

Let $\bar\pi\colon \bar M \to H_{\theta^+}$ with $\bar M$ countable and transitive be such that κ, $\theta \in ran(\bar\pi)$. Let $\bar\kappa$, $\bar\theta = \bar\pi^{-1}(\kappa, \theta)$. In V, we may pick G being $Col(\omega, < \bar\kappa)$-generic over $\bar M$. Because

$$\Vdash^{\bar M}_{Col(\omega,<\bar\kappa)} \text{ "}\{X \in [H_{\check{\bar\theta}}^{V[\dot G]}]^\omega\colon X \text{ condenses remarkably }\} \text{ is stationary,"}$$

inside $\bar M[G] \subset V$ we get some $\bar\sigma\colon H_\rho^{\bar M} \to H_{\bar\theta}^{\bar M}$ with $c.p.(\bar\sigma) < \rho < \bar\kappa$ and such that ρ is a regular cardinal in $\bar M$.

Now set $M = H_\rho^{\bar M}$, $N = H_{\bar\theta}^{\bar M}$, $\sigma = \bar\sigma$, and $\pi = \bar\pi \circ \bar\sigma$. Then π, M, $\bar\kappa$, σ, N, and $\bar\theta$ are as in 1.1. As θ was arbitrary, this proves "$\Leftarrow$."

$\square$ (1.6)

Lemma 1.7 *Let κ be remarkable. Then $L \models$ "κ is remarkable."*

PROOF. Let $\theta > \kappa$ be a regular cardinal in L. Let G be $Col(\omega, < \kappa)$-generic over V, and let $\mathcal{M} = (L_\theta[G]; \in, \vec{R}) \in L[G]$ be any model of finite type. Let $\mathcal{N} = (H_{\theta^+}^{V[G]}; \in, H_{\theta^+}^V, G, L_\theta[G], \vec{R})$. As κ is remarkable, in $V[G]$ we may pick some

$$\pi\colon (H_\beta^{V[G\cap H_\alpha]}; \in, H_\beta^V, G \cap H_\alpha, L_{\bar\theta}[G \cap L_\alpha], \vec{\bar{R}}) \to \mathcal{N}$$

where $\alpha = crit(\pi) < \beta < \kappa$ and β is a regular cardinal in V. Then

$$\pi \restriction L_{\bar\theta}[G \cap L_\alpha]\colon (L_{\bar\theta}[G \cap L_\alpha], \in, \vec{\bar{R}}) \to \mathcal{M},$$

and $\bar\theta$ is a regular cardinal in L. Because $L_{\bar\theta}[G \cap L_\alpha] \in L[G]$ and is countable there, 0.2 and the existence of $\pi \restriction L_{\bar\theta}[G \cap L_\alpha]$ yield that inside $L[G]$ there are predicates $\vec{\bar{S}}$ on $L_{\bar\theta}[G \cap L_\alpha]$ together with an elementary embedding

$$\sigma\colon (L_{\bar\theta}[G \cap L_\alpha]; \in, \vec{\bar{S}}) \to \mathcal{M}.$$

I.e., $(ran(\sigma); \in, \vec{R} \restriction ran(\sigma)) \prec \mathcal{M}$ where $ran(\sigma) \in L[G]$ and is countable there. As θ and then $\mathcal{M}$ were arbitrary we have shown that in L does κ satisfy the characterization of remarkability from 1.6.

$\square$ (1.7)

2 Getting $L(\mathbb{R})$ absoluteness.

Lemma 2.1 *Let κ be remarkable in L. Let G be $Col(\omega, < \kappa)$-generic over L. Let $P \in L[G]$ be a proper poset, and let H be P-generic over $L[G]$. Then for every real x in $L[G][H]$ there is a poset $Q_x \in L_\kappa$ such that x is Q_x-generic over L.*

