A NEW CONDENSATION PRINCIPLE

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Abstract. We generalize $\nabla(A)$, which was introduced in [Sch∞], to larger cardinals. For a regular cardinal $\kappa > \aleph_0$ we denote by $\nabla_\kappa(A)$ the statement that $A \subseteq \kappa$ and for all regular $\theta > \kappa$,

$$\{X \in [\,\mathbf{L}_\theta[A]\,]^{<\kappa} \;:\; X \cap \kappa \in \kappa \wedge \mathrm{otp}(X \cap \mathrm{Ord}) \in \mathrm{Card}^{\mathbf{L}[A \cap X \cap \kappa]}\}$$

is stationary in $[\,\mathbf{L}_\theta[A]\,]^{<\kappa}$.

It was shown in [Sch∞] that $\nabla_{\aleph_1}(A)$ can hold in a set-generic extension of $\mathbf{L}$. We here prove that $\nabla_{\aleph_2}(A)$ can hold in a set-generic extension of $\mathbf{L}$ as well. In both cases we in fact get equiconsistency theorems. This strengthens results of [Rä00] and [Rä∞].

$\nabla_{\aleph_3}(\emptyset)$ is equivalent with the existence of $0^\#$.

1. Introduction.

The current paper is concerned with condensation properties of models of the form $\mathbf{L}[A]$ where A is a set of ordinals. If $\mathbf{V} = \mathbf{L}$ (or just if $0^\#$ does not exist) and if

$$\pi \,:\, \mathbf{L}_\alpha \to \mathbf{L}_\beta \tag{1}$$

is an elementary embedding then $\pi{\restriction}\mathrm{Card}^{\mathbf{V}}(\alpha) = \mathrm{id}$ (*cf.* [Je78, Lemma 32.12]; in fact, $\pi{\restriction}\mathrm{Card}^{\mathbf{L}}(\alpha) = \mathrm{id}$ unless $\alpha < \aleph_2$, *cf.* [Fr00, Theorem 3.13 (i)]); in particular, α cannot be a cardinal $\geq \aleph_2$ unless $\pi = \mathrm{id}$. On the other hand it is consistent that $0^\#$ does not exist and there is a non-trivial elementary embedding as in (1) with $\alpha \in \mathrm{Card}^{\mathbf{L}} \cap \aleph_2$ (and then $\mathbf{V} \neq \mathbf{L}$); this is the kind of situation that will be studied here.

Let $A \subseteq \omega_1$. In [Sch∞] the second author introduced the assertion, denoted by $\nabla(A)$, that

$$\{X \in [\,\mathbf{L}_{\omega_2}[A]\,]^\omega \colon \exists \alpha < \beta \in \mathrm{Card}^{\mathbf{L}[A\cap\alpha]}\ \exists \pi\ \pi\colon \mathbf{L}_\beta[A \cap \alpha] \cong X \prec \mathbf{L}_{\omega_2}[A]\}$$

be stationary in $[\,\mathbf{L}_{\omega_2}[A]\,]^\omega$. We shall consider generalizations of $\nabla(A)$ to larger cardinals.

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Definition 1. *Let κ and θ both be regular cardinals, $\aleph_0 < \kappa < \theta$. Then by $\nabla^\theta_\kappa(A)$ we denote the statement that $A \subseteq \kappa$ and*

$$\{X \in [\,\mathbf{L}_\theta[A]\,]^{<\kappa} \ : \ X \cap \kappa \in \kappa \wedge \mathrm{otp}(X \cap \mathrm{Ord}) \in \mathrm{Card}^{\mathbf{L}[A \cap X \cap \kappa]}\}$$

is stationary in $[\,\mathbf{L}_\theta[A]\,]^{<\kappa}$. By $\nabla_\kappa(A)$ we denote the statement that $\nabla^\theta_\kappa(A)$ holds for all regular $\theta > \kappa$.

Moreover, we write ∇^θ_κ for $\nabla^\theta_\kappa(\emptyset)$, and ∇_κ for $\nabla_\kappa(\emptyset)$.

It is clear that $\nabla(A)$ is $\nabla^{\aleph_2}_{\aleph_1}(A)$. The principles $\nabla^\theta_\kappa(A)$ come up naturally in several contexts.

