Mutual stationarity in the core model

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Abstract

Foreman and Magidor in [3] study mutual stationarity in Gödel's constructible universe L. We shall extend their analysis to the core model. We include a discussion of what it is that turns an extender model into a core model.

The present paper links mutual stationarity with core model theory. The concept of mutual stationarity found a powerful application in recent work of Foreman and Magidor (cf. [3]). The paper [3] also studies mutual stationarity in Gödel's universe L. We shall extend Foreman and Magidor's analysis to higher core models.

Section 1 will recall basic information about mutually stationary sequences. Section 2 contains a general discussion of the concept of a core model. We want to emphasize that only core models, rather than arbitrary extender models, will be amenable to our analysis; we therefore think that this paper provides a reasonable place for a discussion of what it is that turns an extender model into a core model. Section 2 is thus of independent interest. Section 3 will prove a new condensation result for our core model below $0^\P$ (cf. [14]), and in section 4 we shall prove or main result, Theorem 4.6. Section 5 contains a list of open problems.

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1 Mutual stationarity

Let A be a non-empty set of regular uncountable cardinals, and let $(S_\kappa \colon \kappa \in A)$ be such that $S_\kappa \subset \kappa$ for all $\kappa \in A$. Recall that $(S_\kappa \colon \kappa \in A)$ is called *mutually stationary* [3, Definition 22] if and only if for each large enough regular cardinal θ and for each model $\mathfrak{A}$ with universe H_θ there is some $X \prec \mathfrak{A}$ such that for all $\kappa \in X \cap A$, $sup(X \cap \kappa) \in S_\kappa$. We refer the reader to [3] (in particular, to [3, Section 7]) for key results and background information on mutual stationarity.

For a regular cardinal θ let $cf(\theta)$ denote the class of all ordinals of cofinality θ. Foreman and Magidor proved (cf. [3, Theorem 7]) that if A is a non-empty set of regular uncountable cardinals, and if $(S_\kappa\colon \kappa \in A)$ is such that for each $\kappa \in A$, $S_\kappa \subset \kappa \cap cf(\omega)$ and S_κ is stationary in κ, then $(S_\kappa\colon \kappa \in A)$ is mutually stationary. This immediately gives that if λ is a singular cardinal with cofinality θ then the non-stationary ideal on $\mathcal{P}_{\omega_1}(\lambda)$ is never λ^θ-saturated (cf. [3, Corollary 8]).[a] Despite of this application the authors of [3] regard the concept of mutual stationarity to be also of independent interest; it is this approach that will be adopted here. We shall link mutual stationarity with core model theory.

It is known that, provided $2^{\aleph_0} < \aleph_\omega$, $\aleph_\omega$ is Jonsson if and only if there is a sequence $(k_n\colon n < \omega)$ of integers tending to ω such that $(\omega_n \cap cf(\omega_{k_n})\colon n < \omega)$ is mutually stationary. ($\Leftarrow$ is trivial, $\Rightarrow$ is due to Silver.) We shall therefore focus on ordinals of bounded cofinality; inspired by [3, §7.2] we'll in fact focus on ordinals of a fixed uncountable cofinality.

It is an open problem to decide whether there is a model of set theory in which $(S_n\colon n < \omega)$ must be mutually stationary provided each individual $S_n \subset \aleph_n$ is stationary and consists of points of an uncountable cofinality fixed in advance (cf. the last paragraph of [3, §7.1]). On the other hand, Foreman and Magidor have shown that in Gödel's constructible universe L for each $k < \omega$ there is some $(S_n\colon n < \omega)$ such that each $S_n \subset \aleph_n \cap cf(\omega_k)$ is stationary, but $(S_n\colon n < \omega)$ is not mutually stationary (cf. [3, Theorems 24 and 27]). The purpose of this paper is to extend their result [3, Theorem 27] to higher core models.

We shall need the following lemma, which is essentially due to Baumgartner [2].

Lemma 1.1 *Let $m < \omega$, and let A_0, A_1, ..., A_m be non-empty sets of successor cardinals such that $sup(A_l) < min(A_{l+1})$ for all $l < m$. Let, for each $l \leq m$, $(S_\kappa\colon \kappa \in A_l)$ be mutually stationary and such that for all $\kappa \in A_l$ and for all $\alpha \in S_\kappa$, $cf(\alpha) < min(A_0)$. Then $(S_\kappa\colon \kappa \in \bigcup_{l \leq m} A_l)$ is mutually stationary as well.*

Proof. It certainly suffices to prove 1.1 for $m = 1$. Let ν be the cardinal predecessor of $min(A_0)$. Let $\mathfrak{A}$ be a model expanding $(H_\theta; \in)$ for some large regular θ. As $(S_\kappa\colon \kappa \in A_1)$ is mutually stationary we may pick some $X \prec \mathfrak{A}$ such that $sup(A_0) \subset X$ and for all $\kappa \in X \cap A_1$, $sup(\kappa \cap X) \in S_\kappa$. Pick $F\colon (X \cap A_1) \times \nu \to X$ such that for all $\kappa \in X \cap A_1$, $F(\kappa, -)$ is cofinal in $sup(\kappa \cap X)$. Expand X by F to get (X, F). As $(S_\kappa\colon \kappa \in A_0)$ is mutually stationary we may pick some $Y \prec (X, F)$ such that $\nu \subset Y$ and for all $\kappa \in Y \cap A_0$, $sup(\kappa \cap Y) \in S_\kappa$. However, due to the presence of F, we'll also have that for all $\kappa \in Y \cap A_1$, $sup(\kappa \cap Y) = sup(\kappa \cap X) \in S_\kappa$. As $Y \prec \mathfrak{A}$, we are done. $\square$ Lemma 1.1

[a]This in turn subsumes under the striking more general result of [3] that the non-stationary ideal on $\mathcal{P}_\kappa(\lambda)$ is never λ^+-staturated unless $\kappa = \lambda = \omega_1$.

2 Extender models and the core model

This section intends to discuss and stress the difference between *extender models* and *core models*, a difference which is crucial in this paper and elsewhere. We shall also state a technical lemma which will be used later on.

An *extender model* [b] is a premouse $\mathcal{M}$ which is a proper class (equivalently, such that $OR \subset \mathcal{M}$). Unfortunately, the literature knows a handful of formal definitions of what a premouse is. Currently, the two most common ones are [10, Definition 3.5.1] (US style "Mitchell-Steel premice"; see also [16]) and [4, §4 p. 2] (European style "Friedman-Jensen premice"). The conceptual differences arise from how the respective authors choose to index the extenders on the extender sequences of premice.[c] The current paper is based on Friedman-Jensen premice, so that we here take [4, §4 p. 2] to officially define what a premouse is. In fact, the construction to follow will be based on [14], which in turn builds upon [4]; the reader may find a publisized definition of a (Friedman-Jensen) premouse as well as many other background informations in [14, §1].

Roughly speaking, a premouse is a transitive model of the form

$$J_\alpha[\vec{E}] = (J_\alpha[\vec{E}]; \in, \vec{E}, E_\alpha)$$

for some $\alpha \leq OR$ where $\vec{E}^\frown E_\alpha$ is a coherent sequence of extenders witnessing that certain ordinals are "measurable to a certain extent." We call a premouse $\mathcal{M}$ *weakly iterable* (cf. [4, §11 p. 5]) if $\mathcal{P}$ is $\omega_1 + 1$ iterable for all countable transitive $\mathcal{P}$ which (sufficiently) elementarily embed into $\mathcal{M}$. It is an empirical fact that all the extender models constructed so far are weakly iterable. Examples include the models built in [10, §11] or (more recently) [1] and [11].

For premice transcending a fixed smallness condition weak iterability no longer implies full iterability. We call a premouse $\mathcal{M}$ *fully iterable*, or just *iterable* (cf. [4, §4 p. 26]), if $\mathcal{M}$ is α iterable for every $\alpha \in OR$ (we urge the reader to consult [14, §1] for an exact statement). The iterability condition imposed on extender models is one of the key ingredients which creates a core model.

The $\omega_1 + 1$ iterability of a given premouse $\mathcal{M}$ allows to prove facts which can be expressed by first order statements over $\mathcal{M}$. (Prototype examples include the

[b]The term "extender model" has been suggested by S. Friedman and will be adopted here. Other terms in use to denote an object we shall refer to as an extender model are "fine structural inner model," "Mitchell-Steel model," "$L[E]$-model," and "weasel." Some authors even use "core model" to refer to an extender model, a habit which might be found at least a bit confusing in the light of the discussion to follow.