PROOF. Let $\theta > \kappa$ be a regular L-cardinal such that $\mathcal{P}(P) \subset L_\theta[G]$. Let $x \in \mathbb{R} \cap L[G][H]$, and let $\dot{x} \in L_\theta[G]$ be such that $\dot{x}^H = x$. Consider the structure $\mathcal{M} = (L_\theta[G]; \in, P, \dot{x}, H)$. Because κ is remarkable in L and P is proper we may pick an elementary

$$\pi\colon (L_\beta[G \cap L_\alpha]; \in, \bar{P}, \bar{\dot{x}}, \bar{H}) \to \mathcal{M}$$

with the property that $G \cap L_\alpha$ is $Col(\omega, < \alpha)$-generic over L and β is an $L[G \cap L_\alpha]$-cardinal. By elementarity, $\bar{H}$ is $\bar{P}$-generic over $L_\beta[G \cap L_\alpha]$, and hence over $L[G \cap L_\alpha]$, as $\mathcal{P}(\bar{P}) \cap L[G \cap L_\alpha] \subset L_\beta[G \cap L_\alpha]$. Moreover, by the definability of forcing, we get that $n \in \bar{\dot{x}}^{\bar{H}}$ iff $\exists p \in \bar{H}$ $p \Vdash \check{n} \in \bar{\dot{x}}$ iff $\exists p \in H$ $p \Vdash \check{n} \in \dot{x}$ iff $n \in \dot{x}^H$ iff $n \in x$.

So $\dot{\bar{x}}^{\bar{H}} = x$, and we may set $Q_x = Col(\omega, < \alpha) \star \dot{\bar{P}}$ where $\dot{\bar{P}}^{\bar{H}} = \bar{P}$. Notice that $Q_x \in L_\kappa$.

$\square$ (2.1)

Lemma 2.2 *Let κ be remarkable in L. Let G be $Col(\omega, < \kappa)$-generic over L. Let $P \in L[G]$ be a proper poset, and let H be P-generic over $L[G]$. Let E be $Col(\omega, (2^{\aleph_0})^{L[G][H]})$-generic over $L[G][H]$. Then in $L[G][H][E]$ there is some G' being $Col(\omega, < \kappa)$-generic over L such that*

$$\mathbb{R} \cap L[G'] = \mathbb{R} \cap L[G][H].$$

Proof. Let $(e_i : i < \omega) \in L[G][H][E]$ be such that $\{e_i : i < \omega\} = \mathbb{R} \cap L[G][H]$. By working inside $L[G][H][E]$ we may easily use 2.1 to construct $(\alpha_i, G_i : i < \omega)$ such that $\alpha_0 < \alpha_1 < ...$ and for all $i < \omega$ we have that G_i is $Col(\omega, < \alpha_i)$-generic over L, $G_{i-1} \subset G_i$ (with the convention that $G_{-1} = \emptyset$), $G_i \in L[G][H]$, and $e_i \in L[G_i]$. Set

$$G' = \bigcup_i G_i.$$

Because $Col(\omega, < \kappa)$ has the κ-c.c., G' is $Col(\omega, < \kappa)$-generic over L, and every real in $L[G']$ is in $L[G_i]$ for some $i < \omega$. We get that $\mathbb{R} \cap L[G'] = \mathbb{R} \cap L[G][H]$ as desired.

$\square$ (2.2)

Definition 2.3 *Let $\mathcal{F} \subset V$ be a class of posets. We say that* the $L(\mathbb{R})$ embedding theorem holds for forcings of type $\mathcal{F}$ *if for all posets $P \in \mathcal{F}$, for all G being P-generic over V, for all formulae $\Phi(\vec{v})$, for all $\vec{\alpha} \in OR$, and for all $\vec{x} \in \mathbb{R}^V$ do we have that*

$$L(\mathbb{R}^V) \models \Phi(\vec{\alpha}, \vec{x}) \Leftrightarrow L(\mathbb{R}^{V[G]}) \models \Phi(\vec{\alpha}, \vec{x}).$$

We say that the $L(\mathbb{R})$ embedding theorem holds for proper forcings *if $L(\mathbb{R})$ is absolute under forcings of type $\mathcal{F}$ where $\mathcal{F} = \{P \in V : P$ is proper $\}$.*

Theorem 2.4 (Embedding theorem in $L[G]$) *Let κ be remarkable in L. Let G be $Col(\omega, < \kappa)$-generic over L, and write $V = L[G]$. Then in V the $L(\mathbb{R})$ embedding theorem holds for proper forcings.*