Suppose that $0^\#$ does not exist and that ∇^θ_κ holds for regular cardinals κ and θ with $\theta > \kappa$ (Theorem 2 and Theorem 4 will say that this is consistent for $\kappa = \aleph_1$ and for $\kappa = \aleph_2$). It is then easy to see that there can be no closed unbounded set $C \subseteq [\mathbf{L}_\theta]^{<\kappa}$ such that for all $X \in C$, if $\mathbf{L}_\alpha \cong X$ and $\beta \in (\mathrm{Ord} \setminus \alpha) \cup \{\mathrm{Ord}\}$ is largest such that $\mathbf{L}_\beta$ and $\mathbf{L}_\alpha$ have the same bounded subsets of α then the ultrapower of $\mathbf{L}_\beta$ by (the long extender derived from) the uncollapsing map is well-founded. Therefore, a certain version of Jensen's Frequent Extensions of Embeddings Lemma has to fail. Such situations are discussed in the first author's papers [Rä00] and [Rä∞].

The formulation of $\nabla^\theta_\kappa(A)$, though, as has already been indicated, arose out of the second author's work on the strength of $\mathbf{L}(\mathbb{R})$ absoluteness for proper forcings (*cf.* [Sch00] and [Sch∞]). The following theorem is established by the proofs in [Sch∞].

Theorem 2 ([Sch∞])**.** *Equiconsistent are:*

(a) ZFC+ *"$\mathbf{L}(\mathbb{R})$ is absolute for proper forcings,"*
(b) ZFC+ *"$\mathbf{V} = \mathbf{L}[A] + \nabla_{\aleph_1}(A)$," and*
(c) ZFC+ *"there is a remarkable cardinal."*

Let us repeat the definition of a remarkable cardinal for the convenience of the reader.

Definition 3 ([Sch00, Definition 0.4])**.** *A cardinal κ is called* remarkable *iff for all regular cardinals $\theta > \kappa$ there are π, M, $\bar\kappa$, σ, N, and $\bar\theta$ such that the following hold:*

- *$\pi\colon M \to \mathbf{H}_\theta$ is an elementary embedding,*
- *M is countable and transitive,*
- *$\pi(\bar\kappa) = \kappa$,*
- *$\sigma\colon M \to N$ is an elementary embedding with critical point $\bar\kappa$,*
- *N is countable and transitive,*
- *$\bar\theta = M \cap \mathrm{Ord}$ is a regular cardinal in N, $\sigma(\bar\kappa) > \bar\theta$, and*
- *$M = H^N_{\bar\theta}$, i.e., $M \in N$ and $N \models$ "M is the set of all sets which are hereditarily smaller than $\bar\theta$."*

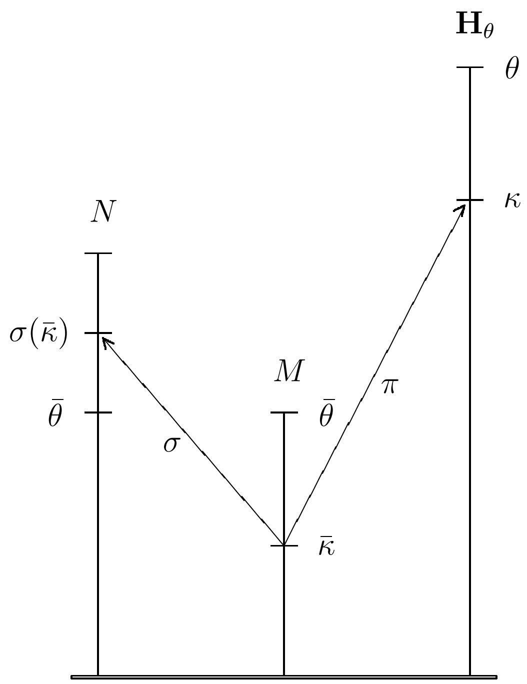

FIGURE 1. Maps witnessing the remarkability of κ

The first author had obtained his results of [Rä00] and [Rä∞] by forcing over $\mathbf{L}$ and exploiting the existence of $0^\#$ in the outer universe. The use of $0^\#$ could not be necessary for this purpose, of course, and the second author realized that the assumption of the existence of a remarkable cardinal would be enough for deriving the conclusions of [Rä00] and [Rä∞]. Along these lines we shall arrive at the following.

