$P \neq NP$ for infinite time Turing machines

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Abstract. We state different versions of the $P = ?NP$ problem for infinite time Turing machines. It is observed that $P \neq NP$ collapses to the fact that there are analytic sets which are not Borel.

In this note we study versions of the $P = ?NP$ problem for infinite time Turing machines. The analytic sets of reals may be construed as an infinite analog to the class NP (cf. for example [3, §3.9]). The $P = ?NP$ problems for infinite time Turing machines can therefore naturally be translated as questions about analytic sets. These questions have classical answers.

1 Analytic sets.

We shall have to consider Polish spaces, i.e., complete separable metric spaces. In particular we shall be interested in the Cantor space ${}^\omega 2$ and in the Baire space ${}^\omega\omega$. We refer the reader to [2] for background information. However, the descriptive set theory which we shall need is pretty elementary indeed. In order to make the paper self-contained modulo [1] this section develops all the necessary descriptive set theoretic tools.

Let $\mathcal{X}$ be a Polish space and let $(\mathcal{O}_n : n < \omega)$ be a recursive enumeration of basic open sets. Let $\mathcal{O}^{\mathcal{X}} \subset \mathcal{X} \times {}^\omega 2$ be defined by

$$(x, y) \in \mathcal{O}^{\mathcal{X}} \Leftrightarrow x \in \bigcup_{y(n)=1} \mathcal{O}_n.$$

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Then $\mathcal{O}^{\mathcal{X}}$ is a universal open set, i.e., $\mathcal{O}^{\mathcal{X}}$ is open in $\mathcal{X} \times {}^{\omega}2$ and if $\mathcal{O}$ is an open subset of $\mathcal{X}$ then there is some $y \in {}^{\omega}2$ with

$$\mathcal{O} = \{x \mid (x,y) \in \mathcal{O}^{\mathcal{X}}\}.$$

A set $A \subset \mathcal{X}$ is analytic if and only if there is a closed $\mathcal{C} \subset \mathcal{X} \times {}^{\omega}\omega$ such that

$$x \in A \Leftrightarrow \exists y \in {}^{\omega}\omega \ (x,y) \in \mathcal{C}.$$

Let $\mathcal{U}^{\mathcal{X}} \subset \mathcal{X} \times {}^{\omega}2$ be defined by

$$(x,y) \in \mathcal{U}^{\mathcal{X}} \Leftrightarrow \exists z \in {}^{\omega}\omega \ ((x,z),y) \notin \mathcal{O}^{\mathcal{X} \times {}^{\omega}\omega}.$$

Then $\mathcal{U}^{\mathcal{X}}$ is a universal analytic set, i.e., $\mathcal{U}^{\mathcal{X}}$ is analytic in $\mathcal{X} \times {}^{\omega}2$ and if A is an analytic subset of $\mathcal{X}$ then there is some $y \in {}^{\omega}2$ with

$$A = \{x \mid (x,y) \in \mathcal{U}^{\mathcal{X}}\}.$$

Now let $\mathcal{X} = {}^{\omega}2$, and let us write $\mathcal{U}$ for $\mathcal{U}^{\mathcal{X}} = \mathcal{U}^{({}^{\omega}2)}$. The set $\Delta = \{x \mid (x,x) \in \mathcal{U}\}$ is analytic. If Δ were coanalytic, i.e., if ${}^{\omega}2 \setminus \Delta$ were analytic, then there would be some $y \in {}^{\omega}2$ with

$$\Delta = \{x \mid (x,y) \notin \mathcal{U}\};$$

but then $y \in \Delta$ iff $(y,y) \notin \mathcal{U}$ iff $y \notin \Delta$. Hence Δ is not coanalytic, and therefore not Borel.

