Problem 1 *Prove the existence of the core model in the theory* BGC + *"there is no inner model with a Woodin cardinal."*

[This problem was contributed by R. Schindler.]

By "core model" I mean an inner model of the form $L[E]$ (i.e., a class-sized premouse, i.e., a weasel) which is iterable for set-sized trees, which computes successors of singular cardinals correctly, and which is recursiveley defined by a definition as in section 6 of [St96] (cf. the discussion in section 2 of [Sch∞∞]). BGC is Bernays-Gödel class theory (with global choice), which is conservative over ZFC.

Traditionally, the existence of the core model K has been shown in theories of the form $\mathsf{BGC} + \Phi$, where Φ is an anti large cardinal assumption. For weaker and weaker Φ this was achieved by the papers [DoJe81], [Koe83], [Je88], [Mi84] and [Mi∞], and [Je91]. The best result known today appears in [Sch01], where the existence of K is shown in the theory BGC + "0-hand grenade doesn't exist."

As of today, if one further weakens Φ we need an additional assumption in order to prove the existence of K. The paper [St96] shows the existence of K in V_κ in the theory ZFC + "there's no inner model with a Woodin cardinal" + "κ is a measurable cardinal in V." The papers [MiScSt97] and [MiSc95] prove the relevant weak covering lemma witnessing that the K of [St96] is a core model. It is known that the measurable cardinal can be replaced by a subtle cardinal here, but it is open as to whether the subtle cardinal can be replaced by a Mahlo cardinal, say.

For some applications of the theory of K this situation is a deficiency. For example, in order to prove by the methods of [St96] that if there is a saturated ideal on ω_1 then there is an inner model with a Woodin cardinal one needs the background assumption that there be a measurable (or a subtle) cardinal in V (cf. [St96, Theorem 7.1]). However, we conjecture that $\mathrm{Con}(\mathsf{ZFC}$ + "there is a saturated ideal on ω_1") $\Rightarrow$ $\mathrm{Con}(\mathsf{ZFC}$ + "there is an inner model with a Woodin cardinal") is provable in Peano Arithmetic.

The paper [Sch∞] proves the existence of a universal weasel W in the theory BGC + "there's no inner model with a Woodin cardinal," where W is universal in the sense that it wins the coiteration against any coiterable (set- or class-sized) premouse. (In subsequent work, [Mi∞∞] has produced premice of height κ which win the coiteration against all coiterable premice of height $< \kappa$, for various κ.) A key step for solving the problem in question may be proving a useful form of weak covering for the weasel which is constructed in [Sch∞]. A serious issue here, however, is that class-sized iteration trees on that weasel need not have cofinal branches (this may or may not indicate that it is the approach of [Mi$\infty\infty$] rather than the one of [Sch$\infty\infty$] which is the right one).

In any event we believe that the methods which are provided by [Sch01] and which are crucial for proving weak covering for a K^c with many overlaps might be highly relevant here.

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