Logic IV

No. 1

Plan:

① General theory of forcing.

② Product forcing.

③ General theory of iterated forcing.

④ c.c.c. forcing.

finite support ~~product~~ iterations.

Martin's Axiom.

supercompact cardinal

⑤ proper forcing,

countable support iterations.

PFA

⑥ Semiproper forcing.

Revised countable support iterations,

~~Martin's Maximum~~

(SPFA)

⑦ Reflection Principle, MM, MM⁺⁺

⑧ Forcing axioms.

Applications of

No. 2

⑨ $MM^{++} \Rightarrow (*)$ (?)

⑩ Theory of subproper forcing + associated forcing axiom

⑪ Theory of subcomple forcing + ——〃——

⑫ Further application's + open questions.

Literature:

K. Kunen Set theory
T. Jech ——〃——
R. Schindler ——〃——
} pure set theory

(+ Large cardinals + forcing axioms.)

Ronald. B. Jensen : hand written notes
(subproper, subcomplete)

① General theory of forcing :

Recall

- $\mathbb{P} = (\mathbb{P}; \leq_{\mathbb{P}}) \left(= (\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}) \right)$
  $= (\mathbb{P}, \leq)$ is a p.o.
  (poset, partial order)

iff ① $\leq$ is reflexive ($p \leq p$ for all $p \in \mathbb{P}$)

② $\leq$ is symmetric ($p \leq q \wedge q \leq p \Rightarrow p = q$)

③ $\leq$ is transitive ($p \leq q \wedge q \leq r \Rightarrow p \leq r$)

(④ $\mathbb{1}_{\mathbb{P}}$ is the largest element w.r.t. $\leq$)

- $p \leq q$: p is stronger than q

- Elements of $\mathbb{P}$ are called forcing conditions

e.g. Cohen forcing : $\mathbb{C} :=$ the set of all finite sequences of natural numbers

$= \omega^{<\omega} = {}^{<\omega}\omega$

$\left(:= \bigcup_{n \in \omega} {}^{n}\omega \right)$

order : $p \leq_{\mathbb{C}} q :\Longleftrightarrow p$ end-extends q ($p \supset q$)

def Let $\mathbb{P} = (\mathbb{P}; \leq)$ be a poset.

$D \subseteq \mathbb{P}$ is called dense (or dense in $\mathbb{P}$)

iff $\forall p \in \mathbb{P} \ \exists q \in D \ (q \leq p)$

e.g. Let $\mathbb{P} = \mathbb{C}$.

(a) Let $n \in \omega$.

$$D_n = \{ p \in \mathbb{C} \mid n \in \mathrm{dom}(p) \}$$ is dense

(∵ just extend the given condition)

(b) Let $x \in {}^{\omega}\omega$ (so $x : \omega \longrightarrow \omega$ is a function)

$$D^{x} = \{ p \in \mathbb{C} \mid \exists n \in \mathrm{dom}(p) \ (p(n) \neq x(n)) \}$$

is dense (∵ Easy)

def Let $\mathbb{P} = (\mathbb{P}, \leq)$ be a poset.

$g \subseteq \mathbb{P}$ is called a filter

$:\Longleftrightarrow$ (a) if $p \in g \wedge q \in \mathbb{P} \wedge p \leq q$, then $q \in g$, and

(b) if $p, q \in g$, then there is some $r \in g$ s.t.

$$r \leq p, q$$

e.g. $\mathbb{P} = \mathbb{C}$:

How do filters look like in case of $\mathbb{C}$?

((a) says "information" in g are closed under "sub information"

(b) says the consistency of "information" in g)

if $g \subseteq \mathbb{C}$ is a filter, then either

① $\exists p \in \mathbb{C}$ $g = \{ p \restriction n \mid n \leq \mathrm{dom}(p) \}$, or

② $\exists x \in {}^{\omega}\omega$ $g = \{ x \restriction n \mid n \in \omega \}$

def $\mathbb{P}$ poset. Let $g \subseteq \mathbb{P}$ be a filter.