Let $G \subset {}^{\omega}2$ be the set of all $x \in {}^{\omega}2$ which are not eventually constant (equivalently, such that there are arbitrary large m and m' with $x(m) = 0$ and $x(m') = 1$). G is a G_δ subset of ${}^{\omega}2$. We may define a bijection $\varphi \colon G \to {}^{\omega}\omega$ by $\varphi(x) = y$ if and only if the n^{th} block of 1's in x contains exacly $y(n)$ 1's. If $A \subset {}^{\omega}2 \times {}^{\omega}\omega$, say, then we may define $A^{\varphi} \subset {}^{\omega}2 \times {}^{\omega}2$ by

$$(x,y) \in A^{\varphi} \Leftrightarrow y \in G \wedge (x, \varphi(y)) \in A.$$

Now let $A \subset {}^{\omega}2$ be lightface analytic which means that there is a recursive $R(-,-)$ such that, if $\mathcal{C}$ denotes the closed set

$$[R] = \{(x,y) \in {}^{\omega}2 \times {}^{\omega}\omega \mid \forall n < \omega \ R(x \restriction n, y \restriction n)\},$$

then we have that

$$x \in A \Leftrightarrow \exists y \in {}^{\omega}\omega \ (x,y) \in \mathcal{C}.$$

We then also have that

$$x \in A \Leftrightarrow \exists y \in {}^{\omega}2 \ (x, y) \in \mathcal{C}^{\varphi}.$$

It is straightforward to verify that $\mathcal{C}^{\varphi}$ is no longer closed. Rather, $\mathcal{C}^{\varphi}$ is a lightface G_{δ} subset of ${}^{\omega}2$.

Now the set $\Delta \subset {}^{\omega}2$ defined above is a lightface analytic set. In fact, the witnessing recursive $R(-,-)$ is easily given by the complement of $\mathcal{O}^{{}^{\omega}2 \times {}^{\omega}\omega}$; we here need that $(\mathcal{O}_n \colon n < \omega)$ be recursive. We then have that

$$x \in \Delta \Leftrightarrow \exists y \in {}^{\omega}2 \ (x, y) \in \mathcal{G},$$

where $\mathcal{G}$ is a lightface G_{δ} subset of ${}^{\omega}2$. It is this latter representation of Δ which we shall need later on.

2 $P \neq NP$ for infinite time Turing machines.

Infinite time Turing machines were introduced in [1]. They have exactly the same hardware as traditional Turing machines. The difference is that one allows transfinite running times. We refer the reader to [1] for exact definitions. An acquaintance with §§1 and 2 (i.e., pp. 569-575) of [1] will basically suffice for our purposes. In what follows, by "Turing machine" we shall always mean an infinite time Turing machine.

It will be convenient to think of a Turing machine to come with *two* halting states, the accept state, and the reject state.

Definition 2.1 *Let $A \subset {}^{\omega}2$. We say that A is decidable in polynomial time, or $A \in P$, if there are a Turing machine T and some $m < \omega$ such that*
(a) T decides A (i.e, $x \in A$ iff T accepts x), and
(b) T halts on all inputs after $< \omega^m$ many steps.

With infinite time Turing machines, all inputs (i.e., elements of the Cantor space ${}^{\omega}2$) may be counted as having the same length, namely ω. So it appears reasonable to have a polynomial time Turing machine being one which always halts after $< \omega^m$ many steps, for some fixed $m < \omega$.

The following just generalizes Definition 2.1.

Definition 2.2 *Let $A \subset {}^{\omega}2$, and let $\alpha \leq \omega_1 + 1$. We say that A is in P_α if there are a Turing machine T and some $\beta < \alpha$ such that*
(a) T decides A (i.e, $x \in A$ iff T accepts x), and
(b) T halts on all inputs after $< \beta$ many steps.

Of course, $P = P_{\omega^\omega}$. Moreover, P_{ω_1+1} is just the class of all $A \subset {}^{\omega}2$ which are decided by some Turing machine.