Proof. Let $P \in V$ be a proper poset, and let H be P-generic over V. By 2.2 (in some further extension) there is G' being $Col(\omega, < \kappa)$-generic over L such that

$\mathbb{R} \cap L[G'] = \mathbb{R} \cap V[H]$. Let $\phi(\vec{v}, \vec{w})$ be a formula, let $\vec{x} \in \mathbb{R} \cap V$, and let $\vec{\alpha} \in OR$. We then have that

$$L(\mathbb{R}^V) \models \phi(\vec{x}, \vec{\alpha}) \Leftrightarrow$$
$$\Vdash^{L[\vec{x}]}_{Col(\omega,<\kappa)} L(\dot{\mathbb{R}}) \models \phi(\check{\vec{x}}, \check{\vec{\alpha}}) \Leftrightarrow$$
$$L(\mathbb{R}^{V[H]}) \models \phi(\vec{x}, \vec{\alpha}).$$

□ (2.4)

Definition 2.5 *Let $\mathcal{F} \subset V$ be a class of posets. We say that* the $L(\mathbb{R})$ anti coding theorem holds for forcings of type $\mathcal{F}$ *if for all posets $P \in \mathcal{F}$, for all G being P-generic over V, and for all $A \subset OR$ with $A \in V$ do we have that*

$$A \in L(\mathbb{R}^V) \Leftrightarrow A \in L(\mathbb{R}^{V[G]}).$$

We say that the $L(\mathbb{R})$ anti coding theorem holds for proper forcings *if the $L(\mathbb{R})$ anti coding theorem holds for forcings of type $\mathcal{F}$ where $\mathcal{F} = \{P \in V \colon P$ is proper $\}$.*

Theorem 2.6 (Anti-coding theorem in $L[G]$) *Let κ be remarkable in L. Let G be $Col(\omega, <\kappa)$-generic over L, and write $V = L[G]$. Then in V the $L(\mathbb{R})$ anti coding theorem holds for proper forcings.*

Proof. Let $P \in V$ be a proper poset, and let H be P-generic over V. By 2.4, it suffices to show that each $A \in L(\mathbb{R}^{V[H]}) \cap V$ is also in $L(\mathbb{R}^V)$. Fix such an A, and let Φ a formula, $\vec{\alpha} \in OR$, and $x \in \mathbb{R}^{V[H]}$ be such that

$$\gamma \in A \Leftrightarrow L(\mathbb{R}^{V[H]}) \models \Phi(\vec{\alpha}, x, \gamma).$$

Let $\dot{x}^H = x$, and assume w.l.o.g. that

$$(\star) \qquad \emptyset \Vdash^V_P \gamma \in \check{A} \Leftrightarrow L(\dot{\mathbb{R}}) \models \Phi(\check{\vec{\alpha}}, \dot{x}, \check{\gamma}).$$

As in the proof of 2.1, we may pick an elementary

$$\pi \colon (L_\beta[G \cap L_\alpha]; \in, \bar{P}, \bar{\dot{x}}, \bar{H}) \to (L_\theta[G]; \in, P, \dot{x}, H)$$

such that β is an $L[G \cap L_\alpha]$-cardinal. Because $L_\beta[G \cap L_\alpha]$ is countable in V we may pick $h \in V$ being $\bar{P}$-generic over $L_\beta[G \cap L_\alpha]$. Of course, h will then also be $\bar{P}$-generic over $L[G \cap L_\alpha]$. Because P is proper we may and shall assume w.l.o.g. that (inside some further forcing extension) for every $p \in \bar{P}$ there is G^p being P-generic over V with $\pi(p) \in G^p$ and such that $\pi^{-1}{}''G^p$ is $\bar{P}$-generic over $L_\beta[G \cap L_\alpha]$ (i.e., over

$L[G \cap L_\alpha])$. Notice that $\dot{x}^{G^p} = \dot{x}^{\bar{G}^p}$ for every $p \in \bar{P}$. In order to prove 2.6 it now clearly suffices to verify the following.