Theorem 4. *Equiconsistent are:*

(a) $\mathsf{ZFC}+$ *" $\mathbf{V} = \mathbf{L}[A] + \nabla_{\aleph_2}(A)$," and*
(b) $\mathsf{ZFC}+$ *" there is a remarkable cardinal."*

We shall also consider $\nabla_\kappa(A)$ for regular $\kappa \geq \aleph_3$. We shall see that for a regular $\kappa \geq \aleph_3$, ∇_κ holds if and only if $0^\#$ exists.

2. The proofs.

[Sch∞, Lemma 1.6] gave an important characterization of remarkable cardinals.

Definition 5 ([Sch∞, Definition 1.5]). *Let κ be a cardinal. Let G be $\mathbf{Col}(\omega, <\kappa)$-generic over $\mathbf{V}$, let $\theta > \kappa$ be a regular cardinal, and let $X \in [\mathbf{H}_\theta^{\mathbf{V}[G]}]^\omega$. We say that X* condenses remarkably *if $X = \operatorname{ran}(\pi)$ for* *some elementary*

$$\pi\colon (\mathbf{H}_\beta^{\mathbf{V}[G\cap\mathbf{H}_\alpha^{\mathbf{V}}]};\in,\mathbf{H}_\beta^{\mathbf{V}},G\cap\mathbf{H}_\alpha^{\mathbf{V}})\to(\mathbf{H}_\theta^{\mathbf{V}[G]};\in,\mathbf{H}_\theta^{\mathbf{V}},G)$$

where $\alpha = crit(\pi) < \beta < \kappa$ *and* β *is a regular cardinal (in* $\mathbf{V}$*).*

Lemma 6 ([Sch∞, Lemma 1.6])**.** *A cardinal* κ *is remarkable if and only if for all regular cardinals* $\theta > \kappa$ *we have that*

$\Vdash^{\mathbf{V}}_{\mathbf{Col}(\omega,<\kappa)}$ "$\{X \in [\mathbf{H}_{\check\theta}^{\mathbf{V}[\dot G]}]^\omega \colon X$ condenses remarkably$\}$ is stationary."

Here is a sufficient criterion for being remarkable in $\mathbf{L}$:

Lemma 7. *Let* κ *be a regular cardinal, and suppose that* ∇_κ *holds. Then* κ *is remarkable in* $\mathbf{L}$*.*

Proof. It is easy to see that $\nabla_\kappa^{\kappa^+}$ implies that κ is an inaccessible cardinal of $\mathbf{L}$.

Fix $\theta > \kappa$, a regular cardinal. By $\nabla_\kappa^{\theta^+}$, we may pick some $\pi\colon \mathbf{L}_\gamma \to \mathbf{L}_{\theta^+}$ such that $\gamma < \kappa$ is a (regular) cardinal in $\mathbf{L}$. Let $\pi(\alpha) = \kappa$ and $\pi(\beta) = \theta$. Let $\bar G$ be $\mathbf{Col}(\omega, <\alpha)$-generic over $\mathbf{V}$ and let $G \supseteq \bar G$ be $\mathbf{Col}(\omega, <\kappa)$-generic over $\mathbf{V}$. Then π extends, in $\mathbf{V}[G]$, to some

$$\tilde\pi \;:\; \mathbf{L}_\gamma[\bar G] \to \mathbf{L}_{\theta^+}[G].$$

Let $\mathfrak{M} \in \mathbf{L}_\gamma[\bar G]$ be a model of finite type with universe $\mathbf{L}_\beta[\bar G]$. We have that

$$\tilde\pi\restriction\mathbf{L}_\beta[\bar G] \;:\; \mathfrak{M} \to \tilde\pi(\mathfrak{M}).$$

Notice that $\gamma < \kappa$, and therefore $\mathbf{L}_\beta[\bar G]$ is countable in $\mathbf{L}[G]$. By absoluteness (*cf.* [Sch∞, Lemma 0.2]), there is hence some $\sigma \in \mathbf{L}_{\theta^+}[G]$ such that $\sigma \;:\; \mathfrak{M} \to \tilde\pi(\mathfrak{M})$.

