Transfer theorems

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What is a transfer theorem?

Here's the ultimate prototype:

Theorem (Harrington, Martin; 1970s).
$\mathbf{\Pi}^1_1$ determinacy $\Rightarrow$ ($< \omega^2$- $\mathbf{\Pi}^1_1$) determinacy.

$\mathbf{\Pi}^1_1 =$ all coanalytic subsets of $\mathbb{R}$, i.e., all complements of projections of Borel subsets of the plane.

($< \omega^2$- $\mathbf{\Pi}^1_1$) $=$ all sets in early levels of the Hausdorff difference hierarchy over $\mathbf{\Pi}^1_1$.

There are more sets in ($< \omega^2$- $\mathbf{\Pi}^1_1$) than there are in $\mathbf{\Pi}^1_1$.

We construe $\mathbb{R} = {}^{\omega}\omega$.

Let $A \subset \mathbb{R}$. Consider $\mathcal{G}(A)$:

$$\begin{array}{c|cccccc} \mathrm{I} & n_0 & & n_2 & & \dots & \\ \hline \mathrm{II} & & n_1 & & n_3 & & \dots \end{array}$$

I wins if $(n_0, n_1, n_2, n_3, ...) \in A$, II wins if $(n_0, n_1, n_2, n_3, ...) \in \mathbb{R} \setminus A$.
I wins $\mathcal{G}(A)$ if I has a winning stategy. Same for II.

Let Γ be a pointclass over $\mathbb{R}$, i.e., a collection of sets of reals.
We say that Γ determinacy holds iff
for all $A \in \Gamma$, either I wins $\mathcal{G}(A)$ or else II wins $\mathcal{G}(A)$.

The Harrington-Martin theorem,
$\mathbf{\Pi}^1_1$ determinacy $\Rightarrow$ ($< \omega^2$- $\mathbf{\Pi}^1_1$) determinacy,
thus says that a given determinacy hypothesis in fact proves an apparently stronger one.

A result of the form

$$\Gamma \text{ determinacy } \Rightarrow \Gamma^* \text{ determinacy},$$

where Γ, Γ^* are pointclasses over $\mathbb{R}$ with $\Gamma \subsetneq \Gamma^*$, is traditionally called a transfer theorem.

However, we shall call the Harrington-Martin theorem a transfer theorem due to the *method by which it is proven*. This method is one typical example for how to prove a result of the above form.

How to prove (the lightface version of) the Harrington-Martin theorem:

Theorem (Harrington).
Π^1_1 determinacy $\Rightarrow 0^\#$ exists.

Theorem (Martin).
$0^\#$ exists $\Rightarrow$ ($< \omega^2$-Π^1_1) determinacy.

Let $L =$ Gödel's constructible universe.
"$0^\#$ exists" says that there is an elementary embedding $\pi: L \to L$ which is not the identity.

The existence of $0^\#$ implies the consistency of ZFC+ "there are weakly compact cardinals" (and much more). "$0^\#$ exists" is thus a large cardinal axiom.

Point is: We don't know another, "direct," proof of the Harrington-Martin theorem.
That is, the only proof known goes through the study of L.

An inner model is a transitive proper class sized model of ZF.

Inner model theory studies inner models.
It constructs inner models which may contain large cardinals, it analyzes these models in great detail, and it uses them to - for instance - prove transfer theorems.

Examples of inner models: L, $L(\mathbb{R})$, HOD, the core model K.

Let's take a brief look at the proof of Harrington's result,
Π^1_1 determinacy $\Rightarrow 0^\#$ exists.

We first define a certain analytic set, S, which is Turing-closed and cofinal in the Turing degrees.
Now assume Π^1_1 determinacy to hold. S then contains a Turing-cone of reals.
Let $R \in \mathbb{R}$ be a base of that cone.
One verifies that every R-admissible ordinal is a cardinal from the point of view of L.