Claim. For all $\gamma \in OR$, $\gamma \in A \Leftrightarrow\ \Vdash^{L[G\cap L_\alpha][h]}_{Col(\omega,<\kappa)}\ L(\dot{\mathbb{R}}) \models \Phi(\check{\vec{\alpha}}, \bar{\dot{x}}^h, \check{\gamma})$.

PROOF. We shall prove "$\Leftarrow$." The proof of "$\Rightarrow$" is almost identical in that it starts from $\neg\ \Phi$ instead of from Φ, and gives $\gamma \notin A$ instead of $\gamma \in A$. Suppose that

$$\Vdash^{L[G\cap L_\alpha][h]}_{Col(\omega,<\kappa)}\ L(\dot{\mathbb{R}}) \models \Phi(\check{\vec{\alpha}}, \bar{\dot{x}}^h, \check{\gamma}).$$

This is itself forced by some $p \in h$, and thus we also get, writing $\bar{G}^p = \pi^{-1}``G^p$, that

$$\Vdash^{L[G\cap L_\alpha][\bar{G}^p]}_{Col(\omega,<\kappa)}\ L(\dot{\mathbb{R}}) \models \Phi(\check{\vec{\alpha}}, \bar{\dot{x}}^{\bar{G}^p}, \check{\gamma}).$$

Because $\bar{G}^p = \pi^{-1}``G^p \in L[G][G^p]$, in much the same way as in the proof of 2.1 we can pick (inside some further forcing extension) some G' being $Col(\omega, <\kappa)$-generic over $L[G \cap L_\alpha][\bar{G}^p]$ such that

$$\mathbb{R} \cap L[G \cap L_\alpha][\bar{G}^p][G'] = \mathbb{R} \cap L[G][G^p].$$

Hence

$$L(\mathbb{R}^{V[G^p]}) \models \Phi(\vec{\alpha}, \bar{\dot{x}}^{\bar{G}^p}, \gamma).$$

But $\bar{\dot{x}}^{\bar{G}^p} = \dot{x}^{G^p}$, so that there is some $q \in G^p$ such that

$$q \Vdash^V_P L(\dot{\mathbb{R}}) \models \Phi(\check{\vec{\alpha}}, \dot{x}, \check{\gamma}).$$

Hence, by $(\star)$, $q \Vdash^V_P \check{\gamma} \in \check{A}$, which implies that $\gamma \in A$.

$\square$ (Claim)
$\square$ (2.6)

Here is an immediate corollary to 2.4 and 2.6, when combined with 1.7.

Corollary 2.7 *Neither the conclusion of the $L(\mathbb{R})$ embedding theorem for proper forcings nor the conclusion of the $L(\mathbb{R})$ anti coding theorem for proper forcings implies Π^1_1-determinacy.*

3 An equiconsistency.

Definition 3.1 *Let $A \subset OR$. We say that A is* good *if $A \subset \omega_1$ and $L_{\omega_2}[A] = H_{\omega_2}$.*

Lemma 3.2 *If $0^\sharp$ does not exist then there is a proper $P \in V$ such that*

$$\Vdash_P \text{"there is a good } A\text{."}$$

PROOF. This uses almost disjoint forcing in its simplest form. Fix δ, a singular cardinal of uncountable cofinality and such that $\delta^{\aleph_0} = \delta$ (for example, let δ be a strong limit). By Jensen's Covering Lemma, we know that $\delta^{+L} = \delta^+$. We may also assume w.l.o.g. that $2^\delta = \delta^+$, because otherwise we may collapse 2^δ onto δ^+ by a δ-closed preliminary forcing. We may hence pick $B \subset \delta^+$ with the property that $H_{\delta^+} = L_{\delta^+}[B]$.

Now let G_1 be $Col(\delta, \omega_1)$-generic over V. Notice that the forcing is ω-closed. Set $V_1 = V[G_1]$. We have that $\omega_2^{V_1} = \delta^+ = \delta^{+L}$. Let $C \subset \omega_1$ code G_1 (in the sense that $G_1 \in L_{\omega_2^{V_1}}[C]$). Using the fact that $Col(\delta, \omega_1)$ has the δ^+-c.c., it is easy to verify that in V_1, $H_{\omega_2} = L_{\omega_2}[B, C]$. Let ω_2 denote $\omega_2^{V_1}$ from now on.