[c]We here suppress that there are good reasons for choosing either of the indexing systems depending on the purpose in mind.

solidity and universality of the standard parameter.) On the other hand, in order to compare two premice $\mathcal{M}$ and $\mathcal{N}$ one needs to have (in general) that both $\mathcal{M}$ and $\mathcal{N}$ are $(max\{Card(\mathcal{M}), Card(\mathcal{N})\})^+ + 1$ iterable. In order to prove significant covering properties of a given extender model $L[E]$ at arbitrary ordinals we seem to need the full iterability of $L[E]$.

A (fully) iterable extender model is called *universal* if it does not loose the coiteration against any other coiterable premouse (cf. [14, Definition 4.3]).

Definition 2.1 *An extender model $L[E]$ is called a* core model *provided the following conditions are met.*[d]
(1) *$L[E]$ is fully iterable,*
(2) *$L[E]$ is universal and it elementarily embeds into any other universal weasel,*
(3) *$L[E]$ is rigid, i.e., there is no non-trivial elementary embedding $\pi\colon L[E] \to L[E]$,*
(4) *$L[E]$ satisfies weak covering in that $cf^V(\kappa^{+L[E]}) \geq Card^V(\kappa)$ for all $\kappa \geq \aleph_2$,*
(5) *$L[E]$ has a forcing absolute definition, i.e., there is a formula $\Phi(-)$ in the language of set theory such that for every poset $\mathbb{P} \in V$ we have that* (a) *for every premouse $\mathcal{M}$,*

$$\mathcal{M} \lhd L[E] \;\Leftrightarrow\; \mathbb{P} \Vdash \Phi(\check{\mathcal{M}}),$$

and (b) *$\mathbb{P}$ forces that for all $\mathcal{M}$ and $\mathcal{N}$ with $\Phi(\mathcal{M}) \wedge \Phi(\mathcal{N})$, $\mathcal{M} \trianglelefteq \mathcal{N} \vee \mathcal{N} \trianglelefteq \mathcal{M}$,*[e] *and*
(6) *$L[E]$ has a uniform local definition, i.e., there is a formula $\Phi(-)$ in the language of set theory*[f] *such that for every large enough*[g] *cardinal κ we have that*

$$\mathcal{M} \lhd L_\kappa[E] \;\Leftrightarrow\; H_\kappa \models \Phi(\mathcal{M}).$$

Notice that by (2) and (3) there can be at most one core model. Woodin has shown that if $L[E]$ is 1-small and $L[E] \models$ "there is a Woodin cardinal," then $L[E]$ satisfies "I am not iterable." In particular, in this situation $L[E]$ thinks that there can be no core model![h] On the other hand, the informed guess is that if there is no inner model with a Woodin cardinal then the core model exists. This view is supported by the seminal [17]. The present author has shown in [14] that the core

[d] Hugh Woodin informs us that his work on $AD_{\mathbb{R}}$ shows that the present conditions might be too demanding in general to yield an appropriate definition of the concept of "core model." However, our conditions (1) through (6) hold for any known core model in a universe in which all premice are tame, i.e., don't have extenders overlapping Woodin cardinals.

[e] $\trianglelefteq$ means "is an initial segment of," and $\lhd$ means "is a proper initial segment of."

[f] We may allow set parameters.

[g] $\kappa > 2^{\aleph_0}$ will do in general.

[h] Cf. also the discussion in [5, §4].

model exists, starting from an assumption being somewhat stronger than "there is no inner model with a Woodin cardinal."

We say that $0^{\P}$ ("zero hand grenade") does not exist (cf. [14, Definition 2.3]) if there is no premouse $\mathcal{M}$ with a measurable cardinal κ such that

$$\mathcal{M}||\kappa \models \text{ "there is a proper class of strong cardinals."}^{i}$$

The paper [14] proves the existence of the core model in the theory $ZFC + 0^{\P}$ does not exist.

If $0^{\P}$ does not exist then we let K denote the core model. We aim to briefly discuss how (1) through (6) in Definition 2.1 materialize for K, if $0^{\P}$ does not exist. (1) is given by [14, Lemma 3.3] (which garantees the existence and iterability of K^c, a preliminary version of K) and the fact that, in [14, §8], K is constructed as the collapse of a hull of K^c. (2) and (3) follow from [14, Corollary 8.17] and (the proofs of) [17, Theorem 8.10] and [17, Theorem 8.8]. (4) is [14, Theorem 8.18] and in fact is given by the proofs of [9] and [8]. (5) is given by the proof of [17, Theorem 5.18 (3)]. We now aim to prove (6) by a lemma which is of independent interest and which generalizes a lemma Jensen has shown to hold under the stronger assumption that $0^{\P}$ does not exist. We shall actually use Lemma 2.2 in the proof of Theorem 4.6.

By $\neg\, 0^{\P}$ we abbreviate the statement that $0^{\P}$ does not exist.

Lemma 2.2 ($\neg\, 0^{\P}$) *Let $\kappa \geq \aleph_2$ be a cardinal in K. Let $\mathcal{M} \trianglerighteq K||\kappa$ be an iterable sound premouse with $\rho_\omega(\mathcal{M}) \leq \kappa$. Then $\mathcal{M} \triangleleft K$.*

PROOF. By standard methods, it suffices to prove that the phalanx $((K, \mathcal{M}), \kappa)$ is iterable (cf. [17, §6]). As $\neg\, 0^{\P}$, [14, Lemma 2.7] shows that for this in turn it suffices to prove that if $\mathcal{N} \trianglerighteq K||\kappa$ is an iterable premouse and $F = E^{\mathcal{N}}_{\nu} \neq \emptyset$ is such that $\mu = c.p.(F) < \kappa \leq \nu$ then $Ult(K; F)$ is well-founded (and therefore iterable). In what follows we may and shall assume that F is the top extender of $\mathcal{N}$.

Set $\tau = \mu^{+K}$ if $cf(\mu^{+K}) > \omega$, and $\tau = \aleph_2$ otherwise. Notice that $\mu < \tau \leq \kappa$ by [14, Theorem 8.18] and $\kappa \geq \aleph_2$.

Let θ be a large enough regular cardinal, and let $\pi \colon \bar{H} \rightarrow H_{\theta^+}$ be elementary and such that $\bar{H}$ is countable and transitive and $ran(\pi)$ contains all the sets of current interest. Let

$$\bar{K} = Ult(\pi^{-1}(K||\theta); \pi \restriction \pi^{-1}(K||\tau))$$

be the ultrapower of $\pi^{-1}(K||\theta)$ by the (long) extender $\pi \restriction \pi^{-1}(K||\tau)$. Also, let

$$\bar{\mathcal{N}} = Ult(\pi^{-1}(\mathcal{N}); \pi \restriction \pi^{-1}(K||\tau))$$

[i] $\mathcal{N}||\alpha$ is $\mathcal{N}$ cut off at α with top extender $E^{\mathcal{N}}_{\alpha}$ (provided $E^{\mathcal{N}}_{\alpha} \neq \emptyset$). We'll also confuse $\mathcal{N}||\alpha$ with its underlying universe.

be the ultrapower of $\pi^{-1}(\mathcal{N})$ by the (long) extender $\pi \restriction \pi^{-1}(K||\tau)$, and let

$$k: \bar{\mathcal{N}} \to \mathcal{N}$$

be the canonical embedding. Set $\bar{\tau} = k^{-1}(\tau)$, and notice that $\bar{\tau} = c.p.(k)$. Let $\bar{F}$ be the top extender of $\bar{\mathcal{N}}$. It is important to notice that $\bar{\mathcal{N}}$ is a premouse (rather than a proto-mouse), as the ultrapower map producing $\bar{\mathcal{N}}$ is continuous at $\pi^{-1}(\mu^{+K})$.

It now suffices to prove that $Ult(\bar{K}; \bar{F})$ is well-founded, as we might have thrown in arbitrary coordinates witnessing the alleged ill-foundedness of $Ult(K; F)$ into the range of π.