The fact that every R-admissible ordinal is an L-cardinal will give the existence of $0^\#$ as follows:

Work in $L[R]$. Let

$$\pi\colon L_\alpha[R] \to L_{\omega_3}[R]$$

be such that $\mathrm{Card}(\alpha) = \aleph_1$ and ${}^\omega L_\alpha[R] \subset L_\alpha[R]$.
Let δ be the critical point of π.
Derive an L_α-ultrafilter U on δ from π by

$$X \in U \Leftrightarrow \delta \in \pi(X),\ X \subset \delta,\ X \in L_\alpha.$$

As α is R-admissible, thus an L-cardinal, U is in fact an L-ultrafilter.
We may now take the ultrapower of L by U, which will be well-founded as ${}^\omega L_\alpha[R] \subset L_\alpha[R]$.
This gives $0^\#$.

Let $\nu = \delta^{+L}$. The amenable structure

$$(L_\nu; \in, U)$$

is a **mouse**. The existence of such a mouse is equivalent to the existence of $0^\#$.

So what is a transfer theorem?

A **transfer theorem** (as we'll take it now) is a result of the form

$$\Phi \Rightarrow \Phi^*$$

such that neither Φ nor Φ^* is allowed to mention any concepts of inner model theory, but which is proven by showing $\Phi \Rightarrow \Psi$ as well as $\Psi \Rightarrow \Phi^*$ for some interpolant Ψ which is a statement of inner model theory.

That is, a transfer theorem is a true implication outside of inner model theory which is *shown by going through inner model theory*. We understand that there is no "direct" proof known for $\Phi \Rightarrow \Phi^*$.

Another transfer theorem:

Theorem (Neeman, Woodin; 1990s).
Let $n \in \mathbb{N}$.
$\mathbf{\Pi}^1_{n+1}$ determinacy $\Rightarrow \Game^{(n)}(<\omega^2\text{-}\ \mathbf{\Pi}^1_1)$ determinacy.

For $n = 0$, this is the Harrington-Martin theorem.

$\mathbf{\Pi}^1_n =$ all subsets of $\mathbb{R}$ obtained by taking projections and complements n times, starting from Borel subsets of $\mathbb{R}^{n+1}$.

For a pointclass Γ, a set of reals is in $\Game^{(n)}(\Gamma)$ if it can be defined from a set in Γ using n successive applications of $\Game$.
$(\Game y)A(x,y) \Leftrightarrow$ I wins $\mathcal{G}(\{y|(x,y) \in A\})$.

There are more sets in $\Game^{(n)}(<\omega^2\text{-}\ \mathbf{\Pi}^1_1)$ than there are in $\mathbf{\Pi}^1_{n+1}$.

How to prove the Neeman-Woodin theorem:

Theorem (Woodin).
$\mathbf{\Pi}^1_{n+1}$ determinacy $\Rightarrow$ there are certain iterable inner models with n Woodin cardinals.

Theorem (Neeman).
If $\mathbf{\Pi}^1_1$ determinacy holds then the conclusion of Woodin's theorem yields $\Game^{(n)}(<\omega^2\text{-}\ \mathbf{\Pi}^1_1)$ determinacy.

We shall only need the following corollary:

Theorem (Martin, Steel, Woodin; 1980s).
The following are equivalent.
(1) For all $n \in \mathbb{N}$, $\mathbf{\Pi}^1_n$ determinacy.
(2) For all $n \in \mathbb{N}$, for all $x \in \mathbb{R}$, $M_n^{\#}(x)$ exists.

Statement (1) is abbreviated by PD ("projective determinacy").

For $n \in \mathbb{N}$ and a set x, $M_n^\#(x)$ is a **mouse**.

Recall the mouse $(L_\nu; \in, U)$ we've seen above. Nowadays, this mouse is considered as *being* $0^\#$. (Or rather, an iterate thereof.)
$M_0^\#(\emptyset) = 0^\#$ in this sense.

