Problem 5 *Suppose that there is no inner model with a Woodin cardinal. Let* $\kappa \geq \aleph_1$*, and suppose that* $X^{\#}$ *exists for every* $X \subset \kappa$*. Is there a cone* $\mathbb{C}$ *of subsets of* κ *such that for all* X*,* $Y \in \mathbb{C}$ *do we have that, setting* $\tau = \min(\kappa^{+K^{L[X]}}, \kappa^{+K^{L[X]}})$*,* $K^{L[X]}|\tau = K^{L[Y]}|\tau$ *?*

[This problem was contributed by R. Schindler]

The hypotheses of problem 5 allow us to construct Steel's core model in $L[X]$, denoted by $K^{L[X]}$, whenver $X \subset \kappa$ (cf. [St96, p. 58]). We write $K^{L[X]}|\tau$ for $K^{L[X]}$ cut off at τ. We say that $\mathbb{C} \subset \mathcal{P}(\kappa)$ is a cone if there is some $X \subset \kappa$ such that $\mathbb{C} = \{Y \subset \kappa : X \in L[Y]\}$.

Problem 5 is tightly connected with problem 4 on this list. The proof of [ScSt99, Theorem 4.2] shows that if κ is a singular cardinal such that problem 5 has an affirmative answer for κ then the failure of $\square_\kappa$ implies the existence of an inner model with a Woodin cardinal (cf. [ScSt99, Theorem 4.2]).

Steel has shown that problem 5 has an affirmative answer for all κ such that $\alpha < \kappa$ implies $\alpha^{\aleph_0} \leq \kappa$ (cf. [ScSt99, Lemma 3.1.1]). Therefore, if κ is a countably closed singular cardinal then the failure of $\square_\kappa$ implies the existence of an inner model with a Woodin cardinal.

There are obvious variants of problem 5 for large cardinal situations which are less restrictive than "there is no inner model with a Woodin cardinal." Any solution of problem 5 as stated should yield solutions to these variants as well.

References

[ScSt99] E. Schimmerling and J.R. Steel, *The maximality of the core model*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3119–3141.

[St96] J.Steel, *The core model iterability problem*, Lecture Notes in Logic 8 (1996)