Let $(\mathcal{T},\mathcal{U})$ denote the coiteration of $((\mathcal{N}, \bar{\mathcal{N}}), \bar{\tau})$ with $\mathcal{N}$. [4, §8 Lemma 1] shows that the last model $\mathcal{M}^{\mathcal{T}}_{\infty}$ of $\mathcal{T}$ will sit above $\bar{\mathcal{N}}$, that there will be no drop along the main branch of $\mathcal{T}$, and that $\mathcal{M}^{\mathcal{T}}_{\infty} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\infty}$. Set $\varphi = \pi^{\mathcal{T}}_{0\infty}$, and $\mathcal{N}^* = \mathcal{M}^{\mathcal{T}}_{\infty}$. We thus have

$$\varphi: \bar{\mathcal{N}} \to \mathcal{N}^*,$$

where $\mathcal{N}^*$ is an iterate of $\mathcal{N}$ and $\varphi \restriction \bar{\tau} = id$. As $\bar{\mathcal{N}}$ has a top extender, namely $\bar{F}$, with critical point $\mu < \bar{\tau}$, $\mathcal{N}^*$ will also have a top extender, call it F^*, with critical point μ. As $\neg\, 0^{\P}$, we'll therefore have that $\mathcal{U}$ can only use extenders with critical point $> \mu$ (cf. Claim 2 in the proof of [14, Lemma 2.4]).

Another application of [4, §8 Lemma 1] shows that the coiteration of $((K, \bar{K}), \bar{\tau})$ with K produces an embedding

$$\chi: \bar{K} \to K^*$$

where K^* is an iterate of K and $\chi \restriction \bar{\tau} = id$. In order to finish the proof of Lemma 2.2 we now split the argument into two cases according to whether $\tau = \mu^{+K}$ or $\tau = \omega_2$.

Let us now first assume that $\tau = \mu^{+K}$. Then $\bar{\tau} = \mu^{+\bar{\mathcal{N}}} = \mu^{+\mathcal{N}^*} < \tau = \mu^{+\mathcal{N}} = \mu^{+K}$. This and the fact that $\mathcal{U}$ only uses extenders with critical point $> \mu$ implies that we may actually construe $\mathcal{U}$ as an iteration of $\mathcal{N}||\tau = K||\tau$. Especially, $\mathcal{N}^*$ is an iterate of K. Therefore (cf. [9, Fact 3.19.1]), $Ult(\mathcal{P}; F^*)$ is well-founded whenever $\mathcal{P}$ is an iterate of K with $\mathcal{P}||\bar{\tau} = K||\bar{\tau}$ and $\bar{\tau} = \mu^{+\mathcal{P}}$. Thus, using φ, $Ult(\mathcal{P}; \bar{F})$ is well-founded whenever $\mathcal{P}$ is an iterate of K with $\mathcal{P}||\bar{\tau} = K||\bar{\tau}$ and $\bar{\tau} = \mu^{+\mathcal{P}}$. We therefore in particular know that $Ult(K^*; \bar{F})$ is well-founded. Hence $Ult(\bar{K}; \bar{F})$ is well-founded, too, using χ.

Let us finally suppose that $\tau = \omega_2$. By [14, Lemma 2.7], $\mathcal{T}$ is really an iteration of $\bar{\mathcal{N}}$. Moreover, $\bar{\mathcal{N}} \cap OR < \omega_2$, so that by [12, Theorem 3.4] $K||\tau = \mathcal{N}||\tau$ wins the coiteration against $\bar{\mathcal{N}}$. In other words, we may again construe $\mathcal{U}$ as an iteration of $\mathcal{N}||\tau = K||\tau$. We may now continue exactly as in the previous case. $\square$ Lemma 2.2

Lemma 2.2 readily implies the following which is now easy to verify.[j]

Theorem 2.3 ($\neg\, 0^\P$) *$K||\kappa$ is uniformly $\Sigma_3^{H_\kappa}$ in the parameter $K||\omega_2$ for all cardinals $\kappa \geq \aleph_3$, i.e., there is a Σ_3 formula $\Phi(v_0, v_1)$ such that for all such κ, $\mathcal{M} \triangleleft K||\kappa$ if and only if $H_\kappa \models \Phi(\mathcal{M}, K||\omega_2)$.*

As a matter of fact, if $\neg\, 0^\P$ then K satisfies that "$0^\P$ does not exist and I am the core model."[k] The proof of our main result, theorem 4.6, will be run in the theory $ZFC\ +\ 0^\P$ does not exist $+\ V = K$. We want to emphasize that we do not know whether Theorem 4.6 remains true if we replace "$0^\P$ does not exist" by a weaker anti large cardinal assumption. *A fortiori*, we don't know whether 4.6 remains true if we moreover replace $V = K$ by "V is a weakly iterable extender model."

We are now going to state a lemma which is implicit in [13]. We formulate it under the assumption that $0^\P$ does not exist, although it is known to hold under much weaker circumstances. The reader should consult [9, §2.3] or [13, Section 2] on proto-mice.[l]

Lemma 2.4 ($\neg\, 0^\P$) *Let $\mathcal{P}$ be an iterable sound premouse with top extender G, and let $\mathcal{Q}$ be a proto-mouse with top extender fragment F. Let $k\colon \mathcal{Q} \rightarrow_{\Sigma_0} \mathcal{P}$. Suppose that $\lambda = max\{\xi\colon \sigma \restriction \xi = id\} = \kappa^{+\mathcal{Q}}$, $\rho_1(\mathcal{P}) \leq \kappa$, and $\mu = c.p.(F) < \kappa$. Suppose also that $\rho_1(\mathcal{Q}) \leq \kappa$ and $\mathcal{Q}$ is sound and solid above κ. Let ρ with $\lambda < \rho < \kappa^{+\mathcal{P}}$ be largest such that F measures the subsets of μ which exist in $\mathcal{Q}||\rho$. Let*

$$\pi\colon \mathcal{Q}||\rho \rightarrow_F \mathcal{R} = Ult_n(\mathcal{Q}||\rho; F),$$

where $n < \omega$ is such that $\rho_{n+1}(\mathcal{Q}||\rho) \leq \mu < \rho_n(\mathcal{Q}||\rho)$. Then $\rho_{n+1}(\mathcal{R}) \leq \kappa$, $\mathcal{R}$ is sound above κ, and in fact $\mathcal{R} \triangleleft \mathcal{P}$. Moreover, $\mathcal{Q} \in \mathcal{P}$.

Proof sketch. We may define $l\colon \mathcal{R} \rightarrow \mathcal{P}||i_G(\rho)$ by setting

$$[a, f]_F^{\mathcal{Q}||\rho} \ \mapsto\ [k(a), f]_G^{\mathcal{P}||\mu^{+\mathcal{P}}},$$

where a and f are appropriate. It can be verified that $\rho_{n+1}(\mathcal{R}) = \rho_1(\mathcal{Q}) \leq \kappa$ and that $\mathcal{R}$ is sound above κ. We may then apply the condensation lemma, [4, §8 Lemma 4], to get that $\mathcal{R} \triangleleft \mathcal{P}$. We shall also have that F is definable over $\mathcal{R}$, and therefore $\mathcal{Q} \in \mathcal{P}$. The reader may find a full proof of Lemma 2.4 along the lines of the proof of [13, Lemma 2.19]. $\square$ Lemma 2.4

[j] Steel and the author (independently from each other) have shown for the K of [17] that $K||\kappa$ is uniformly lightface $\Sigma_3^{H_\kappa}$ for all cardinals $\kappa > 2^{\aleph_0}$. On the other hand, it is open whether Lemma 2.2 holds for the K of [17].

[k] We shall let $V = K$ abbreviate the statement "I am the core model." It is in fact true that if $\neg\, 0^\P$ then any weakly iterable extender model satisfies "$0^\P$ does not exist and $V = K$."

[l] Proto-mice are called fragments in [13].

3 Condensation and collapsing structures

Let κ be a cardinal, and let $\pi: \bar{K} \to K||\kappa$ be elementary. We may ask: under which circumstances is $\bar{K}$ an iterate of K? This section will provide a sufficient critierion for when the answer is "yes." This criterion will be used in the next section.

Theorem 3.1 *Suppose that $0^\P$ does not exist and that $V = K$. Let κ be a cardinal, and let $\pi: \bar{K} \to K||\kappa$ be elementary. Suppose that*

$$cf(sup(\pi"\alpha^{+\bar{K}})) > \omega$$

whenever α is an infinite cardinal in $\bar{K}$ (we understand that $\alpha^{+\bar{K}} = \bar{K} \cap OR$ if α is the largest cardinal in $\bar{K}$). Then $\bar{K}$ is a normal iterate of K, i.e., there is a normal iteration tree $\mathcal{T}$ on K with a last model $\mathcal{M}^{\mathcal{T}}_\infty$ such that $\bar{K} \trianglelefteq \mathcal{M}^{\mathcal{T}}_\infty$.