In general, $M_n^\#(x) = (L_\gamma[E, x]; \in, x, E, U)$, where

- U is an $L[E, x]$-ultrafilter,
- E is a sequence of "extenders" (= a system of ultrafilters) witnessing that $L_\gamma[E, x] \models$ "there are n Woodin cardinals,"
- $M_n^\#(x)$ is iterable.

Iterability is one of the main concerns of inner model theory; here, we can't discuss it, though.

"$M_n^{\#}(x)$ exists" is a large cardinal axiom which, for $n > 0$, is much stronger than "$0^{\#}$ exists."

For the record:
A cardinal κ is **Woodin** iff for all $A \subset \kappa$ there are arbitrarily large $\delta < \kappa$ such that for all $\alpha < \kappa$ there is some elementary embedding $\pi: V \to M$ with M transitive and with critical point δ such that $V_\alpha \subset M$ and $A \cap \alpha = \pi(A) \cap \alpha$.

By the Martin-Steel-Woodin theorem, PD can be recast as the statement that certain mice with Woodin cardinals exist. To show PD (and therefore its consequences) it thus suffices to prove the existence of these mice.

We shall now be interested in transfer theorems of the form

$$\Phi \Rightarrow \text{PD}.$$

By the Martin-Steel-Woodin theorem, our interpolant will be "for all $n \in \mathbb{N}$, for all $x \in \mathbb{R}$, $M_n^{\#}(x)$ exists."
That is, using a hypothesis Φ, we'll have to prove that for all $n \in \mathbb{N}$ the **mouse operator**

$$x \mapsto M_n^{\#}(x)$$

is *total*.
There is one key method for how to achieve this: by induction on n.

This method is Woodin's **core model induction**.*

*In fact there is a more complicated extension of this method, also due to Woodin. It is this extension which is the real core model induction. What we call "core model induction" here provides only the first ω many steps of that real one.

Why is PD a nice consequence to have in a transfer theorem?

(1) PD itself has nice consequences: all projective sets of reals are Lebesgue measurable, have the property of Baire, Projective Uniformization holds, etc.

(2) Determinacy hypotheses are canonical measures for the "consistency strength" of set-theoretical assertions:
Every natural set-theoretical assumption which is known to imply a given determinacy hypothesis consistency-wise also implies it outrightly. Moreover, the natural pointclasses Γ form a natural hierarchy.

The transfer theorems $\Phi \Rightarrow$ PD we shall be interested in will be such that Φ is no determinacy assumption.
We'll verify the remarkable empirical fact that PD is implied by an amazing variety of hypotheses. We take this as supporting the view that PD is actually *true*.

Theorem (Woodin; 1980s). Suppose that there is an ω_1-dense ideal on ω_1. Then PD holds.

Woodin in fact developed his core model induction in order to prove a strengthening of this result.
He shows that the existence of an ω_1-dense ideal on ω_1 implies $AD^{L(\mathbb{R})}$ (and is equiconsistent with it).

Other transfer theorems:

Theorem (Martin, Mitchell, Schimmerling, Steel, Todorcevic, Woodin, Zeman; 1990s).
PFA (i.e., the Proper Forcing Axiom) $\Rightarrow$ PD.
In fact, suppose that κ is a singular countably closed cardinal such that $\Box_\kappa$ fails. Then PD holds.

Theorem (Sch; 1997).
Suppose that every uncountable cardinal is singular. Then PD holds.

Theorem (Foreman, Magidor, Sch; 1997).
Suppose that $\aleph_\omega$ is a strong limit cardinal and $\aleph_n$ has the tree property for all $2 \leq n < \omega$. Then PD holds.

Let us now turn towards a transfer theorem we actually want to sketch the proof of.
In it, the hypothesis is a statement of cardinal arithmetic.

