Problem 4 *Does the existence of a singular strong limit cardinal κ such that $\Box_\kappa$ fails imply that AD holds in $L(\mathbb{R})$?*

[This problem was contributed by E. Schimmerling and J.R. Steel]

The real problem here is to improve as much as possible the existing lower bound on the consistency strength of the assertion that $\neg\Box_\kappa$ holds for some singular κ. Upper bounds on the consistency strength of $\neg\Box_\kappa$ for κ a singular strong limit have been established by Solovay, Shelah, and Magidor. These upper bounds are in the neighborhood of a supercompact cardinal.

We would guess this upper bounds are close to optimal, and that in fact the existence of any singular cardinal κ such that $\neg\Box_\kappa$ implies there is an inner model with a superstrong cardinal. We have stated the problem as above because it would take a new idea to push the lower bound even as high as AD in $L(\mathbb{R})$, even in the strong limit case, and this new idea will probably figure in any significant improvement of the lower bound. Making do without the "strong limit" assumption on κ seems essentially the same as problem 5 on our list.

The problem is interesting because many natural statements seem to get logical strength from the fact they imply $\neg\Box_\kappa$ for some singular κ. (For example, PFA, Rado's conjecture, PMEA, ...). By [3], core models below many superstrongs satisfy $\forall\kappa\Box_\kappa$. Thus one gets strength from $\neg\Box_\kappa$, κ singular, by constructing core models satisfying a weak covering theorem. On a technical level, this problem represents an interesting test of our ability to do that.

The following theorem is the best partial result on the problem.

Theorem 1 *Let κ be a singular cardinal such that $\alpha < \kappa \Rightarrow \alpha^\omega < \kappa$, and suppose that $\Box_\kappa$ fails. Then $V^{Col\ (\omega,\kappa)} \models \mathsf{PD}$.*

The proof of this theorem is a local core model induction: one shows by induction on n that for all $X \subseteq \kappa$, $M_n(X)^\sharp$ exists and is κ^+-iterable. (This is actually equivalent to PD holding after collapsing κ to be countable.) The induction is "local" because of the restriction that $X \subseteq \kappa$; one cannot show even that $X^\sharp$ exists for all $X \subseteq \kappa^+$ under the hypotheses of Theorem 1, of course. This local nature makes the problem of getting strength more delicate than in the core model induction which gets AD in $L(\mathbb{R})$ from PFA plus an inaccessible, or from $L(\mathbb{R})$ generic absoluteness. (See [1], [4], and

[6] for these "global" core model inductions, which are due to Woodin.) See [6] and [1] for examples of local inductions. There, however, the strength comes from the existence of certain ideals, rather than from a failure of weak covering.

The cases $n = 0$ and (essentially) $n = 1$ of Theorem 1 are done in [2]; if you have seen a core model induction before, then it is a reasonable exercise to complete the proof of Theorem 1. Presumably, the problem will be solved by extending the induction, showing by induction on α that $J_\alpha(\mathbb{R}) \models \mathsf{AD}$ holds after collapsing κ. Steel has done the "end-of-gap" cases in such an induction; the main remaining problem is the case α has uncountable cofinality but is not $\mathbb{R}$-admissible.

Steel and Schimmerling have used the "poor man's core model induction" technique of [5] to prove inductive determinacy holds after collapsing κ under the hypotheses of 1. This approach looks like a dead end at the moment, however.

It should be possible to drop the hypothesis that $\forall\alpha < \kappa(\alpha^\omega < \kappa)$ from 1. The natural attempt to do so, if one follows the line of [2], involves solving problem 5 on this list.

References

[1] R. Ketchersid, *Towards $AD_{\mathbb{R}}$ from an ω_1-dense ideal plus* CH, Ph.D. thesis, Berkeley 2000, available at `http://www.math.ucla.edu/~ketchers/publications/thesis.pdf`

[2] E. Schimmerling and J.R. Steel, *The maximality of the core model*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3119–3141.

[3] E. Schimmerling and M. Zeman, *Square in core models*, Bull. Symbolic Logic **7** (2001) 305–314.

[4] R.Schindler and J.R. Steel, *The strength of AD*, preprint, available at `http://www.logic.univie.ac.at/~rds/`

[5] J.R. Steel, *Core models with more Woodin cardinals*, preprint, available at `http://www.math.berkeley.edu/~steel/`

[6] A.S. Zoble, *Stationary reflection and the determinacy of inductive games*, Ph.D. thesis, Berkeley 2000, available at `http://www.math.toronto.edu/~larson/zoblethe.ps`