Let $\mathcal{D}$ be a collection of dense subsets of $\mathbb{P}$.

g is called *$\mathcal{D}$-generic*

$\Longleftrightarrow$ $\underset{\neq \emptyset}{g \cap D}$ for all $D \in \mathcal{D}$

lemma Let $\mathbb{P}$ be a poset. Let $\mathcal{D}$ be a <u>countable</u> collection of dense subsets of $\mathbb{P}$

Then there is a $\mathcal{D}$-generic filter.

(proof)

Let $\mathcal{D} = \langle D_n \mid n \in \omega \rangle$

Construct $\langle p_n \mid n \in \omega \rangle$ inductively as follows :

$$\begin{array}{l}
p_0 \in D_0 \\
\ \ \mathbin{\rotatebox[origin=c]{90}{$\leq$}} \\
p_1 \in D_1 \\
\ \ \mathbin{\rotatebox[origin=c]{90}{$\leq$}} \\
p_2 \in D_3 \\
\quad \vdots
\end{array}$$

- $p_0 \geq p_1 \geq p_2 \geq \cdots$
- $p_n \in D_n$

(just use the density)

Note By construction, we may show $\forall p \in \mathbb{P}\ \exists g$: $\mathcal{D}$-generic s.t. $p \in g$

Let g be the filter generated by $\{p_n \mid n \in \omega\}$.

Then g is a $\mathcal{D}$-generic filter.

(We have used DC.) ■

↑

$g = \{p \in \mathbb{P} \mid \exists n \in \omega\ p_n \leq p$

e.g.

$\mathbb{P} = \mathbb{C}$

$D_n = \{ p \in \mathbb{P} \mid n \in \mathrm{dom}(p) \}$ for $n < \omega$

$D^x = \{ p \in \mathbb{C} \mid \exists n \in \mathrm{dom}(p) \ \ p(n) \neq x(n) \}$ for $x \in {}^\omega\omega$

if $g \subseteq \mathbb{C}$ is $\{ D_n \mid n \in \omega \}$-generic,

then $x_g := \bigcup g$ is a function from ω to ω.

and there must be some $x \in {}^\omega\omega$ s.t. $x = x_g$.

$(g = \{ x \restriction n \mid n \in \omega \})$

⇝ Hence. let $X \subseteq {}^\omega\omega$ be countable.

let $\mathcal{D} = \{ D_n \mid n \in \omega \} \cup \{ D^x \mid x \in X \}$:

Then $\mathcal{D}$ is ctble.

Let $g \subseteq \mathbb{C}$ be $\mathcal{D}$-generic.

(ctble := countable
unctble := uncountable)

Then $x_g = \bigcup g \notin X$ (since $g \cap D^x \neq \emptyset$ for all $x \in X$)

This shows there are unctbly many reals.

(More precisely: since $g \cap D^x \neq \emptyset$, let $p \in g \cap D^x$

Note that $p \sqsubseteq \chi_g$ (χ_g end-extends p)

Since $p \neq x \restriction \mathrm{dom}(p)$, $\chi_g \neq x$)

ZFC := the standard axiomatization of set theory.

= Zermelo-Frankel with the $\underbrace{\text{Axiom of Choice}}_{(\text{AC})}$

From now on, we will frequently talk about models of ZFC

$(M; E)$

$\quad\ \ \cap|$

M^2 : interpretation of $\dot{\in}$

In particular, we will frequenty talk about <u>transitive</u> models of ZFC, i.e., $(M; E)$ models of ZFC where

M is transitive $(\forall x \in M\ \forall y \in x\ \ y \in M)$

and

$E = \in \restriction (M \times M)$

- ZFC doesn't prove the existence of a model of ZFC.

- ZFC + "there is a model of ZFC"

  doesn't prove the existence of a transitive model of ZFC.

lemma (special case of downward Löwenheim-Skolem)

(a) if there is a model $(M;E)$ of ZFC,
then there is such a model where M is ctble.

(b) if there is a transitive model $(M;E)$ of ZFC,
then there is such a transitive model where
M is ctble. //

For the development of the theory of forcing,
we will assume that there is a (ctble) transitive model
of ZFC.

def Let $\mathbb{P} \in M$, where M is a transitive model of ZFC and $\mathbb{P}$ is a poset.