In L we may pick $(A'_\xi : \xi < \delta^+)$, a sequence of almost disjoint subsets of δ. In $L_{\omega_2}[C]$ we may pick a bijective $g: \omega_1 \to \delta$. Then if we let $\alpha \in A_\xi$ iff $g(\alpha) \in A'_\xi$ for $\alpha < \omega_1$ and $\xi < \omega_2$, we have that $(A_\xi : \xi < \omega_2) \in L[C]$ is a sequence of almost disjoint subsets of ω_1.

In V_1, we may pick $D \subset \omega_2$ with $H_{\omega_2} = L_{\omega_2}[B, C] = L_{\omega_2}[D]$ (for example, $D = B \oplus C$). We let P_2 be the forcing for coding D by a subset of ω_1, using the almost disjoint sets A_ξ.

To be specific, P_2 consists of pairs $p = (l(p), r(p))$ where $l(p): \alpha \to 2$ for some $\alpha < \omega_1$ and $r(p)$ is a countable subset of ω_2. We have $p = (l(p), r(p)) \leq_{P_2} q = (l(q), r(q))$ iff $l(p) \supset l(q)$, $r(p) \supset r(q)$, and for all $\xi \in r(q)$, if $\xi \in D$ then

$$\{\beta \in dom(l(p)) \setminus dom(l(q)) : l(p)(\beta) = 1\} \cap A_\xi \;=\; \emptyset.$$

By a Δ-system argument, P_2 has the ω_2-c.c. It is clearly ω-closed, so no cardinals are collapsed. Moreover, if G_2 is P_2-generic over V_1, and if we set

$$A' \;=\; \bigcup_{p \in G_2} \{\beta \in dom(l(p)) : l(p)(\beta) = 1\},$$

then $A' \subset \omega_1$ and we have that for all $\xi < \omega_2$,

$$\xi \in D \text{ iff } Card(A' \cap A_\xi) \leq \aleph_0.$$

This means that D is an element of any inner model containing $(A_\xi : \xi < \omega_2)$ and A'. (Of course, much more holds.) An example of such a model is $L[C, A']$. Set $V_2 = V_1[G_2]$, and let $A = C \oplus A'$. Because P_2 has the ω_2-c.c., we get that in V_2 do we have $H_{\omega_2} = L_{\omega_2}[A]$.

Recall that all the forcings we have used to obtain V_2 were ω-closed. In particular, V_2 is a proper set-generic extension of V.

$\square$ (3.2)

It is easy to see that the conclusion of 3.2 is actually equivalent with the property that V is not closed under $\sharp$'s.

Definition 3.3 *Let $A \subset \omega_1$. By $\nabla(A)$ we denote the assertion that*

$$\{X \in [L_{\omega_2}[A]]^\omega \colon \exists \alpha < \beta \in Card^{L[A\cap\alpha]}\ \exists \pi\ \pi\colon L_\beta[A \cap \alpha] \cong X \prec L_{\omega_2}[A]\}$$

is stationary in $[L_{\omega_2}[A]]^\omega$.

Theorem 3.4 *Suppose that $L(\mathbb{R})$ is absolute under proper forcings. Then*

$$\forall A\ (A \text{ good } \Rightarrow \nabla(A))$$

holds in all proper set-generic extensions of V.

Proof. Let Ψ denote the statement that the reals can be well-ordered in $L(\mathbb{R})$. By adding ω_1 Cohen reals with finite support, which is proper, one obtains an extension of V in which Ψ fails. Hence if $L(\mathbb{R})$ is supposed to be absolute under proper forcings, Ψ has to fail in V to begin with, and it has to fail in every proper set-forcing extension of V.

Let us now fix a good A such that $\nabla(A)$ fails. We shall define a proper forcing $P \in V$ such that

$$\Vdash_P\ \Psi.$$

This will give a contradiction, and prove 3.4 in V; of course, by replacing V by a proper set-forcing extension of itself, the very same argument will prove the full 3.4.

