Problem 8. *Let M be a model of "$\mathsf{ZFC} + K$ exists." Does the iterability of K^M imply the iterability of M ?*

[This problem was contributed by I. Neeman]

Work with a model M and suppose that $M \models$"$\mathsf{ZFC} + K$ exists." (There are various ways to reach this situation. For example: Suppose z is a real, $M_1^{\#}(z)$ exists, and there are no class models with two Woodin cardinals. Take M equal to $M_1^{\sharp}(z)$ cut at its Woodin cardinal.)

Suppose $\mathcal{T}$ is an iteration tree of length ω on M. Let $\langle M_n, j_{m,n} \mid m \leq n < \omega\rangle$ be the models and embeddings of $\mathcal{T}$. Let K_n denote K^{M_n}. Observe that $j_{m,n}{\restriction}K_m$ is an elementary embedding of K_m into K_n.

Let $\mathcal{P} = K^M$ and let $\mathcal{U}$ be an iteration tree on K^M. We use $\langle \mathcal{P}_\xi, k_{\zeta,\xi} \mid \zeta \leq \xi < \mathrm{lh}(\mathcal{U})\rangle$ to denote the models and embeddings of $\mathcal{U}$.

A **frame** for $\mathcal{T}$ is an iteration tree $\mathcal{U}$ on K^M together with ordinals $\langle \alpha_n \mid n < \omega\rangle$ and embeddings $\langle \pi_n \mid n < \omega\rangle$ so that:

1. $\alpha_n < \mathrm{lh}(\mathcal{U})$ for each $n < \omega$ and π_n is an elementary embedding of K_n into $\mathcal{P}_{\alpha_n}$;
2. $m <_{\mathcal{T}} n$ iff $\alpha_m <_{\mathcal{U}} \alpha_n$ for all $m < n < \omega$; and
3. The following diagram commutes:

$$\begin{tikzcd} \mathcal{P}_{\alpha_m} \arrow[r, "k_{\alpha_m,\alpha_n}"'] & \mathcal{P}_{\alpha_n} \\ K_m \arrow[u, "\pi_m"'] \arrow[r, "j{\restriction}K_m"'] & K_n \arrow[u, "\pi_n"'] \end{tikzcd}$$

A **solid frame** consists of the above objects together with a sequence $\langle \rho_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle$ so that:

4. $k_{\zeta,\xi}(\rho_\zeta) > \rho_\xi$ strictly whenever $\zeta <_{\mathcal{U}} \xi$, $(\zeta,\xi) \cap \{\alpha_n \mid n < \omega\} \neq \emptyset$, and $(\zeta,\xi)_{\mathcal{U}} \cap \{\alpha_n \mid n < \omega\} = \emptyset$.

The sequence $\langle \rho_\xi \mid \xi < \mathrm{lh}(\mathcal{U})\rangle$ witnesses that $\mathcal{U}$ is continuously illfounded off $\{\alpha_n \mid n < \omega\}$. A wellfounded cofinal branch through a solid frame must contain infinitely many of the α_n-s. Such a branch therefore induces a branch through $\mathcal{T}$. It is easy to see, using condition (3), that the induced branch through $\mathcal{T}$ leads to a wellfounded direct limit.

Problem 8.* *Is there a (definable over M) continuous map which associates to each length ω iteration tree on M a solid frame?*

Continuity is interpreted in the natural way: knowledge of $\mathcal{T}{\restriction}n+1$ should suffice to determine $\langle \alpha_0, \ldots, \alpha_n\rangle$, $\mathcal{U}{\restriction}\alpha_n + 1$, $\langle \pi_0, \ldots, \pi_n\rangle$, and $\langle \rho_\xi \mid \xi \leq \alpha_n\rangle$.

In the case that $\mathcal{T}$ is a linear iteration, the answer to Problem 8* is yes. $\mathcal{U}$ in this case breaks into a linear composition of iteration trees. The methods used to obtain $\mathcal{U}$ in this case trace directly to [St96].

A positive answer to Problem 8*, either as stated or with some variant of condition (4), should help settle Problem 9 on this list. (There are other scenarios for tackling that problem, more descriptive set theoretic in nature.)

At the moment the best approximation to a positive answer is given by the results of [Sch?].

References

[Sch?] Schindler, R., *Iterations of the core model*, submitted.
[St96] Steel, J., *The core model iterability problem*, LNL #8, 1996.