(note that $\mathbb{P}$ is a poset $\Leftrightarrow$ $M \models \mathbb{P}$ is a poset)

Let $\mathcal{G} \subseteq \mathbb{P}$ be a filter.

We say $\mathcal{G}$ is <u>M-generic</u> (or $\mathbb{P}$-generic, generic over M)

$:\Leftrightarrow$ $\mathcal{G}$ is $\mathcal{D}$-generic, where

$$\mathcal{D} = \{ D \in M \mid D \text{ is dense in } \mathbb{P} \}$$

(Note that for $D \in M$,

D is dense in $\mathbb{P}$

$\Leftrightarrow$ $M \models D$ is dense in $\mathbb{P}$)

lemma In this situation, If M is ctble, then there is a M-generic filter.

To do

$$\mathbb{P} \in M \xrightarrow{\text{Add } g} \underset{\text{Want to define}}{\underline{M[g]}}$$

Today's goal : Fix a poset $\mathbb{P} \in M$

Fix a ctm M of ZFC

ctm $\equiv$ ctble transitive model

Let $g \subseteq \mathbb{P}$ be M-generic,

i.e. $g \cap D \neq \phi$ for all $D \subseteq \mathbb{P}$: dense in $\mathbb{P}$ with $\underline{\underline{D \in M}}$

Define

the "forcing extension" $M[g]$ of M,

and show $M[g] \models \mathrm{ZFC}$

def

Let $\mathbb{P}$ be a poset.

$p \in \mathbb{P}$ is an <u>atom</u> $:\Longleftrightarrow$ for all $q, r \leq p$, then there is some $s \leq q, r$.

$\mathbb{P}$ is called <u>atomless</u> $:\Longleftrightarrow$ $\mathbb{P}$ doesn't have atoms

e.g. $\mathbb{C}$: Cohen forcing.

$\mathbb{C}$ is atomless, since for any $p \in \mathbb{C}$,

$p^\frown\langle(\mathrm{lh}(p), 0)\rangle$, $p^\frown\langle(\mathrm{lh}(p), 1)\rangle$

are <u>incompatible</u>.

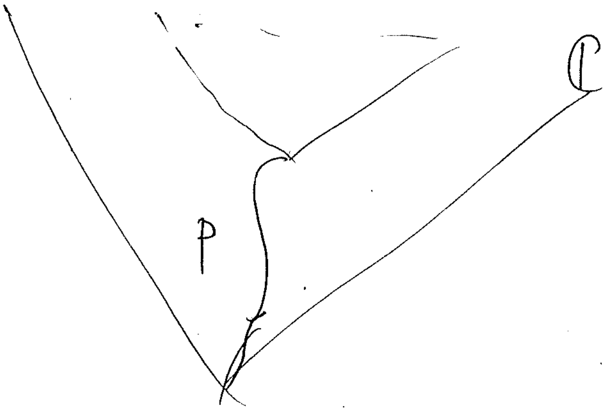
$(\mathrm{lh}(p) = \mathrm{dom}(p))$ (It means there is no common extension)

lemma Let M be a transitive model of ZFC,

Let $\mathbb{P} \in M$ be atomless.

If g is M-generic,

then $g \notin M$



(proof) Suppose not. Assume $g \in M$.

Then $\mathbb{P} \setminus g \in M$, since $M \models \mathrm{ZFC}$.

But $\mathbb{P} \setminus g$ is also dense, since $\mathbb{P}$ is atomless.

∵ Given $p \in \mathbb{P}$. Since $\mathbb{P}$ is atomless, let $q, r \leq p$ be incompatible condition. Then either $q \notin g$ or $r \notin g$ (g: filter)
i.e. either $q \in \mathbb{P} \setminus g$ or $r \in \mathbb{P} \setminus g$

Since g was generic, $(\mathbb{P} \setminus g) \cap g \neq \phi$

This is a contradiction !! $\blacksquare$

We now fix $\underset{\sim}{M}$, $\overset{\in M}{\mathbb{P}}$, and $g \subseteq \mathbb{P}$ M-generic.
((ctble) trans model of ZFC)

We aim to define $M[g]$:

def For $x \in M$, let

$$x^g = \{ y^g \mid \exists p \in g \ (p, y) \in x \},$$ called the g-interpretation of x.