The key observation here is that $\neg\ \nabla(A)$ implies that "reshaping" our good A is proper. We let P_1 consist of functions $p\colon \alpha \to 2$ with $\alpha < \omega_1$ and such that for all $\xi \leq \alpha$ we have that

$$L[A \cap \xi, p \restriction \xi] \models Card(\xi) \leq \aleph_0.$$

This is Jensen's classical forcing for reshaping A (cf. [1]). We need the following.

Claim. P_1 is proper.

PROOF. Let us consider $\mathcal{M} = (L_{\omega_2}[A]; \in, A)$. Because $\nabla(A)$ fails, there is a club $C \subset [H_{\omega_2}]^\omega$ such that for all $X \in C$, if

$$\pi\colon (L_\beta[A \cap \alpha]; \in, A \cap \alpha) \cong (X; \in, A \cap X) \prec \mathcal{M}$$

then β is not a cardinal in $L[A \cap \alpha]$. Let us fix some such X. We have to show that for any $p \in P_1 \cap X$ there is $q \leq_{P_1} p$ which is (P, X)-generic.

For this we use an argument of [8]. Let $(\dot{\alpha}_i\colon i < \omega)$ enumerate the ordinal names in X. We shall produce $q \leq_{P_1} p$ such that for all $i < \omega$ we have that $q \Vdash \dot{\alpha}_i \in X$. We may assume w.l.o.g. that $\alpha = \omega_1^{L[A \cap \alpha]}$, as otherwise the task of constructing q turns out to be an easier variant of what is to follow. Now as β has size α in $L[A \cap \alpha]$ we may pick a club $E \subset \alpha$ in $L[A \cap \alpha]$ which grows faster than all clubs in $L_\beta[A \cap \alpha]$, i.e., whenever $\bar{E} \subset \alpha$ is a club in $L_\beta[A \cap \alpha]$ then $E \setminus \bar{E}$ is bounded in α.

We are now going to construct a sequence $(p_i : i < \omega)$ of conditions below p such that $p_{i+1} \leq_{P_1} p_i$ and $p_{i+1} \Vdash \dot{\alpha}_i \in X$. We also want to maintain inductively that $p_{i+1} \in L_\beta[A \cap \alpha]$. (Notice that $p \in L_\beta[A \cap \alpha]$ to begin with.) In the end we also want that setting $q = \bigcup_{i<\omega} p_i$, we have that $q \in P_1$, which of course is the the non-trivial part. For this purpose, we also pick $(\bar{\alpha}_i\colon i < \omega)$ cofinal in α.

To commence, let $p_0 = p$. Now suppose that p_i is given, $p_i \in L_\beta[A \cap \alpha]$. Set $\gamma = dom(p_i) < \alpha$. Work inside $L_\beta[A \cap \alpha]$ for a minute. For all δ such that $\gamma \leq \delta < \alpha$ we may pick some $p^\delta \leq_{P_1} p_i$ such that: $p^\delta \Vdash \pi^{-1}(\dot{\alpha}_i) \in L_\beta[A \cap \alpha]$, $dom(p^\delta) > max\{\bar{\alpha}_i, \delta\}$, and for all limit ordinals λ, $\gamma \leq \lambda \leq \delta$, $p^\delta(\lambda) = 1$ iff $\lambda = \delta$. Then there is $\bar{E}$ club in α such that for any $\eta \in \bar{E}$, $\delta < \eta \Rightarrow dom(p^\delta) < \eta$.

Now back in $L[A \cap \alpha]$, we may pick $\delta \in E$ such that $E \setminus \bar{E} \subset \delta$. Set $p_{i+1} = p^\delta$, and let for future reference $\delta = \delta_{i+1}$. Of course $p_{i+1} \Vdash \dot{\alpha}_i \in X$. We also have that $dom(p_{i+1}) < min\{\epsilon \in E : \epsilon > \delta\}$, so that for all limit ordinals $\lambda \in E \cap (dom(p_{i+1}) \setminus dom(p_i))$ we have that $p_{i+1}(\lambda) = 1$ iff $\lambda = \delta_{i+1}$.