Therefore, $\Vdash^{\mathbf{L}_{\theta^+}}_{\mathbf{Col}(\omega,<\kappa)}$ "there is some countable $X \prec \tilde\pi(\mathfrak{M})$ such that $X \cap \kappa \in \kappa$ and $\mathrm{otp}(X \cap \mathrm{Ord})$ is a cardinal in $\mathbf{L}[\dot G \cap \mathbf{L}_{X\cap\kappa}]$."

Pulling this assertion back by $\tilde\sigma$ yields that $\Vdash^{\mathbf{L}_\gamma}_{\mathbf{Col}(\omega,<\alpha)}$ "there is some countable $X \prec \mathfrak{M}$ such that $X \cap \alpha \in \alpha$ and $\mathrm{otp}(X \cap \mathrm{Ord})$ is a cardinal in $\mathbf{L}[\dot G \cap \mathbf{L}_{X\cap\alpha}]$." As $\mathfrak{M}$ was arbitrary, we thus have $\Vdash^{\mathbf{L}_\gamma}_{\mathbf{Col}(\omega,<\alpha)}$ "the set of all $X \in [\,\mathbf{L}_\beta[\dot G]\,]^\omega$ such that X condenses remarkably is stationary." Lifting this up by π yields $\Vdash^{\mathbf{L}}_{\mathbf{Col}(\omega,<\kappa)}$ "the set of all $X \in [\,\mathbf{L}_\theta[\dot G]\,]^\omega$ such that X condenses remarkably is stationary."

We have shown that κ is remarkable in $\mathbf{L}$, using Lemma 6.

$\square$ (Lemma 7)

It is easy to see that for no κ can $\nabla_\kappa^{\kappa^+}$ hold in $\mathbf{L}$. We shall now consider the task of forcing $\nabla_\kappa(A)$ to hold in a set-generic extension of $\mathbf{L}$. As to $\nabla_{\aleph_1}(A)$, Con(3) $\Rightarrow$ Con(2) in Theorem 2 is shown by proving that if κ is remarkable in $\mathbf{L}$ and $G \subseteq \kappa$ is (induced by some) $\mathbf{Col}(\omega, {<}\kappa)$-generic filter over $\mathbf{L}$ (via some simple coding) then $\nabla_\kappa(G)$ holds in $\mathbf{L}[G]$. Let us now turn towards $\nabla_{\aleph_2}(A)$.

Theorem 8. *Let κ be remarkable in $\mathbf{L}$, and suppose that there is no $\lambda < \kappa$ such that $\mathbf{L}_\kappa \models$ "λ is remarkable". There is then a forcing $\mathbb{P} \in \mathbf{L}$ with the property that in $\mathbf{V}^{\mathbb{P}}$ there is some A such that $\nabla_{\aleph_2}(A)$ holds.*

Proof. Let $\mathbf{Nm}$ denote Namba forcing. Let $\theta > \omega_2$ be regular. By $\mathbb{CN}_\theta$ we shall denote the forcing

$$\mathbf{Col}(\omega_2, \theta) \star \mathbf{Nm}.$$

The key idea will be to iterate this 2-step forcing iteration. Notice that $\mathbf{Col}(\omega_2, \theta)$ turns the cofinality of each cardinal $\xi \in [\omega_2, \theta]$ with former cofinality $\geq \omega_2$ into ω_2, and therefore $\mathbb{CN}_\theta$ turns the cofinality of each such cardinal into ω.