(define x^g by the transfinite induction on $\in$-relation)

Remark

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- the relation $y \mathrel{E} x :\Longleftrightarrow \exists p\ (p, y) \in x$ is well-founded relation.

~~e.g.~~

~~• $\emptyset^g = \emptyset$~~

~~• Let $p, q \in \mathbb{P}$ be incompatible.~~

~~Let $x =$~~

def

$$M[g] := \{ x^g \mid x \in M \}$$

$M[g]$ is called the $\mathbb{P}$-generic extension of M via g

Lemma $M[g]$ is transitive

(proof) Clear.

Let $y^g \in x^g \in M[g]$.

Since $y \in x \in M$, $y \in M$,

Hence $y^g \in M[g]$. //

■

<u>lemma</u> $M \subseteq M[g]$

(proof)

For $x \in M$, recursively define

$$\check{x} = \{(\underset{\curvearrowleft}{\check{y}}, p) \mid y \in x, p \in \mathbb{P}\}. \quad (\in M, \text{ since } M \models ZFC)$$

(again, this is well-defined by the transfinite induction)

And by def, $\check{x}^g = x$. ■

We call $\check{x}$ the <u>check name</u> for x

$$\left(\check{x}^g = \{\check{y}^g \mid y \in x\} \underset{\substack{\uparrow \\ \text{by induction}}}{=} \{y \mid y \in x\} = x \right)$$

<u>lemma</u> $g \in M[g]$

(proof)

$$\dot{g} := \{(p, \check{p}) \mid p \in \mathbb{P}\}$$: the canonical name for M-generic

Then $\dot{g}^g = g$ ■

We have

$\rightsquigarrow \quad M \cup \{g\} \subseteq M[g].$

Lemma

the class of ordinals.

$$M[g] \cap ON = M \cap ON,$$

i.e. M and $M[g]$ have the same ordinals.

(proof) (Note that $M \models \alpha$ is ordinal $\Leftrightarrow M[g] \models \alpha$ is ordinal.)
for $\alpha \in M, M[g]$

$M \cap ON \subseteq M[g] \cap ON$: Clear.

$M[g] \cap ON \subseteq M \cap ON$: Let $x^g \in ON$, where $x \in M$.

We want to see $x^g \in M$.

For that, it suffices to prove inductively

$$rk_\in(x^g) \leq rk_\in(x)$$

for all $x \in M$,

$rk_\in(z) = \sup\{rk_\in(u)+1 \mid u \in z\}$

$=$ the least α s.t. $z \subseteq V_\alpha$

if $x^g \in ON$, $x^g = rk_\in(x^g) \leq rk_\in(x) \boxed{\in M \cap ON}$

And this is done by induction.

$$rk_\in(x^g) = \sup\{rk_\in(y^g)+1 \mid (p,y) \in x, \ p \in g\}$$

$$\leq \sup\{rk_\in(y)+1 \mid (p,y) \in x, \ p \in g\} \leq rk_\in(x). \quad //$$

With M, $\mathbb{P} \in M$, $g \subseteq \mathbb{P}$ still as before,

we now want to show $\underline{\underline{M[g] \models ZFC}}$.

<u>thm</u> $M[g] \models ZFC$

In order to prove this,

<u>def</u> A formula φ in the language of set theory is $\underline{\Sigma_0}$ iff all quantifiers are bounded.

$$\begin{pmatrix} \exists x \in y\ \psi \\ \forall x \in y\ \psi \end{pmatrix}$$

- if $\varphi, \psi : \Sigma_0$,

  then $\neg\varphi$, $\varphi \vee \psi$, $\varphi \wedge \psi$, $\exists x \in y\ \varphi$
  $\forall x \in y\ \varphi$ are Σ_0.

- $x = y$, $x \in y$ are Σ_0

A formula φ is $\underline{\Sigma_1}$

$:\Longleftrightarrow$ φ is of the form $\exists x_1 \cdots \exists x_k \bar{\varphi}$,

where $\bar{\varphi}$ is Σ_0

A formula φ is Π_1

$:\Longleftrightarrow$ φ is of the form $\forall x_1 \cdots \forall x_k \bar{\varphi}$,

where $\bar{\varphi}$ is Σ_0.