Now set $q = \bigcup_{i<\omega} p_i$. We are done if we can show that q is a condition. Well, it is easy to see that we have arranged that $dom(q) = \alpha$, so that the only problem here is to show that

$$L[A \cap \alpha, q] \models Card(\kappa) \leq \aleph_0.$$

But by the construction of the p_i's we have that

$$\{\lambda \in E \cap (dom(q) \setminus dom(p)) : \lambda \text{ is a limit ordinal and } q(\lambda) = 1\}$$

$$= \{\delta_{i+1} : i < \omega\},$$

being a cofinal subset of E. But E is an element of $L[A \cap \alpha, q]$, so that $\{\delta_{i+1} : i < \omega\} \in L[A \cap \alpha, q]$ witnesses that $Card(\alpha) \leq \aleph_0$.

We have shown that $q \in P_1$ is (P, X)-generic, as desired.

$\square$ (Claim)

Now let G be P_1-generic over V, and pick $D \subset \omega_1$ such that $L_{\omega_2}[D] = L_{\omega_2}[A, G]$. We may now "code down to a real" by using almost disjoint forcing. By the fact that D is "reshaped," there is a (unique) sequence $(a_\beta : \beta < \omega_1)$ of subsets of ω such that for each $\beta < \omega_1$, a_β is the $L[D \cap \beta]$-least subset of ω being almost disjoint from any $a_{\bar\beta}$ for $\bar\beta < \beta$.

We then let P_2 consist of all pairs $p = (l(p), r(p))$ where $l(p)\colon n \to 2$ for some $n < \omega$ and $r(p)$ is a finite subset of ω_1. We let $p = (l(p), r(p)) \leq_{P_2} q = (l(q), r(q))$ iff $l(p) \supset l(q)$, $r(p) \supset r(q)$, and for all $\beta \in r(q)$, if $\beta \in D$ then

$$\{\gamma \in dom(r(p)) \setminus dom(r(q)) \,:\, r(p)(\gamma) = 1\} \cap a_\beta \;=\; \emptyset.$$

By a Δ-system argument, P_1 has the c.c.c.. Moreover, if H is P_2-generic over $V[G]$, and if we set

$$a = \bigcup_{p \in H} \{\gamma \in dom(l(p)) \,:\, l(p)(\gamma) = 1\},$$

then we have that for $\gamma < \omega_1$,

$$\gamma \in D \text{ iff } Card(a \cap a_\gamma) < \aleph_0.$$

Moreover, because P_2 has the c.c.c., we get that in $V[G][H]$ we have that $H_{\omega_2} = L_{\omega_2}[a]$.

In particular, $\mathbb{R} \cap V[G][H] \subset L[a]$ which implies that in $V[G][H]$ there is a $\Delta^1_2(a)$-well-ordering of the reals. Thus, if we set $P = P_1 \star \dot{P}_2$ then P is proper and

$$\Vdash_P \;\Psi.$$

$\square$ (3.4)

Lemma 3.5 *Suppose that $L(\mathbb{R})$ is absolute under proper forcings. Then ω_1 is remarkable in L.*

Proof. By 1.3, we may assume that $0^\sharp$ does not exist. Let $\theta > \kappa$ be a regular L-cardinal. Using 3.2 we may easily find a proper set-forcing extension of V in which there is a good B and in which $\theta < \omega_2$ (just primarily force with $Col(\omega_1, \theta)$, which is ω-closed). By finally forcing with $Col(\omega, < \omega_1)$ (which has the c.c.c.) we get a proper set-forcing extension of V in which we may pick a good A such that

$A_{odd} = \{2\delta + 1 \in A \colon \delta < \omega_1\}$ essentially is $Col(\omega, < \omega_1)$-generic over L, and $\theta < \omega_2$. By 3.4 we know that in that extension,

$$\{X \in [L_{\omega_2}[A]]^\omega \colon \exists \alpha < \beta \in Card^{L[A \cap \alpha]} \; \exists \pi \; \pi \colon L_\beta[A \cap \alpha] \cong X \prec L_{\omega_2}[A]\}$$

is stationary in $[L_{\omega_2}[A]]^\omega$. We may now argue exactly as in the proof of 1.7 to see that this implies that ω_1 has to be remarkable in L.