We shall now define a suitable RCS iteration $\langle \langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \rangle : i < \kappa \rangle$ as follows. We let $\mathbb{P}_0 := \emptyset$, we let $\mathbb{P}_{i+1} = \mathbb{P}_i \star \dot{\mathbb{Q}}_i$ for $i < \kappa$, and for limit ordinals $\lambda < \kappa$ we let $\mathbb{P}_\lambda$ be the revised limit (Rlim) of $\langle \langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \rangle : i < \lambda \rangle$. The definition of $\mathbb{Q}_i$ splits into two cases according to whether i is even or odd. Let us deal with the odd case first. In order to apply the theory of RCS iterations introduced in [Sh98, XI : §1] we shall set $\mathbb{Q}_{2i+1} := \mathbf{Col}(\omega_1, < 2^{|\mathbb{P}_{2i+1}| + |i|})$ for $i < \kappa$. Let us now discuss the even case, i.e., let us define $\mathbb{Q}_{2i}$ for $i < \kappa$. It will be easy to verify that inductively, $\Vdash^{\mathbf{L}}_{\mathbb{P}_{2i}} \dot{G} \subseteq \omega_2$. (Here and in what follows we confuse generic objects with sets of ordinals obtained via some simple coding.) By Lemma 7 and our assumption that no $\lambda < \kappa$ is remarkable in $\mathbf{L}_\kappa$, for each $p \in \mathbb{P}_{2i}$ there is some (least) $\theta_p < \kappa$ such that

$$\neg\ (\, p \Vdash^{\mathbf{L}}_{\mathbb{P}_{2i}} \nabla^{\theta_p}_{\aleph_2}(\dot{G})\,).$$

Letting $\theta^\star := \sup \{\, \theta_p \mid p \in \mathbb{P}_{2i} \,\} < \kappa$, we then obviously have that

$$\Vdash^{\mathbf{L}}_{\mathbb{P}_{2i}} \neg \nabla^{\theta}_{\aleph_2}(\dot{G}).$$

Now let $\dot{\mathbb{Q}}_{2i}$ be a name for $\mathbb{CN}_{\theta^\star}$ as being defined in $\mathbf{L}^{\mathbb{P}_{2i}}$. In particular, $\mathbb{CN}_{\theta^\star}$ satisfies Shelah's condition that we need for applying the relevant theorems about RCS iterations. The definition of this condition can be found in [Sh98, XI : §2] (especially Definition 2.4). Although Shelah's condition is not in general preserved under iterations we are able to prove it for our forcing iteration due to the results in [Sh98, XI : §4,§5]. More precisely, the proof of [Sh98, Lemma 4.4] which shows that $\mathbf{Nm}$ satisfies the desired condition will yield that in fact every $\mathbb{CN}_\theta$ satisfies it as well (*cf.* [Sh99]).

We finally let $\mathbb{P}$ be the revised limit of $\langle\langle\mathbb{P}_i, \dot{\mathbb{Q}}_i\rangle : i < \kappa\rangle$. Let G be $\mathbb{P}$-generic over $\mathbf{L}$. We have $G \subseteq \kappa$ and moreover by how we have defined our forcing iteration we can apply the theorems of [Sh98, XI] to see that $\mathbb{P}$ has the κ-c.c. and $\omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}}$. It is moreover easy to see that $\kappa = \omega_2^{\mathbf{L}[G]}$. We are left with having to verify that $\nabla_\kappa(G)$ holds in $\mathbf{L}[G]$.

Suppose not. In fact suppose that there are $p \in \mathbb{P}$ and some (least) θ such that

$$\neg\ (\, p \Vdash^{\mathbf{L}}_{\mathbb{P}} \nabla^\theta_{\aleph_2}(\dot{G})\,).$$

Because κ is remarkable in the ground model, inside $\mathbf{L}$ we may pick $\pi\colon \mathbf{L}_\gamma \to \mathbf{L}_{\theta^+}$ and $\sigma\colon \mathbf{L}_\gamma \to \mathbf{L}_{\tilde{\gamma}}$ such that $\gamma < \tilde{\gamma} < \omega_1$, $\kappa \in \operatorname{ran}(\pi)$, $\alpha = \pi^{-1}(\kappa)$ is the critical point of σ, $\sigma(\alpha) > \gamma$ and γ is a regular cardinal in $\mathbf{L}_{\tilde{\gamma}}$.