<u>e.g.</u>

<u>Extensionality</u>:

$$\forall x \forall y \left(\forall z \in x (z \in y) \wedge \forall z \in y (z \in x) \to x = y \right)$$

is Π_1

<u>Foundation</u>

$$\forall x \left(\underbrace{x \neq \phi}_{(\exists y \in x)(y=y)} \longrightarrow \exists \overset{y}{\cancel{\ }} \in x \, \neg \exists z \in y (z \in x) \right)$$

: Π_1

Pairing — $z=\{x,y\} \swarrow \Sigma_0$

$\iff x\in z \wedge y\in z \wedge \forall_{u\in z}(u=x \vee u=y)$

— $\forall_x \forall_y \exists_z \left(z=\{x,y\} \right) : \Pi_2$

Union

— $y=\bigcup x$

$\iff \forall_{z\in y} \exists_{u\in x} (z\in u) \wedge$

$\forall_{u\in x} \forall_{z\in u} (z\in y) \quad : \Sigma_0$

— $\forall_x \exists_y \left(y=\bigcup x \right) : \Pi_2$

being transitive x is transitive $\iff \forall_{y\in x}\forall_{z\in y}(z\in x) : \Sigma_0$

Being ordinal :

x is ordinal

$\iff x$ is linearly ordered by $\in$ and trans.

$\iff \forall_{y\in x}\forall_{z\in x}\left(y\in z \vee z\in y \vee y=z \right)$

$\rceil$ + x is transitive: Σ_0

Cor

$M[g] \models$ extensionality , foundation , ~~pairing~~ ,
infinity , ~~union~~ .

(proof)

- extensionality , Foundation : Π_1 ✓ok
- Infinity : $M \models$ Infinity and

$\textcircled{\Sigma_1}$ $\underset{\underset{\Sigma_1}{\curvearrowright}}{M \subseteq M[g]}$ ✓ok

~~∘ Pairing :~~ ■

lemma $M[g] \models$ Pairing , Union .

(proof)

- Pairing : let $x^g , y^g \in M[g]$.

Let $Z = \{(p, x) ; p \in \mathbb{P}\} \cup \{(p, y) : p \in \mathbb{P}\}$

Then $Z \in M$, since $M \models ZFC$.

Then $Z^g = \{x^g, y^g\} \in M[g]$.

Since $Z = \{x, y\}$ is Σ_0 , ~~hence~~ hence $M[g] \models Z^g = \{x^g, y^g\}$.

Infinity

$$\exists x\,\big(\phi\in x\wedge\forall y\in x\,(y\cup\{y\}\in x)\big)$$

$$\Leftrightarrow\ \exists x\,\big(\exists y\in x\,\underbrace{\forall z\in y\,(z\neq z)}_{y=\phi}\wedge\forall y\in x\,\underbrace{(y\cup\{y\}\in x)}_{\Sigma_0}\big)$$

$: \Sigma_1$

$$\forall y\in x\,\exists z\in x\,\left(\begin{array}{l}\forall u\in y\,(u\in z)\wedge y\in z\wedge\\ \forall u\in z\,(u=y\vee u\in y)\end{array}\right)$$

$: \Sigma_0$

lemma (absoluteness)

Let $M\subseteq N$ both be transitive, and let φ be a formula, and let $x_1\cdots x_k\in M$.

Then

① if φ is Σ_0, then $(M\models\varphi[x_1\cdots x_k]\Leftrightarrow N\models\varphi[x_1\cdots x_k])$

② if φ is Σ_1, then $(M\models\varphi[x_1\cdots x_k]\Rightarrow N\models\varphi[x_1\cdots x_k])$

③ if φ is Π_1, then $(N\models\varphi[x_1\cdots x_k]\Rightarrow M\models\varphi[x_1\cdots x_k])$

∘ Union : let $x^g \in M[g]$.

Let $z = \bigcup \sim$ (Same)

$\blacksquare$

Proved

- $M[g]$ is transitive
- $M \cup \{g\} \subseteq M[g]$
- $M[g] \cap ON = M \cap ON$
- $M[g] \models$ Extensionality, Foundation, Infinity. Pairing. Union.