Cardinal arithmetic studies the possible values of κ^λ, or rather, possible patterns of values of κ^λ.
Example:

Theorem (Easton; end of 1960s).
Restricted to regular κ's, $\kappa \mapsto 2^\kappa$ can be anything which is monotone and obeys $\mathrm{cf}(2^\kappa) > \kappa$.

However, SCH = the Singular Cardinal Hypothesis holds in Easton's models.

SCH says that $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+ \cdot 2^{\mathrm{cf}(\kappa)}$ for all cardinals κ.

In particular, if SCH holds and κ is a singular strong limit cardinal then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. For instance, GCH = the Generalized Continuum Hypothesis cannot fail for the first time at $\aleph_\omega$, provided that SCH holds true.

Magidor, in the 1970s, was the first one to produce a model of set theory in which SCH fails, in fact in which GCH fails for the first time at $\aleph_\omega$.

We know that the failure of SCH implies Π^1_1 determinacy:

Theorem (Jensen; 1970s).
Suppose that $0^{\#}$ doesn't exist. Then for each set X of ordinals there is some $Y \in L$ with $Y \supset X$ and $\mathrm{Card}(Y) \leq \mathrm{Card}(X) \cdot \aleph_1$.

This is Jensen's Covering Lemma. It gives:

Corollary (Jensen).
If SCH fails then $0^{\#}$ exists.

Suppose that $0^{\#}$ doesn't exist.
To get the idea, look at $\aleph_\omega$. For each $X \subset \aleph_\omega$ of size $\aleph_0$ there is some $Y \in L$ of size $\aleph_1$ with $Y \supset X$, $Y \subset \aleph_\omega$. As GCH holds in L, there are $\leq \aleph_{\omega+1}$ many such Y. Each such Y has $\leq 2^{\aleph_0}$ many countable subsets.
So $\aleph_\omega^{\aleph_0} \leq \aleph_{\omega+1} \cdot 2^{\aleph_0}$, i.e. SCH holds at $\aleph_\omega$.

Gitik has determined the consistency strength of $\neg$SCH.

Theorem (Gitik; 1980s).
$\neg$SCH is equiconsistent with the existence of a cardinal δ with $o(\delta) = \delta^{++}$.

By Jensen and Martin, $\neg$SCH implies Π^1_1 determinacy.
By Gitik, $\neg$SCH does *not* imply Π^1_2 determinacy (and a fortiori *not* PD).

The hypothesis in the transfer theorem stated 3 slides from this one is the first statement in cardinal arithmetic known to imply PD. This statement will express a very strong violation of SCH.

Silver had shown that SCH cannot fail for the first time at a singular cardinal of *uncountable* cofinality.
For example:

Theorem (Silver; 1970s).
Let κ be a singular strong limit cardinal of uncountable cofinality. If

$$\{\alpha < \kappa | 2^\alpha = \alpha^+\}$$

is stationary then $2^\kappa = \kappa^+$.

It was this result in reaction to which Jensen proved his Covering Lemma.

What about $\{\alpha < \kappa | 2^\alpha > \alpha^+\}$ in the situation of Silver's theorem? It might be non-stationary (it might even be $\emptyset$). But *can it be stationary* (with its complement still being stationary)? This is an open question.

Lemma (Gitik).
Let κ be a singular strong limit cardinal of uncountable cofinality. Let $\{\alpha < \kappa | 2^\alpha = \alpha^+\}$ be stationary as well as co-stationary.
Then for each club $C \subset \kappa$ there is some limit point $\mu \in C$ such that

$$\mathrm{cf}(\prod(\{\alpha^+ | \alpha \in C \cap \mu\})) > \mu^+.$$

Here, $\prod(\mathsf{a})$ is the set of all choice functions from a which is supposed to be ordered by $f < g$ iff for all $\alpha \in \mathsf{a}$, $f(\alpha) < g(\alpha)$.
$\mathrm{cf}(\prod(\mathsf{a}))$ is the smallest size of a subset of $\prod(\mathsf{a})$ cofinal in it.

