Problem 6 *Suppose that W, V are inner models of set theory with the same cardinals, $V = W[r]$ for some real r, $W \models \mathsf{GCH}$, but $V \models \neg\mathsf{CH}$. Must $0^\dagger$ exist?*

[This problem was contributed by R. Schindler]

By "inner model of set theory" I mean a transitive class-sized model of ZFC. $V = W[r]$ means that V is the least inner model of set theory containing all sets in $W \cup \{r\}$. We may construe the question as being formulated in Bernays-Gödel class theory, BGC.

The paper [ShWo84] shows that $\mathsf{Con}(\mathsf{BGC}+$ "there are $\aleph_2$ many measurable cardinals") $\Rightarrow \mathsf{Con}(\mathsf{BGC}+$ "there are W, V as in the hypothesis of problem 6").

On the other hand, [Sh82] shows that if the hypothesis of problem 6 holds then there is an inner model with a measurable cardinal. The proof uses the following "strong covering lemma."

Theorem 1 (Shelah) *Suppose that there is no inner model with a measurable cardinal, and let K denote the core model. Let $\kappa \geq \aleph_1$ be regular, and $\alpha \geq \kappa$. Then $[\alpha]^\kappa \cap K$ is stationary in $[\alpha]^\kappa$.*

Theorem 1 can be generalized to:

Theorem 2 (Jensen, Mitchell) *Suppose that every mouse is "short" in the sense of [Koe88], and let K denote the core model. There is then a maximal Prikry system $\mathbb{C}$ for K with the following property. Let $\kappa \geq \aleph_1$ be regular, and $\alpha \geq \kappa$. Then $[\alpha]^\kappa \cap K[\mathbb{C}]$ is stationary in $[\alpha]^\kappa$.*

It is conceivable that Theorem 2 can be used to prove that $\mathsf{Con}(\mathsf{BGC}+$ "there are W, V as in the hypothesis of problem 6") $\Rightarrow \mathsf{Con}(\mathsf{BGC}+$ "there are $\aleph_2$ many measurable cardinals"). Any positive solution to problem 6 should in fact yield this result.

On the other hand, it is conceivable that a variant of the proof of [ShWo84, Theorem 1] will give a proof of $\mathsf{Con}(\mathsf{BGC}+$ "there is a measurable cardinal") $\Rightarrow \mathsf{Con}(\mathsf{BGC}+$ "there are W, V as in the hypothesis of problem 6").

Therefore, any solution to problem 6 should in fact determine the consistency strength of the the statement that there be a pair (W, V) of inner models of set theory with the same cardinals, $V = W[r]$ for some real r, $W \models \mathsf{GCH}$, and $V \models \neg\mathsf{CH}$.

References

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[ShWo84] Shelah, S., and Woodin, H., *Forcing the failure of* CH *by adding a real*, J.S.L. **49** (1984), pp. 1185 - 1189.