PROOF. We may assume without loss of generality that $\pi \neq id$. Let $\delta = c.p.(\pi)$, and let $\eta \leq \delta$ be least such that $(\mathcal{P}(\eta) \cap K) \setminus \bar{K} \neq \emptyset$. Note that $(\mathcal{P}(\delta) \cap K) \setminus \bar{K} \neq \emptyset$, because otherwise (as there are no superstrong extenders) $\pi \upharpoonright \mathcal{P}(\delta) \cap K$ would be an extender which collapses the cardinal $\pi(\delta)$.

Let $(\mathcal{U}, \mathcal{T})$ denote the (padded) coiteration of $\bar{K}$ with K. We'll have that $[0, \infty)_U \cap \mathcal{D}^{\mathcal{U}} = \emptyset$ by the universality of K (alternatively, by Dodd-Jensen). By $\neg\, 0^\P$ we shall also have that $\pi^{\mathcal{U}}_{0\infty} \upharpoonright \delta = id$ (cf. the proof of [14, Lemma 5.2]).

We want to show that $\mathcal{U}$ is trivial (i.e., that $\bar{K}$ doesn't move in the comparison with K), and in fact that $\mathcal{M}^{\mathcal{T}}_\infty$ is set-sized and $\rho_\omega(\mathcal{M}^{\mathcal{T}}_\infty) < \bar{K} \cap OR$.

Claim 1. Let μ be such that $\pi^{\mathcal{U}}_{0\infty} \upharpoonright \mu = id$. Then for no $F = E^{\mathcal{T}}_\nu$ do we have that $\bar{\mu} = c.p.(F) < \eta$ and $F(\bar{\mu}) \leq \mu$.

PROOF. Suppose otherwise. Notice first that $F(\bar{\mu})$ is a cardinal in $\mathcal{M}^{\mathcal{T}}_\infty$, hence in $\mathcal{M}^{\mathcal{U}}_\infty$, and hence in $\bar{K}$. Therefore, $\pi \circ F(\bar{\mu})$ is a cardinal in K $(= V)$. However, $\pi \circ F$ would be an extender which collapses the cardinal $\pi \circ F(\bar{\mu})$, again as there are no superstrong extenders. Contradiction! $\square$ Claim 1

Claim 2. $\mathcal{U}$ is trivial.

PROOF. Assume not. Let $F = E^{\mathcal{U}}_\epsilon$, where $\epsilon + 1$ is least in $(0, \infty]_U$, and let $\mu = c.p.(F) \geq \delta$. (Remember that $\pi^{\mathcal{U}}_{0\infty} \upharpoonright \delta = id$.) Let $\beta < lh(\mathcal{U}) = lh(\mathcal{T})$ be minimal with $\mathcal{M}^{\mathcal{T}}_\beta \trianglerighteq \bar{K}||\mu^{+\bar{K}}$. We let $(\kappa_\gamma: \gamma \leq \theta)$ enumerate the cardinals of $\bar{K}$ in the closed interval $[\eta, \mu]$, and we let $\lambda_\gamma = \kappa_\gamma^{+\bar{K}}$ for $\gamma \leq \theta$. For each $\gamma \leq \theta$ we let $\delta(\gamma) \leq \beta$ be the least δ such that $\mathcal{M}^{\mathcal{T}}_\delta||\lambda_\gamma = \bar{K}||\lambda_\gamma$, and we let $\mathcal{P}_\gamma$ be the largest initial segment $\mathcal{P}$ of $\mathcal{M}^{\mathcal{T}}_{\delta(\gamma)}$ such that all bounded subsets of λ_γ which are in $\mathcal{P}$ are in $\bar{K}$ as well. In particular, $\lambda_\gamma = \kappa_\gamma^{+\mathcal{P}_\gamma}$. By Claim 1 we shall have that $\rho_\omega(\mathcal{P}_\gamma) \leq \kappa_\gamma$ for all $\gamma \leq \theta$. Moreover, $\mathcal{P}_\gamma$ is sound above κ_γ.

We let $\vec{\mathcal{P}}$ denote the phalanx

$$((K^\frown(\mathcal{P}_\gamma : \gamma \leq \theta)^\frown \mathcal{M}_\epsilon^{\mathcal{U}}), \eta^\frown(\lambda_\gamma : \gamma \leq \theta)).$$

In the language of [9, Definition 2.4.5], $\vec{\mathcal{P}}$ is a special phalanx of premice.

Let $\mathcal{U}^*$ and $\mathcal{V}$ be the padded iteration trees arising from the comparison of $\vec{\mathcal{P}}$ with K. We understand that $\mathcal{U}^*$ either has a last ill-founded model, or else that $\mathcal{M}_\infty^{\mathcal{U}^*}$ and $\mathcal{M}_\infty^{\mathcal{V}}$ are lined up. The following says that it is the latter which will hold.

Subclaim. $\vec{\mathcal{P}}$ is coiterable with K.

Proof. Suppose that $\mathcal{U}^*$ has a last ill-founded model. Let $\sigma \colon \bar{H} \rightarrow H_\Omega$ be such that Ω is regular and large enough, $\bar{H}$ is countable and transitive, and $\{\vec{\mathcal{P}}, \mathcal{U}^*\} \subset ran(\sigma)$. Then $\sigma^{-1}(\mathcal{U}^*)$ witnesses that $\sigma^{-1}(\vec{\mathcal{P}})$ is not iterable (in a special respect).

Fix $\gamma \in \theta + 1 \cap ran(\sigma)$ for a moment. Let

$$\mathcal{Q}_\gamma = Ult_n(\sigma^{-1}(\mathcal{P}_\gamma); \sigma \restriction \sigma^{-1}(\mathcal{P}_\gamma || \lambda_\gamma)),$$

where $n < \omega$ is such that $\rho_{n+1}(\mathcal{P}_\gamma) \leq \kappa_\gamma < \rho_n(\mathcal{P}_\gamma)$. Let $\sigma_\gamma \supset \sigma \restriction \sigma^{-1}(\mathcal{P}_\gamma || \lambda_\gamma)$ denote the canonical embedding from $\sigma^{-1}(\mathcal{P}_\gamma)$ into $\mathcal{Q}_\gamma$. Notice that by $cf(\lambda_\gamma) > \omega$ it is clear that $sup\ \sigma"\sigma^{-1}(\lambda_\gamma)$ is not cofinal in λ_γ. Hence if k_γ denotes the canonical embedding from $\mathcal{Q}_\gamma$ into $\mathcal{P}_\gamma$ then $sup\ \sigma"\sigma^{-1}(\lambda_\gamma) = \kappa_\gamma^{+\mathcal{Q}_\gamma} = k_\gamma^{-1}(\lambda_\gamma)$ is the critical point of k_γ.

Unfortunately, $\mathcal{Q}_\gamma$ might not be premouse but rather a proto-mouse; this will in fact be the case if $\mathcal{P}_\gamma$ has a top extender with critical point $\bar{\mu} < \kappa_\gamma$ and $\rho_1(\mathcal{P}_\gamma) \leq \kappa_\gamma$, as then σ_γ is discontinuous at $\sigma_\gamma^{-1}(\bar{\mu}^{+\mathcal{P}_\gamma})$ and the top extender fragment of $\mathcal{Q}_\gamma$ will not measure all the subsets of $\bar{\mu}$ which exist in $\mathcal{Q}_\gamma$. Let us therefore define an object $\mathcal{R}_\gamma$ as follows. We set $\mathcal{R}_\gamma = \mathcal{Q}_\gamma$ if $\mathcal{Q}_\gamma$ is a premouse. Otherwise, if $\bar{\mu} < \kappa_\gamma$ is the critical point of the top extender of $\mathcal{P}_\gamma$, we let

$$\mathcal{R}_\gamma = Ult_n(\mathcal{Q}_\gamma || \rho; G),$$

where G is the top extender fragment of $\mathcal{Q}_\gamma$, ρ is maximal with

$$\bar{\mu}^{+\mathcal{Q}_\gamma || \rho} = sup\ \sigma_\gamma"\sigma^{-1}(\bar{\mu})^{+\sigma^{-1}(\mathcal{P}_\gamma)},$$

and $n < \omega$ is such that $\rho_{n+1}(\mathcal{Q}_\gamma) \leq \bar{\mu} < \rho_n(\mathcal{Q}_\gamma)$.