Lemma 2.3 ([1, Theorem 2.6]) *Let $A \subset {}^{\omega}2$. Then $A \in P_{\omega^2}$ if and only if A is an arithmetic set.*

Lemma 2.4 *Let $A \subset {}^{\omega}2$. Then $A \in P_{\omega+2}$ if and only if A is a lightface G_δ set.*

Proof. This is straightforward. The less trivial direction is given by the proof of [1, Theorem 2.6]. $\square$

Lemma 2.5 *Let $A \subset {}^{\omega}2$. Then $A \in P_{\omega_1^{CK}}$ if and only if A is a hyperarithmetic set. If $A \in P_{\omega_1}$ then A is a Borel set.*

Proof. The first part is [1, Theorem 2.7]. The second part is an immediate consequence of the proof thereof. $\square$

It is on the other hand not true that every Borel set is in P_{ω_1}. Lemma 2.7 will characterize P_{ω_1}.

Definition 2.6 *Let $A \subset {}^{\omega}2$. If $\alpha < \omega_1$, then we say that $A \in \Delta^1_1(\alpha)$ if $A \in \Delta^1_1(x)$ uniformly for every real x coding α. We say that A is Δ^1_1 in a countable ordinal if there is some $\alpha < \omega_1$ such that $A \in \Delta^1_1(\alpha)$.*

Lemma 2.7 [1] *$A \in P_{\omega_1}$ if and only if A is Δ^1_1 in a countable ordinal. In fact, if α is admissible then $A \in P_\alpha$ if and only if $A \in \Delta^1_1(\beta)$ for some $\beta < \alpha$.*

Proof. This follows from revisiting the proof of [1, Theorem 2.7]. "$\Rightarrow$" is immediate. As to "$\Leftarrow$," note that we may pick a real x coding β such that $\alpha \geq \omega_1^x$ (= the least x-admissible $> \omega$). The Borel code for A is the a tree with rank $< \omega_1^x \leq \alpha$. $\square$

We now turn to the class NP.

[1] This was observed independently by J.D. Hamkins.

Definition 2.8 *Let $A \subset {}^{\omega}2$. We say that A is verifiable in polynomial time, or $A \in NP$, if there are a Turing machine T and some $m < \omega$ such that*
(a) $x \in A$ if and only if $(\exists y \;\; T$ accepts $x \oplus y)$, and
(b) T halts on all inputs after $< \omega^m$ many steps.

Definition 2.9 *Let $A \subset {}^{\omega}2$, and let $\alpha \leq \omega_1 + 1$. We say that A is in NP_α, if there are a Turing machine T and some $\beta < \alpha$ such that*
(a) $x \in A$ if and only if $(\exists y \;\; T$ accepts $x \oplus y)$, and
(b) T halts on all inputs after $< \beta$ many steps.

Again, $NP = NP_{\omega^\omega}$. NP_α is the class of all projections of sets in P_α. It is now immediate that $P \neq NP$.

Theorem 2.10 $NP_{\omega+1} \setminus P_{\omega_2} \neq \emptyset$.

Proof. Let Δ and $\mathcal{G}$ be as in section 1. In particular, Δ is a lightface analytic subset of ${}^{\omega}2$ which is not Borel, $\mathcal{G}$ is a lightface G_δ set, and

$$x \in \Delta \Leftrightarrow \exists y \in {}^{\omega}2 \; (x, y) \in \mathcal{G}.$$

By Lemma 2.4, $\mathcal{G} \in P_{\omega+2}$. Hence $\Delta \in NP_{\omega+2}$. However, by Lemma 2.5, Δ cannot be in P_{ω_1}, as it is not Borel. $\square$

Another version of the $P =?NP$ problem counts an input $x \in {}^{\omega}2$ as having length ω_1^x (= the least x-admissible $> \omega$). Note that no admissible ordinal is clockable (cf. [1, Theorem 8.8]. This leads to:

Definition 2.11 *Let $A \subset {}^{\omega}2$. We say that $A \in P^+$ if there is a Turing machine T such that*
(a) $x \in A$ if and only if T accepts x, and
(b) T halts on all inputs x after $< \omega_1^x$ many steps.

