Problem 3 *Does* BMM + *"every set of reals in* $L(\mathbb{R})$ *is Lebesgue measurable" imply* PD*?*

[This problem was contributed by R. Schindler]

Here BMM is Bounded Martin's Maximum, which says that the boldface Σ_1 theory of H_{ω_2} is frozen w.r.t. stationary preserving set-forcing extensions. PD is Projective Determinacy.

The following is known. Let BPFA denote the Bounded Proper Forcing Axiom. Then ZFC + BPFA + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable" is equiconsistent with ZFC + "there is a reflecting cardinal above a remarkable cardinal" (cf. [Sch∞, Theorem 2]). In particular, BPFA + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable" does not implay that $0^\#$ exists. On the other hand, BMM + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable" implies that V is closed under $x \mapsto x^\#$ (cf. [Sch∞, Corollary 26]). Moreover, by methods which are similar to (but not identical with) the ones developed by [Sch00] one can prove that BMM + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable" implies that there is an inner model with a strong cardinal ([Sch$\infty\infty$]). It is completely open, though, how to derive the existence of $0^\P$ from BMM + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable." New techniques for "coding sets into K" are called for. The best known upper bound for the strength of BMM + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable" is a proper class of Woodin cardinals (cf. [W99, Theorem 10.99]).

I conjecture that the question asked has an affirmative answer. In fact, I conjecture that BMM + "every set of reals in $L(\mathbb{R})$ is Lebesgue measurable" implies that the Axiom of Determinacy holds in $L(\mathbb{R})$.

References

[Sch00] R. Schindler, *Coding into K by reasonable forcing*, Trans. Amer. Math. Soc. 353 (2000), pp. 479-489.

[Sch∞] R. Schindler, *Forcing axioms and projective sets of reals*, Proc. FotFS III, submitted.

[Sch$\infty\infty$] R. Schindler, unpublished.

[W99] H. Woodin, The axiom of determinacy, forcing axioms, and the nonstationary ideal, Berlin + New York, 1999.