A recent transfer theorem:

Theorem (Gitik, Sch; 2002).
Let κ be a singular strong limit cardinal of uncountable cofinality. Let $\{\alpha < \kappa | 2^\alpha = \alpha^+\}$ be stationary as well as co-stationary. Then PD holds.

The proof will use Woodin's core model induction. It will exploit a covering argument for certain core models K (which is a generalization of the argument yielding Jensen's Covering Lemma) to provide a canonical set of functions from K witnessing $\mathrm{cf}(\prod(C \cap \mu)) = \mu^+$, where C and μ are as in Gitik's lemma. This will show covering fails for K which in turn allows us to perform the induction step.

Fix κ, a singular strong limit cardinal of uncountable cofinality.
Set $S = \{\alpha < \kappa | 2^{\alpha} = \alpha^{+}\}$. We suppose S to be stationary as well as co-stationary.

We show:

Claim. For each $n \in \mathbb{N}$ and for each $x \in H_{\kappa}$, $M_n^{\#}(x)$ exists.

This is shown by induction.

$n = 0$: Here we only use $\kappa \setminus S$ is stationary.
Let α be a singular strong limit cardinal which is a member of $\kappa \setminus S$. In particular, $2^{\alpha} > \alpha^{+}$. Thus SCH fails.
As α may be chosen arbitrarily large below κ, H_{κ} is closed under #'s (in much the same way as $\neg$SCH $\Rightarrow$ "$0^{\#}$ exists").

$n \mapsto n+1$: Here we use the conclusion of Gitik's lemma.

Illustrative case: $x = \emptyset$. I.e., we aim to prove $M_{n+1}^{\#} = M_{n+1}^{\#}(\emptyset)$ exists.
We assume $M_{n+1}^{\#} = M_{n+1}^{\#}(\emptyset)$ doesn't exist and derive a contradiction.

As H_κ is closed under the mouse operator $x \mapsto M_n^{\#}(x)$ whereas $M_{n+1}^{\#}$ doesn't exist, we may build the **core model** K of height κ.

K is a mouse of the form $(L_\kappa[E]; \in, E)$, where
- E is a *maximal* sequence of extenders, i.e., whenever an extender can be put onto the sequence it will be put onto the sequence,
- K is iterable.

The idea behind the maximality requirement is that we want to be able to prove a covering lemma for K.

Now fix a "nice" club C.
The choice of C actually depends on K. We have to choose C in such a way that we don't run into a dead end at some later point.

Let μ be a limit point of C such that

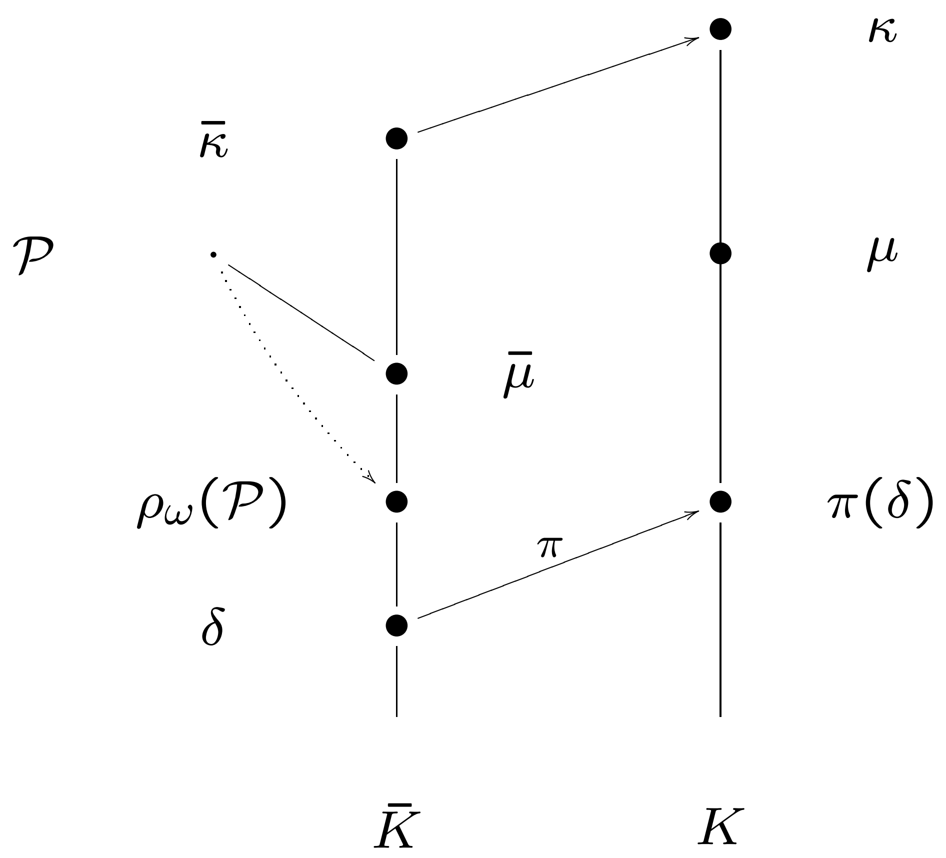
$$\mathrm{cf}(\prod(\{\alpha^+ | \alpha \in C \cap \mu\})) = \theta > \mu^+.$$