Definition 2.12 *Let $A \subset {}^{\omega}2$. We say that $A \in NP^+$ if there is a Turing machine T such that*
(a) $x \in A$ if and only if $(\exists y \;\; T$ accepts $x \oplus y)$, and
(b) T halts on all inputs $x \oplus y$ after $< \omega_1^x$ many steps.

Again we'll have that $P \neq NP$.

Theorem 2.13 $P^+ = P_{\omega_1^{\mathrm{CK}}} = \Delta^1_1$.

PROOF. Let $A \in P^+$. It is straightforward that there is then a Σ_1 formula Ψ (saying that there is a certain sequence of snapshots) such that

$$x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \Psi(x).$$

This implies that A is coanalytic (i.e., Π^1_1). Of course, we also have that ${}^\omega 2 \setminus A \in P^+$, so that by the same argument ${}^\omega 2 \setminus A \in \Pi^1_1$. Therefore, $P^+ \subset \Delta^1_1$.

On the other hand, we have $\Delta^1_1 = P_{\omega_1^{\mathrm{CK}}} \subset P^+$. $\square$

Corollary 2.14 *$NP^+ \setminus P^+ \neq \emptyset$.*

3 Some open problems.

We may allow a Turing machine to take even more time to reach its decision. Recall that if $\lambda + n$ is clockable for $n < \omega$ then so is λ. We arrive at:[2]

Definition 3.1 *Let $A \subset {}^\omega 2$. We say that $A \in P^{++}$ if there is a Turing machine T such that*
(a) $x \in A$ if and only if T accepts x, and
(b) T halts on all inputs x after $\leq \omega_1^x + \omega$ many steps.

Definition 3.2 *Let $A \subset {}^\omega 2$. We say that $A \in NP^{++}$ if there is a Turing machine T such that*
(a) $x \in A$ if and only if $(\exists y \;\; T$ accepts $x \oplus y)$, and
(b) T halts on all inputs $x \oplus y$ after $\leq \omega_1^x + \omega$ many steps.

P^{++} is a larger class than P^+:

Theorem 3.3 *Every lightface analytic set is in P^{++}.*

PROOF. Let A be a lightface analytic set. There is a recursive $R(-,-)$ such that

$$x \in A \Leftrightarrow \exists y \in {}^\omega\omega \; \forall n < \omega \; R(x \restriction n, y \restriction n).$$

[2] In spirit this has been suggested by P. Welch.

For $x \in {}^\omega 2$ consider the tree

$$T_x = \{s \mid R(x \upharpoonright lh(s), s)\}.$$

Then

$$x \in A \Leftrightarrow T_x \text{ is illfounded.}$$

We can design a Turing machine $\mathcal{T}$ which, on input x, first produces T_x and then crosses out the wellfounded part of T_x. This wellfounded part has rank $\leq \omega_1^x$ (as every wellfounded tree which is recursive in x has rank $< \omega_1^x$). The machine $\mathcal{T}$ is finally suppose to check if there is something left after crossing out the wellfounded part of T_x. This will take another ω many steps of computation. On input x, $\mathcal{T}$ has therefore a running time $\leq \omega_1^x + \omega$. $\square$

Of course, P^{++} is also closed under complements.

Question. $P^{++} \neq NP^{++}$?

Definition 3.4 *Let $f\colon \mathcal{D} \to \omega_1$. Let $A \subset {}^\omega 2$. We say that $A \in P^f$ if there is a Turing machine T such that*
(a) $x \in A$ if and only if T accepts x, and
(b) T halts on all inputs x after $< f(x)$ many steps.

Definition 3.5 *Let $f\colon \mathcal{D} \to \omega_1$. Let $A \subset {}^\omega 2$. We say that $A \in NP^f$ if there is a Turing machine T such that*
(a) $x \in A$ if and only if $(\exists y \;\; T$ accepts $x \oplus y)$, and
(b) T halts on all inputs $x \oplus y$ after $< f(x)$ many steps.

Question. For which $f\colon \mathcal{D} \to \omega_1$ is $P^f \neq NP^f$?

References

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