Let $\bar{\mathbb{P}} := \pi^{-1}(\mathbb{P})$ and $\tilde{\mathbb{P}} = \sigma(\bar{\mathbb{P}})$. It is easy to see that $\bar{\mathbb{P}} = \tilde{\mathbb{P}}\restriction\alpha$ (with the obvious meaning). Let $\beta := \pi^{-1}(\theta)$. Notice that, using π, there is some $q \in \bar{\mathbb{P}}$ such that β is least with

$$\neg\ (\, q \Vdash^{\mathbf{L}_\gamma}_{\bar{\mathbb{P}}} \nabla^\beta_{\aleph_2}(\dot{G})\,).$$

Therefore, there is some $\beta^\star \geq \beta$ such that forcing with $\mathbb{CN}_{\beta^\star}$, as defined in $\mathbf{L}^{\bar{\mathbb{P}}}_{\tilde{\gamma}}$, is the next step right after forcing with $\bar{\mathbb{P}}$ in the iteration $\tilde{\mathbb{P}}$.

Let $\bar{G} \in \mathbf{L}$ be $\bar{\mathbb{P}}$-generic over $\mathbf{L}_\gamma$ (and hence over $\mathbf{L}_{\tilde{\gamma}}$, too) such that $q \in \bar{G}$, and let $\tilde{G} \supseteq \bar{G}$ be $\tilde{\mathbb{P}}$-generic over $\mathbf{L}_{\tilde{\gamma}}$. Then σ lifts to $\tilde{\sigma}\colon \mathbf{L}_\gamma[\bar{G}] \to \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$. In order to derive a contradiction it now suffices to prove that $\nabla^\beta_\alpha(\bar{G})$ holds in $\mathbf{L}_\gamma[\bar{G}]$.

Let $\mathfrak{M} \in \mathbf{L}_\gamma[\bar{G}]$ be a model of finite type with universe $\mathbf{L}_\beta[\bar{G}]$. We have $\tilde{\sigma}\restriction\mathbf{L}_\beta[\bar{G}] : \mathfrak{M} \to \tilde{\sigma}(\mathfrak{M})$ and we would now like to build a tree $T \in \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$ searching for an embedding like this one.

Claim 1. In $\mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$, $\mathbf{L}_\beta[\bar{G}] = \bigcup_{n<\omega} X_n$, where for each $n < \omega$, $X_n \subseteq X_{n+1}$, $X_n \in \mathbf{L}_\gamma[\bar{G}]$, and $\operatorname{Card}(X_n) = \alpha$ in $\mathbf{L}_\gamma[\bar{G}]$.

Proof. Let $F\colon \alpha \to \beta$, $F \in \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$, be surjective, and let $f\colon \omega \to \alpha$, $f \in \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$, be cofinal, where F, f are the objects adjoined by forcing with $\mathbb{CN}_{\beta^\star}$, as defined in $\mathbf{L}^{\bar{\mathbb{P}}}_{\tilde{\gamma}}$. Let

$$X'_n := F\text{''}f(n), \text{ for } n < \omega.$$

Notice that $F\restriction\xi \in \mathbf{L}_{\tilde{\gamma}}[\bar{G}]$ (and hence $\in \mathbf{L}_\gamma[\bar{G}]$) for each $\xi < \alpha$. In particular, $X'_n \in \mathbf{L}_\gamma[\bar{G}]$ for each $\xi < \alpha$. The rest is easy. $\square$ (Claim 1)

Now fix $\langle X_n : n < \omega\rangle$ as provided by Claim 1. We may and shall assume that furthermore $X_n \prec \mathfrak{M}$ for all $n < \omega$.

Claim 2. $\tilde{\sigma}\restriction X_n \in \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$ for each $n < \omega$.

Proof. Let $f\colon \alpha \to X_n$ be bijective, $f \in \mathbf{L}_\gamma[\bar{G}]$. For $x \in X_n$ we'll then have that $y = \tilde{\sigma}(x)$ iff there is some $\xi < \alpha$ with $x = f(\xi) \wedge y = \tilde{\sigma}(f)(\xi)$. But f and $\tilde{\sigma}(f)$ are both in $\mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$. Therefore, $\tilde{\sigma}\restriction X_n \in \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$. $\square$ (Claim 2)