We can apply [4, §8 Lemma 4] (if $\mathcal{Q}_\gamma$ is a premouse) and Lemma 2.4 (if $\mathcal{Q}_\gamma$ is not a premouse) and deduce that $\rho_\omega(\mathcal{R}_\gamma) \leq \kappa_\gamma$ and $\mathcal{R}_\gamma \lhd \mathcal{P}_\gamma$. But as $\kappa_\gamma^{+\mathcal{R}_\gamma} < \lambda_\gamma = \kappa_\gamma^{+\mathcal{P}_\gamma}$, we'll of course have that $\mathcal{R}_\gamma \lhd \mathcal{P}_\gamma||\lambda_\gamma = \mathcal{M}^{\mathcal{U}}_\epsilon||\lambda_\gamma$. Therefore $\mathcal{R}_\gamma$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_\epsilon$.

Now let $\vec{\mathcal{Q}}$ denote the special phalanx of proto-mice

$$(K^\frown(\mathcal{Q}_\gamma\colon \gamma \in \theta+1 \cap ran(\sigma))^\frown \mathcal{M}^{\mathcal{U}}_\epsilon, \eta^\frown(\lambda_\gamma\colon \gamma \in \theta+1\cap ran(\sigma))),$$

and let $\vec{\mathcal{R}}$ denote the special phalanx of premice

$$(K^\frown(\mathcal{R}_\gamma\colon \gamma \in \theta+1 \cap ran(\sigma))^\frown \mathcal{M}^{\mathcal{U}}_\epsilon, \eta^\frown(\lambda_\gamma\colon \gamma \in \theta+1\cap ran(\sigma))).$$

Due to the existence of the family of maps

$$\sigma \restriction \sigma^{-1}(K||\kappa), (\sigma_\gamma\colon \gamma \in \theta+1\cap ran(\sigma)), \sigma \restriction \sigma^{-1}(\mathcal{M}^{\mathcal{U}}_\epsilon)$$

we know that $\vec{\mathcal{Q}}$ cannot be iterable, as $\sigma^{-1}(\vec{\mathcal{P}})$ is not iterable. But then, arguing axactly as for [9, Lemma 3.18], $\vec{\mathcal{R}}$ cannot be iterable. However, any iteration of $\vec{\mathcal{R}}$ can be construed as an iteration of $((K, \mathcal{M}^{\mathcal{U}}_\epsilon), \delta)$, and thus in turn of $((K, \bar{K}), \delta)$. But $((K, \bar{K}), \delta)$ is iterable. Contradiction! $\square$ Subclaim

Now notice that $\mathcal{U}^* \restriction \epsilon$ is trivial, $\mathcal{V} \restriction \epsilon = \mathcal{T} \restriction \epsilon$, and that $F = E^{\mathcal{U}}_\epsilon = E^{\mathcal{U}^*}_\epsilon$ will be the first extender used in $\mathcal{U}^*$. By [14, Lemma 2.7], we'll then[m] in fact have that no extender from $\mathcal{U}^*$ will be applied to (an inital segment of) the last model of the phalanx $\vec{\mathcal{P}}$, $\mathcal{M}^{\mathcal{U}}_\epsilon$. We may therefore finally argue as in the proof of [17, Theorem 8.6] to derive a contradiction. $\square$ Claim 2

We have shown Theorem 3.1 $\square$ Theorem 3.1

We do not know how to prove Theorem 3.1 if the assumption that $V = K$ is removed from its statement.

We aim to continue the discussion which was begun in the proof of Theorem 3.1. Specifically, we want to see how the argument leads to collapsing structures.

Definition 3.2 ($\neg\, 0^\P$) *Let $\alpha \leq \gamma \in OR$. We say that $K||\gamma$ is a collapsing structure for α provided the following holds true. For all $\bar\gamma \leq \gamma$, $\bar\gamma = \gamma$ if and only if there are $n < \omega$, $\delta < \alpha$, and $\vec{p} \in {}^{<\omega}\bar\gamma$ such that*

$$Hull_n^{K||\bar\gamma}(\delta \cup \{\vec{p}\}) \cap \alpha$$

is cofinal in α.

[m] This is the only place in this proof where we really use the assumption that K is below $0^\P$ in a way which does not seem to be avoidable. If the assumption that K is below $0^\P$ is dropped then at the time of writing I don't see how to prove the iterability of the relevant phalanx (which would then have to be longer than the phalanx $\vec{\mathcal{P}}$ defined above) needed to verify Claim 2.

Of course, collapsing structures are unique so that we may and shall talk about *the* collapsing structure for a given α.

Now let $\pi \colon \bar{K} \to K||\kappa$ and everything else be as in the proof of Theorem 3.1. We wish to isolate collapsing structures for $sup(\pi"\alpha^{+\bar{K}})$ whenever $\alpha \geq \eta$.

Let $(\kappa_\gamma \colon \gamma < \theta)$ enumerate the cardinals of $\bar{K}$ in the half-open interval $[\eta, \bar{K} \cap OR)$, and let $\lambda_\gamma = \kappa_\gamma^{+\bar{K}}$ for $\gamma < \theta$. For each $\gamma < \theta$ we let $\delta(\gamma) < lh(\mathcal{T})$ be the least δ such that $\mathcal{M}_\delta^\mathcal{T}||\lambda_\gamma = \bar{K}||\lambda_\gamma$, we let $\mathcal{P}_\gamma$ be the largest initial segment $\mathcal{P}$ of $\mathcal{M}_{\delta(\gamma)}^\mathcal{T}$ such that all bounded subsets of λ_γ which are in $\mathcal{P}$ are in $\bar{K}$ as well, and we let $n(\gamma)$ be the $n < \omega$ such that

$$\rho_{n+1}(\mathcal{P}_\gamma) \leq \kappa_\gamma < \rho_n(\mathcal{P}_\gamma).$$

Notice that $n(\gamma)$ will always be defined by Claims 1 and 2. Let for each $\gamma < \theta$,

$$\mathcal{Q}_\gamma = Ult_{n(\gamma)}(\mathcal{P}_\gamma; \pi \restriction \bar{K}||\lambda_\gamma).$$

Let σ_γ denote the canonical embedding from $\mathcal{P}_\gamma$ into $\mathcal{Q}_\gamma$. We also set $\tilde{\lambda}_\gamma = sup\, \pi"\lambda_\gamma$. Notice that if $\mathcal{Q}_\gamma$ is well-founded then $\tilde{\lambda}_\gamma = \sigma_\gamma(\lambda_\gamma) = \pi(\kappa_\gamma)^{+\mathcal{Q}_\gamma} < \pi(\lambda_\gamma)$, $\rho_{n+1}(\mathcal{Q}_\gamma) \leq \pi(\kappa_\gamma)$, and $\mathcal{Q}_\gamma$ is sound above $\pi(\kappa_\gamma)$. Unfortunately, again even if it is well-founded, $\mathcal{Q}_\gamma$ might not be premouse but rather a proto-mouse, namely if $\mathcal{P}_\gamma$ has a top extender with critical point $\bar{\mu} < \kappa_\gamma$ and $n(\gamma) = 0$.

Let us therefore define, inductively for $\gamma < \theta$, objects $\mathcal{R}_\gamma$ as follows. We understand that we let the construction break down as soon as one of the models defined is ill-founded. It will be clear from the construction that if $\mathcal{R}_\gamma$ is well-defined then $\mathcal{R}_\gamma$ is a premouse, $\mathcal{R}_\gamma \trianglerighteq K||\tilde{\lambda}_\gamma$, $\tilde{\lambda}_\gamma = \pi(\kappa_\gamma)^{+\mathcal{R}_\gamma}$, and $\rho_\omega(\mathcal{R}_\gamma) \leq \kappa_\gamma$; we shall then let $n^*(\gamma)$ be the least $n < \omega$ such that $\rho_{n+1}(\mathcal{R}_\gamma) \leq \kappa_\gamma < \rho_n(\mathcal{R}_\gamma)$.

Fix $\gamma < \theta$, and suppose $\mathcal{R}_{\bar{\gamma}}$ to be given for all $\bar{\gamma} < \gamma$. We set $\mathcal{R}_\gamma = \mathcal{Q}_\gamma$ if $\mathcal{Q}_\gamma$ is a premouse. Otherwise, if $\kappa_{\bar{\gamma}}$ is the critical point of the top extender of $\mathcal{P}_\gamma$, we let

$$\mathcal{R}_\gamma = Ult_{n^*(\bar{\gamma})}(\mathcal{R}_{\bar{\gamma}}; G),$$

where G is the top extender fragment of $\mathcal{Q}_\gamma$.