$\square$ (3.5)

Corollary 3.6 *The following are equiconsistent.*
(1) $L(\mathbb{R})$ is absolute for proper forcings.
(2) The $L(\mathbb{R})$ embedding theorem holds for proper forcings.
(3) $V \neq L(\mathbb{R})$, and the $L(\mathbb{R})$ anti coding theorem holds for proper forcings.
(4) There is a remarkable cardinal.

Proof. Con(1) $\Rightarrow$ Con(4) is 3.5. Con(4) $\Rightarrow$ Con(2) and Con(4) $\Rightarrow$ Con(3) are 2.4 and 2.6. Con(3) $\Rightarrow$ Con(4) follows from the proofs of 3.2 and 3.4. Con(2) $\Rightarrow$ Con(1) is trivial.

$\square$ (3.6)

4 A derived model theorem.

We have shown in 2.4 that there is a model of $L(\mathbb{R})$ absoluteness for proper forcing which is of the form $L[G]$ where G is $Col(\omega, < \kappa)$-generic over L for some inaccessible κ in L. We are now going to show that – under some genericity assumption – every model of $L(\mathbb{R})$ absoluteness for proper forcing is of this form.

Definition 4.1 *We let* ($\natural$) *denote the assertion that every real is set-generic over L, i.e., that for every $x \in \mathbb{R}$ there is some poset $P \in L$ and some $G \in V$ being P-generic over L such that $x \in L[G]$.*

Theorem 4.2 (Derived model theorem) *Assume that* ($\natural$) *holds and that $L(\mathbb{R})$ is absolute for proper forcing. Then (in some set-generic extension of V) there is G being $Col(\omega, < \omega_1^V)$-generic over L such that $L(\mathbb{R}^V) = L(\mathbb{R}^{L[G]})$.*

Proof. By 3.2 and 3.4 there is $V[H]$, a proper set-generic extension of V, in which there is a good A, and $\triangledown(A)$ holds. By ($\natural$), for every $x \in \mathbb{R}^V$ we may pick a poset $P_x \in L$ and some $K_x \in V$ being P_x-generic over L such that $x \in L[K_x]$.

Let θ_x be such that $P_x \in H_{\theta_x}$. By primarily forcing with $Col(\omega_1, sup_{x \in \mathbb{R}}(Card(P_x)))$ we may assume w.l.o.g. that any P_x is hereditarily smaller than ω_2 in $V[H]$, i.e., $P_x \in L_{\omega_2}[A]$ for every $x \in \mathbb{R}^V$.

Now fix $x \in \mathbb{R}^V$, and set $\mathcal{M} = (L_{\omega_2}[A]; \in, A, P_x, K_x, \dot{x})$ where $\dot{x}^{K_x} = x$. Using $\nabla(A)$ there is some

$$\pi\colon (L_\beta[A \cap \alpha]; \in, A \cap \alpha, \bar{P}_x, \bar{K}_x, \bar{\dot{x}}) \to \mathcal{M}$$

such that β is a cardinal in $L[A \cap \alpha]$, and hence so in L. We get that $x = (\bar{\dot{x}})^{\bar{K}_x} \in L[\bar{K}_x]$ where $\bar{K}_x$ is $\bar{P}_x$-generic over L, and $\bar{P}_x$ is countable. Notice that π only exists in $V[H]$. However, by 0.2 we may then also find, inside V, some

$$\sigma\colon (L_\beta; \in, \tilde{P}_x, \tilde{K}_x, \tilde{\dot{x}}) \to \mathcal{M},$$

so that $x = (\tilde{\dot{x}})^{\tilde{K}_x} \in L[\tilde{K}_x]$ where $\tilde{K}_x$ is $\tilde{P}_x$-generic over L, and $\tilde{P}_x$ is countable.

But now, as in the proof of 2.2, in a $Col(\omega, (2^{\aleph_0})^V)$-generic extension of V we may construct G being $Col(\omega, < \omega_1^V)$-generic over L such that $L(\mathbb{R}^V) = L(\mathbb{R}^{L[G]})$.

$\square$ (4.2)

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