Such μ exists by Gitik's lemma.
Let $(f_i | i < \theta)$ witness $\mathrm{cf}(\prod(\{\alpha^+ | \alpha \in C \cap \mu\})) = \theta$.

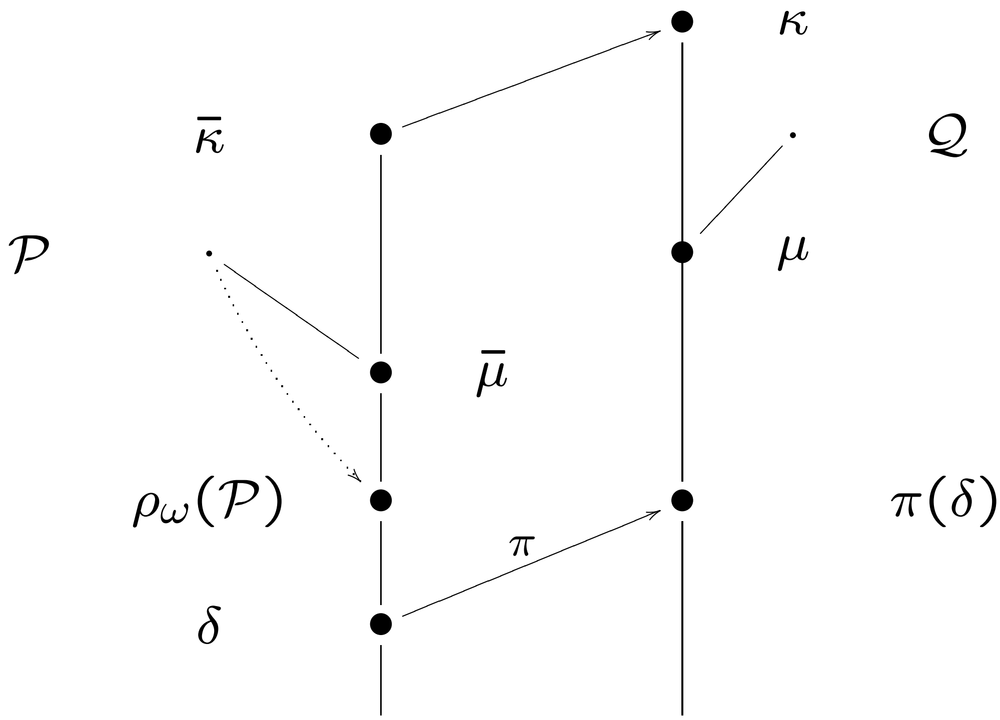
Idea: We aim to replace each individual f_i by some canonical $\tilde{f}_i$ with $f_i < \tilde{f}_i$. "Canonical" here means: provided by K.
As K satisfies GCH, there can be only $\leq \mu^+$ many such $\tilde{f}_i$. Thus $\mathrm{cf}(\prod(\{\alpha^+ | \alpha \in C \cap \mu\})) \leq \mu^+$ after all. Contradiction!