Lemma 3.3 *For each $\gamma < \theta$ we have that $\mathcal{R}_\gamma$ is well-defined premouse. In fact, $\mathcal{R}_\gamma \triangleleft K$, and $\mathcal{R}_\gamma$ is the collapsing structure for $\pi(\kappa_\gamma)$.*

Proof. We shall verify that $\mathcal{R}_\gamma$ is an iterable premouse. This will suffice via Lemma 2.2.

Let $\sigma \colon \bar{H} \to H_\Omega$ be such that Ω is regular and large enough, $\bar{H}$ is countable and transitive, and $ran(\sigma)$ contains all the sets of current interest. For $\gamma \in \theta \cap ran(\sigma)$ we shall inductively choose $\mathcal{Q}_\gamma^* \in \bar{K}||\lambda_\gamma$ and $\mathcal{R}_\gamma^* \triangleleft \bar{K}||\lambda_\gamma$ together with embeddings

$$\bar{\varphi}_\gamma \colon \mathcal{Q}_\gamma \cap ran(\sigma) \to \pi(\mathcal{Q}_\gamma^*) \text{ and}$$

$$\varphi_\gamma\colon \mathcal{R}_\gamma \cap ran(\sigma) \to \pi(\mathcal{R}^*_\gamma)$$

such that $\bar{\varphi}_\gamma \restriction (\tilde{\lambda}_\gamma \cap ran(\sigma)) = \varphi_\gamma \restriction (\tilde{\lambda}_\gamma \cap ran(\sigma)) = id$. We shall also inductively maintain that $\mathcal{Q}^*_\gamma$ is a premouse or a proto-mouse with $\rho_{n(\gamma)+1}(\mathcal{Q}^*_\gamma) \leq \kappa_\gamma < \rho_{n(\gamma)}(\mathcal{Q}^*_\gamma)$ which is sound and solid above κ_γ, that $\mathcal{R}^*_\gamma$ is a premouse with $\rho_{n^*(\gamma)+1}(\mathcal{R}^*_\gamma) \leq \kappa_\gamma < \rho_{n^*(\gamma)}(\mathcal{R}^*_\gamma)$ which is sound and solid above κ_γ, and that there is some $\lambda'_\gamma < \lambda_\gamma$ such that $\mathcal{Q}^*_\gamma \trianglerighteq K||\lambda'_\gamma$, $\mathcal{R}^*_\gamma \trianglerighteq K||\lambda'_\gamma$, and $\lambda'_\gamma = \kappa_\gamma^{+\mathcal{Q}^*_\gamma} = \kappa_\gamma^{+\mathcal{R}^*_\gamma}$. As we might have thrown in potential witnesses to $\mathcal{R}_\gamma$ not being iterable (for some fixed γ) into the range of σ, it will be clear from the construction that this does the job.

Fix $\gamma \in \theta \cap ran(\sigma)$, and let us suppose that $\mathcal{Q}^*_{\bar{\gamma}}$, $\mathcal{R}^*_{\bar{\gamma}}$, $\bar{\varphi}_{\bar{\gamma}}$, and $\varphi_{\bar{\gamma}}$ have already been defined for all $\bar{\gamma} \in \gamma \cap ran(\sigma)$. We let

$$\mathcal{Q}^*_\gamma = Ult_{n(\gamma)}(\sigma^{-1}(\mathcal{P}_\gamma); \sigma \restriction \sigma^{-1}(\mathcal{P}_\gamma||\lambda_\gamma)),$$

and we let k_γ be the canonical embedding from $\mathcal{Q}^*_\gamma$ into $\mathcal{P}_\gamma$. As $cf(\lambda_\gamma) > \omega$, the critical point of k_γ will be $k_\gamma^{-1}(\lambda_\gamma) = \lambda'_\gamma$. If $\mathcal{Q}^*_\gamma$ is a premouse (which will be the case if and only if $\mathcal{Q}_\gamma$ is a premouse, which in turn is the case if and only if $\mathcal{P}_\gamma$ does not have a top extender with critical point $\bar{\mu} < \kappa_\gamma$ or else $n(\gamma) > 0$) then we set $\mathcal{R}^*_\gamma = \mathcal{Q}^*_\gamma$. Otherwise, let $\kappa_{\bar{\gamma}}$ be the critical point of the top extender of $\mathcal{P}_\gamma$, $\bar{\gamma} < \gamma$, and let G^* be the top extender fragment of $\mathcal{P}^*_\gamma$; then set

$$\mathcal{R}^*_\gamma = Ult_{n^*(\bar{\gamma})}(\mathcal{R}^*_{\bar{\gamma}}; G^*)$$

(notice that by our inductive hypotheses $\mathcal{R}^*_{\bar{\gamma}}$ is also the longest initial segment of $\mathcal{R}^*_\gamma$ to which G^* could be applied).

By using [4, §8, Lemma 4] and Lemma 2.4 we get that $\mathcal{Q}^*_\gamma \in \mathcal{P}_\gamma||\lambda_\gamma = \bar{K}||\lambda_\gamma$ and $\mathcal{R}^*_\gamma \triangleleft \mathcal{P}_\gamma||\lambda_\gamma = \bar{K}||\lambda_\gamma$. It is easy to see that

$$[a, f]^{\mathcal{P}_\gamma}_{\pi \restriction \bar{K}||\lambda_\gamma} \mapsto \pi \circ k_\gamma^{-1}(f)(a),$$

where a and f are appropriate, defines an embedding from $\mathcal{Q}_\gamma \cap ran(\sigma)$ into $\pi(\mathcal{Q}^*_\gamma)$. This is our embedding $\bar{\varphi}_\gamma$.

If $\mathcal{Q}^*_\gamma$ is a premouse then we let $\varphi_\gamma = \bar{\varphi}_\gamma$. We are left with having to define $\bar{\varphi}_\gamma$ in the case that $\mathcal{Q}^*_\gamma$ is not a premouse. Let G be the top extender fragment of $\mathcal{Q}_\gamma$, and let $\tilde{G}$ be the top extender fragment of $\pi(\mathcal{Q}^*_\gamma)$. Notice that

$$\pi(\mathcal{R}^*_\gamma) = Ult_{n^*(\bar{\gamma})}(\pi(\mathcal{R}^*_{\bar{\gamma}}); \tilde{G}).$$

It is thus straightforward to verify that

$$[a, f]^{\mathcal{R}_{\bar{\gamma}}}_G \mapsto [\bar{\varphi}_\gamma(a), \varphi_{\bar{\gamma}}(f)]^{\pi(\mathcal{R}^*_{\bar{\gamma}})}_{\tilde{G}} = i_{\tilde{G}} \circ \varphi_{\bar{\gamma}}(f)(\bar{\varphi}_\gamma(a)),$$

for appropriate a and f, defines an embedding from $\mathcal{R}_\gamma \cap ran(\sigma)$ into $\pi(\mathcal{R}^*)$. This is our embedding φ_γ.

It is easy to check our inductive hypotheses for γ. $\square$ Lemma 3.3

We remark that in order for Lemma 3.3 to hold true it is not important how the models $\mathcal{P}_\gamma$ were actually obtained to begin with. Lemma 3.3 remains true if the models $\mathcal{R}_\gamma$ are defined starting from any sequence $(\mathcal{P}_\gamma \colon \gamma < \theta)$ of iterable premice such that $\mathcal{P}_\gamma \trianglerighteq \bar{K}||\lambda_\gamma$, $\mathcal{P}_\gamma$ is sound and solid above κ_γ, and $\rho_\omega(\mathcal{P}_\gamma) \leq \kappa_\gamma$. It is this observation which we shall make use of in the next section.

4 Mutual stationarity in the core model

We shall need yet another condensation result in the proof of Theorem 4.6. This result, however, only holds under an additional assumption.

Definition 4.1 (cf. [14, Definition 1.2]) *Let $\mathcal{M}$ be a premouse, and let $\kappa < \kappa^{+\mathcal{M}} < \tau < \mathcal{M} \cap OR$ be such that κ and τ are cardinals of $\mathcal{M}$. Then κ is said to be $< \tau$-strong in $\mathcal{M}$ if for all $\alpha < \tau$ there is some $E^{\mathcal{M}}_\beta \neq \emptyset$ with critical point κ and such that $\alpha \leq \beta < \tau$.*

Definition 4.2 *Let $\mathcal{M}$ be a premouse. We say that $\mathcal{M}$ does not reach $o(\kappa) = \kappa^{++}$ if there is no $\lambda \in \mathcal{M}$ such that λ is $< \lambda^{++\mathcal{M}}$-strong in $\mathcal{M}$.*

William Mitchell, in [6] and [7], had shown that K exists if no extender model reaches $o(\kappa) = \kappa^{++}$. Of course, if no extender model reaches $o(\kappa) = \kappa^{++}$ then $\neg\, 0^{¶}$ holds. However, an extender model W which does not reach $o(\kappa) = \kappa^{++}$ can be such that $0^{¶} \in W$, or $0^{⁋} \in W$, or even $M_1^{\#} \in W$, etc.