Fix $i < \theta$. Pick $\pi\colon \bar{H} \to H_\kappa$ such that $\bar{H}$ is transitive, ${}^\omega\bar{H} \subset \bar{H}$, $\mathrm{ran}(f_i) \subset \mathrm{ran}(\pi)$, and $\mathrm{Card}(\bar{H})$ is "small."
Set $\bar{K} = \pi^{-1}(K)$, $\bar{\mu} = \pi^{-1}(\mu)$.

We coiterate K with $\bar{K}$. This gives a mouse $\mathcal{P} \triangleright \bar{K}||\bar{\mu}$ with a small projectum.



We may "lift up" $\mathcal{P}$, using π, getting $\mathcal{Q} \rhd K||\mu$.



Set $g(\xi^{+K}) = \text{Hull}^{\mathcal{Q}}(\xi + 1) \cap \xi^{+} < \xi^{+}$. Construed as a function from μ to μ, this actually exists in K, as $\mathcal{Q} \lhd K$.
Set $\tilde{f}_i = g \restriction \{\alpha^{+} | \alpha \in C \cap \mu\}$. Then $\tilde{f}_i$ is as desired, if C was chosen accurately, as K computes successors of singular cardinals correctly.

Recall that we don't know if the hypothesis of this last transfer theorem is consistent.

Question 1. Let κ be a singular strong limit cardinal of uncountable cofinality.
Can $\{\alpha < \kappa | 2^\alpha = \alpha^+\}$ be stationary as well as co-stationary?

A result related to the last transfer theorem:

Theorem (Sch; 2002). Let α be a limit ordinal. Let $2^{|\alpha|^+} < \aleph_\alpha$, but $\aleph_\alpha^{|\alpha|} > \aleph_{|\alpha|^+}$. Suppose further that there is a measurable cardinal. Then Π^1_2 determinacy holds.

Question 2. Let α be a limit ordinal. Suppose that $2^{|\alpha|^+} < \aleph_\alpha$, but $\aleph_\alpha^{|\alpha|} > \aleph_{|\alpha|^+}$. Does PD hold? Does $0 = 1$ hold?

Refinements of the core model induction have shown various hypotheses to imply

- $AD^{L(\mathbb{R})}$,
- the existence of inner models with $\omega \cdot 2$ Woodin cardinals,
- the existence of inner models in which the "$AD_{\mathbb{R}}$ hypothesis" holds true,
- or even the existence of an inner model in which $AD^{+} + \theta_0 < \theta$ holds (which implies the existence of a non-tame mouse)

Practically everything in this area is due to Woodin.

Question 3. Is there an alternative to Woodin's core model induction which can also be used for deriving "significant"* consistency strengths from given hypotheses?

*i.e., being at the level of PD and higher up

An idea would be to use K^c, a "preliminary" version of the true core model K, rather than K itself in order to derive consistency strength. The advantage of K^c over K is that it "always exists." (Take this cum grano salis.)

Question 4. Let α be a singular strong limit cardinal. Does K^c compute α^+ correctly? I.e., is $\alpha^{+K^c} = \alpha^+$?

This is known to be true below a measurable limit of strong cardinals.
If the answer to this question turns out to be "yes," this should then lead to a positive answer to:

Question 5. Let κ be a singular strong limit cardinal of uncountable cofinality.
Let $\{\alpha < \kappa | 2^\alpha = \alpha^+\}$ be stationary as well as co-stationary.
Must there be a non-tame mouse?