Lemma 4.3 ($\neg\, 0^{⁋}$; cf. [14, Corollary 1.3]) *Let $\mathcal{M}$ be a 0-iterable premouse which has a top extender F with critical point κ. Let $\kappa^{+\mathcal{M}} < \tau \leq \rho_1(\mathcal{M})$, where τ is a cardinal in $\mathcal{M}$. Then κ is $< \tau$-strong in $\mathcal{M}$.*

Corollary 4.4 *Let $\mathcal{M}$ be a 0-iterable premouse which does not reach $o(\kappa) = \kappa^{++}$ and which has a top extender F with critical point κ. Then $\rho_1(\mathcal{M}) \leq \kappa^{+\mathcal{M}}$.*

It can be true, though, that the top extender F as in Corollary 4.4 has more than one generator. The following lemma is part of the folklore.

Lemma 4.5 *Let $\mathcal{M}$ be an iterable premouse which does not reach $o(\kappa) = \kappa^{++}$. Let $n < \omega$, and let λ be a cardinal in $\mathcal{M}$ with $\rho_{n+1}(\mathcal{M}) \leq \lambda < \rho_n(\mathcal{M})$. Let $\vec{p} \in {}^{<\omega}(\mathcal{M} \cap OR)$ be the standard parameter of $\mathcal{M}$. Let*

$$\pi \colon \bar{\mathcal{M}} \cong Hull_{n+1}^{\mathcal{M}}(\lambda \cup \{\vec{p}\}) \prec \mathcal{M},$$

where $\bar{\mathcal{M}}$ is transitive. Then $\lambda^{+\bar{\mathcal{M}}} = \lambda^{+\mathcal{M}}$, and $\bar{\mathcal{M}}||\lambda^{+\mathcal{M}} = \mathcal{M}||\lambda^{+\mathcal{M}}$.

Proof sketch. By [4, §8 Lemma 4] we know that $\bar{\mathcal{M}}||\lambda^{+\bar{\mathcal{M}}} = \mathcal{M}||\lambda^{+\bar{\mathcal{M}}}$, and that if $(\mathcal{U}, \mathcal{T})$ denotes the coiteration of $((\mathcal{M}, \bar{\mathcal{M}}), \lambda)$ with $\mathcal{M}$ then $\mathcal{M}_\infty^{\mathcal{U}}$ sits above $\bar{\mathcal{M}}$, neither $[0,\infty]_U$ nor $[0,\infty]_T$ contains any drop, and $\mathcal{M}_\infty^{\mathcal{U}} = \mathcal{M}_\infty^{\mathcal{T}}$. We have that $\mathcal{P}(\lambda) \cap \bar{\mathcal{M}} = \mathcal{P}(\lambda) \cap \mathcal{M}_\infty^{\mathcal{U}}$. Let us now assume that $\lambda^{+\bar{\mathcal{M}}} < \lambda^{+\mathcal{M}}$.

It is easy to see that there can be then no $E_\nu^{\mathcal{M}} \neq \emptyset$ with $c.p.(E_\nu^{\mathcal{M}}) < \lambda$ and $\lambda^{+\bar{\mathcal{M}}} \leq \nu < \lambda^{+\mathcal{M}}$. The reason is that otherwise we'd have that $c.p.(E_\nu^{\mathcal{M}})$ would be $< \lambda^{+\bar{\mathcal{M}}}$-strong in $\bar{\mathcal{M}}$ and hence $< \lambda^{+\mathcal{M}}$-strong in $\mathcal{M}$, contradicting the assumption that $\mathcal{M}$ does not reach $o(\kappa) = \kappa^{++}$. It is also easy to see that this fact is inherited by all iterates of $\mathcal{M}||\lambda^{+\mathcal{M}}$; more precisely, if $\mathcal{P}$ is an iterate of $\mathcal{M}||\lambda^{+\mathcal{M}}$ via an iteration which only uses extenders with indices $\geq \lambda^{+\bar{\mathcal{M}}}$ then there can be no $E_\nu^{\mathcal{P}} \neq \emptyset$ with $c.p.(E_\nu^{\mathcal{P}}) < \lambda$ and $\lambda^{+\bar{\mathcal{M}}} \leq \nu \leq \mathcal{P} \cap OR$.

Now $\mathcal{U}$ certainly only uses extenders with indices $\geq \lambda^{+\bar{\mathcal{M}}}$. By the preceding paragraph, we thus know that $\lambda^{+\bar{\mathcal{M}}} < \lambda^{+\mathcal{M}}$ implies that $\mathcal{U}$ only uses extenders with critical points $\geq \lambda$, and therefore that there must be a drop along the main branch of $\mathcal{U}$. Contradiction! $\square$ Lemma 4.5

We can now state and prove our main result. Our proof will closely follow [3, Section 7.2] to a certain extent. By an interval (of ordinals) we mean a set of ordinals of the form $[\alpha, \beta)$.

Theorem 4.6 *Suppose that $0^\P$ does not exist and that $V = K$. Suppose that K does not reach $o(\kappa) = \kappa^{++}$. Let $k \in OR \setminus 1$. There is then a sequence $(S_i^n : i \in OR \setminus k, 0 < n < \omega)$ such that*
- *for all $i \in OR \setminus k$ and $n < \omega \setminus 1$ we have that $S_i^n \subset \aleph_{i+1}$ is stationary in $\aleph_{i+1}$ and $\alpha \in S_i^n \Rightarrow cf(\alpha) = \aleph_k$, and*
- *for all limit ordinals λ and for all $f \colon \lambda \to \omega \setminus 1$ we have that $(S_i^{f(i)} : i \in \lambda \setminus k)$ is mutually stationary if and only if we can split the domain $\lambda \setminus k$ of f into a finite partition $D_1 \cup ... \cup D_m$ of intervals such that $f \restriction D_l$ is constant whenever $1 \leq l \leq m$.*

Proof. We commence by defining the sequence $(S_i^n : i \in OR \setminus k, 0 < n < \omega)$. If α is a singular ordinal, then we let $(\gamma(\alpha), n(\alpha), \delta(\alpha), \vec{p}(\alpha))$ be the lexicographically least tuple $(\gamma, n, \delta, \vec{p})$ such that

$$Hull_n^{K||\gamma}(\delta \cup \vec{p}) \cap \alpha$$

is cofinal in α. In particular, $K||\gamma(\alpha)$ is the collapsing structure for α. For an ordinal $i \geq k$ and a natural number n we define

$$S_i^n = \{\alpha < \aleph_{i+1} : cf(\alpha) = \aleph_k \wedge n(\alpha) = n\}.$$

We are now going to show that $(S_i^n : i \in OR \setminus k, 0 < n < \omega)$ witnesses the truth of Theorem 4.6. Let λ be a limit ordinal.

We shall first prove that if $f : \lambda \setminus k \to \omega \setminus 1$ is such that we can split the domain $\lambda \setminus k$ of f into a finite partition $D_1 \cup ... \cup D_m$ of intervals with $Card(f" D_l) = 1$ for all l then $(S_i^{f(i)} : i \in \lambda \setminus k)$ is mutually stationary. By Lemma 1.1 it will be enough if we prove this under the assumption that $ran(f) = \{n\}$ for some $n \in \omega \setminus 1$.

Let $\mathfrak{A}$ be a model with universe $K||\kappa$, where κ is a large regular cardinal. Let γ_0 be least such that $\mathfrak{A} \in K||\gamma_0$, and let γ be the $\aleph_k^{th}$ ordinal β such that $\beta > \gamma_0$ and $K||\beta \prec_{\Sigma_{n-1}} K||\kappa^+$. Let

$$X = Hull_n^{K||\gamma}(\aleph_k \cup \{\mathfrak{A}\}).$$

One can then argue exactly as for [3, Lemma 25] that for every $i \in X \cap \lambda$, $sup(X \cap \aleph_{i+1}) \in S_i^n$. The only thing to notice is that the use of the condensation lemma for L can be replaced by [4, §8, Lemma 4] in a straightforward way.

Let us now fix some $f : \lambda \setminus k \to \omega \setminus 1$ such that $(S_i^{f(i)} : i \in \lambda \setminus k)$ is mutually stationary. We aim to prove that we can split the domain $\lambda \setminus k$ of f into a finite partition $D_1 \cup ... \cup D_m$ of intervals with $Card(f" D_l) = 1$ for all l. We shall exploit the covering argument of section 3.

Let $\kappa = \aleph_\lambda$, and let

$$N \prec (K||\kappa; \in, ...)$$

be such that for all $i \in N \cap \lambda$ we have that $sup(N \cap \aleph_{i+1}) \in S_i^{f(i)}$. In particular, $cf(sup(N \cap \aleph_{i+1})) = \aleph_k > \omega$ for each such i. Let

$$\pi : \bar{K} \cong N \prec K||\kappa$$

be such that $\bar{K}$ is transitive.

We shall now apply the results of section 3. Let us adopt the notation from there. In particular, by the proof of Theorem 3.1, $\mathcal{T}$ is a normal iteration tree on K such that $\mathcal{M}_\infty^{\mathcal{T}} \trianglerighteq \bar{K}$.

Let $\mathcal{D} = \mathcal{D}^{\mathcal{T}} \cap (0, \infty]_T$, and let $\mathcal{D} = \{\alpha_0 + 1 < ... < \alpha_N + 1\}$ where $N < \omega$. Let α_i^* be the T-predecessor of $\alpha_i + 1$ for $i \leq N$. Notice that $\alpha_0^* = 0$. Let $\kappa^{-1} = \eta$, let $\kappa^i = c.p.(E_{\alpha_i}^{\mathcal{T}})$ for $0 < i \leq N$, and let $\kappa^{N+1} = \bar{K} \cap OR$. For any $\gamma < \theta$, let $i(\gamma)$ be the unique $i \leq N$ such that $\kappa_\gamma \in [\kappa^i, \kappa^{i+1})$. Notice that $\mathcal{M}_{\alpha_{i+1}^*}^{\mathcal{T}} \trianglerighteq \bar{K}||\kappa^{i+1}$ and, by the proof of Theorem 3.1, $\rho_\omega(\mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}) \leq \kappa^i$. For any $\gamma < \theta$, let $\eta(\gamma)$ be the largest $\eta \leq \mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}} \cap OR$ such that

$$\mathcal{P}(\kappa_\gamma) \cap \bar{K} = \mathcal{P}(\kappa_\gamma) \cap \mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}||\eta,$$

and let $m(\gamma)$ be the least $m < \omega$ with $\rho_{m+1}(\mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}||\eta(\gamma)) \leq \kappa_\gamma$. If $\kappa^i \leq \kappa_\gamma \leq \kappa_{\gamma'} < \kappa^{i+1}$ then $(\eta(\gamma'), m(\gamma')) \leq_{\text{lex}} (\eta(\gamma), m(\gamma))$, where $\leq_{\text{lex}}$ denotes the lexicographical ordering. This shows:

Claim 1. We can split θ into a finite partition I_0, ..., I_p of intervals such that whenever $\{\gamma, \gamma'\} \subset I_i$ then $i(\gamma) = i(\gamma')$, $\eta(\gamma) = \eta(\gamma')$, and $m(\gamma) = m(\gamma')$.

For $\gamma < \theta$ we may let $\mathcal{P}'_\gamma$ be the unique transitive $\mathcal{P}$ such that

$$\mathcal{P} \cong Hull_{m(\gamma)+1}^{\mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}||\eta(\gamma)}(\kappa_\gamma \cup \{\vec{p}\}) \prec \mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}||\eta(\gamma),$$

where $\vec{p}$ is the standard parameter of $\mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}||\eta(\gamma)$. As K does not reach $o(\kappa) = \kappa^{++}$, by Lemma 4.5 we shall have that

$$\mathcal{P}(\kappa_\gamma) \cap \mathcal{P}'_\gamma = \mathcal{P}(\kappa_\gamma) \cap \mathcal{M}^{\mathcal{T}}_{\alpha^*_{i+1}}||\eta(\gamma) = \mathcal{P}(\kappa_\gamma) \cap \bar{K}.$$

Moreover, we'll clearly have that $\rho_{m(\gamma)+1}(\mathcal{P}'_\gamma) \leq \kappa_\gamma < \rho_{m(\gamma)}(\mathcal{P}'_\gamma)$.

We now use the models $\mathcal{P}'_\gamma$ to define $\mathcal{Q}'_\gamma$, $\mathcal{R}'_\gamma$, and $m^*(\gamma)$ in exactly the same way as we had defined $\mathcal{Q}_\gamma$, $\mathcal{R}_\gamma$, and $n^*(\gamma)$ using the models $\mathcal{P}_\gamma$ in the previous section.[n] We set

$$\mathcal{Q}'_\gamma = Ult_{m(\gamma)}(\mathcal{P}'_\gamma; \pi \restriction \bar{K}||\lambda_\gamma).$$

We set $\mathcal{R}'_\gamma = \mathcal{Q}'_\gamma$ and $m^*(\gamma) = n^*(\gamma)$ if $\mathcal{Q}'_\gamma$ is a premouse. If $\mathcal{Q}'_\gamma$ is a proto-mouse rather than a premouse then we set

$$\mathcal{R}'_\gamma = Ult_{m^*(\bar{\gamma})}(\mathcal{R}'_{\bar{\gamma}}; G),$$

where $\kappa_{\bar{\gamma}}$ is the critical point of the top extender of $\mathcal{P}'_\gamma$ and G is the top extender fragment of $\mathcal{Q}'_\gamma$.

Suppose that $\{\gamma, \gamma'\} \subset I_i$, where I_i is an interval as in Claim 1. Then $\mathcal{P}'_\gamma$ has a top extender if and only if $\mathcal{P}'_{\gamma'}$ has a top extender. Suppose this is the case, and let us further suppose that $\kappa_{\bar{\gamma}} < \kappa_\gamma \leq \kappa_{\gamma'}$, where $\kappa_{\bar{\kappa}}$ is the critical point of the top extender of $\mathcal{P}'_\gamma$ as well as of $\mathcal{P}'_{\gamma'}$. Then $\mathcal{Q}'_\gamma$ is a proto-mouse if and only if $\mathcal{Q}'_{\gamma'}$ is a proto-mouse, and in fact $m^*(\gamma) = m^*(\gamma')$. This shows:

[n]It is well possible that $\mathcal{P}'_\gamma \neq \mathcal{P}_\gamma$. We'll have to have that $\mathcal{R}'_\gamma = \mathcal{R}_\gamma$, though.

Claim 2. We can split θ into a finite partition I'_0, ..., $I'_{p'}$ of intervals such that whenever $\{\gamma, \gamma'\} \subset I'_i$ then and $m^*(\gamma) = m^*(\gamma')$.

By the remark right after the proof of Lemma 3.3 we shall now have the following.

Claim 3. For each $\gamma < \theta$ we have that $\mathcal{R}'_\gamma$ is a well-defined premouse. In fact, $\mathcal{R}'_\gamma \lhd K$, and $\mathcal{R}'_\gamma$ is the collapsing structure for $\pi(\kappa_\gamma)$. We have that $\rho_{m^*(\gamma)+1}(\mathcal{R}'_\gamma) \leq \kappa_\kappa < \rho_{m^*(\gamma)}(\mathcal{R}'_\gamma)$.

By Claim 3,

$$n(sup(\pi"\lambda_\gamma)) = m^*(\gamma) + 1$$

whenever $\gamma < \theta$. Now suppose that we couldn't split the domain $\lambda \setminus k$ of f into a finite partition $D_1 \cup ... \cup D_m$ with $Card(f"D_l) = 1$ for all l. We could then have expanded $(K||\kappa; \in, ...)$ so as to make sure that there is a sequence $(\gamma_q : q < \omega) \in {}^\omega\lambda$ such that $f \restriction \{\pi(\gamma_q) : q < \omega\}$ is not eventually constant. This would contradict Claim 2. $\square$ Theorem 4.6

5 Open problems.

Many questions remain open. Can Theorem 4.6 be extended to the core model of [14], or to the one of [17] (cf. [15, Problem # 10])? Or can Theorem 4.6 even be extended to extender models which do not know how to fully iterate themselves? Finally: Can one use methods provided by the current paper to get a reasonable lower bound for the consistency strength of the assumption that $(S_n : n < \omega)$ must be mutually stationary provided every individual $S_n \subset \aleph_n$ is stationary?

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