Now let T be the tree of height ω consisting of all (X_n, τ), where $n < \omega$ and $\tau\colon X_n \to \tilde{\sigma}(\mathfrak{M})$ is elementary, ordered by $(X_n, \tau) \leq (X_m, \tau')$ if and only if $n \geq m$ and $\tau \supseteq \tau'$. Of course, $T \in \mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$. Claim 2 witnesses that T is illfounded in $\mathbf{V}$. T is hence illfounded in $\mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$ as well. This buys us that in $\mathbf{L}_{\tilde{\gamma}}[\tilde{G}]$, there is some elementray

$$\tau\ :\ \mathfrak{M} \to \tilde{\sigma}(\mathfrak{M}).$$

We thus have that $\mathbf{L}_{\tilde{\gamma}}[\tilde{G}] \models$ "there is some $X \prec \tilde{\sigma}(\mathfrak{M})$ such that $\mathrm{Card}(X) < \sigma(\alpha)$, $X \cap \sigma(\alpha) \in \sigma(\alpha)$, and $\mathrm{otp}(X \cap \mathrm{Ord}) \in \mathrm{Card}^{\mathbf{L}[\tilde{G} \cap X \cap \sigma(\alpha)]}$." Pulling this back by $\tilde{\sigma}$ gives that $\mathbf{L}_\gamma[\bar{G}] \models$ "there is some $X \prec \mathfrak{M}$ such that $\mathrm{Card}(X) < \alpha$, $X \cap \alpha \in \alpha$, and $\mathrm{otp}(X \cap \mathrm{Ord}) \in \mathrm{Card}^{\mathbf{L}[\dot{G} \cap X \cap \alpha]}$," yielding the desired contradiction.

As $\mathfrak{M}$ was arbitrary, this shows that $\nabla^\beta_\alpha(\bar{G})$ holds in $\mathbf{L}_\gamma[\bar{G}]$. $\square$ (Theorem 8)

The forcing iteration $\mathbb{P}$ which leads to the proof of Theorem 8 can also be reorganized along the lines of [Rä00]. In that paper the task is divided into two parts: in the first part we iterate the collapse forcing using ideas of the Easton forcing construction, and in the second part we shoot reasonable countable sequences through certain ordinals using Namba forcing combined with the above-mentioned Levy collapse, again by an RCS iteration—in this case we know that Shelah's condition holds for Namba forcing (*cf.* [Sh98, XI : §1]). Nevertheless, this simplified construction from [Rä00] and the construction used in the current paper are in a sense equivalent. Following the approach of [Rä00] would lead to many tedious though elementary details which would have to be checked; this is why we chose the present construction for this paper.

Our Theorem 8 strengthens a result which is proved in Chapter 7 of [Rä∞] and which (in the terminology provided by Definition 1) shows that if $0^{\#}$ exists then there is a set-generic forcing extension $\mathbf{V}$ of $\mathbf{L}$ in which there is some $A \subseteq \omega_2$ such that $\mathbf{V} = \mathbf{L}[A]$ and $\nabla^{\gamma^{+(\omega+1)}}_{\aleph_2}$ holds for arbitrary Silver indiscernibles γ.

We get the following corollary to Lemma 7 and Theorem 8, which is just a restatement of Theorem 4.

Corollary 9. *Equiconsistent are:*

(a) ZFC+ *“* $\mathbf{V} = \mathbf{L}[A] + \nabla_{\aleph_2}(A)$*,” and*
(b) ZFC+ *“ there is a remarkable cardinal.”*

We finally turn towards ∇_κ for $\kappa \geq \aleph_3$.

Lemma 10. *Let κ be a regular cardinal, $\kappa \geq \aleph_3$. Suppose that $\nabla_\kappa^{\kappa^+}$ holds. Then $0^\#$ exists.*

Proof. Suppose not. Pick $\pi\colon \mathbf{L}_\beta \to \mathbf{L}_{\kappa^+}$ such that $\omega_2 < \alpha = \mathrm{crit}(\pi) < \kappa$ and β is a cardinal of $\mathbf{L}$. We have that $\mathcal{P}(\alpha) \cap \mathbf{L} \subseteq \mathbf{L}_\beta$, and we may hence define the ultrapower $\mathrm{Ult}(\mathbf{L}; U)$, where $X \in U$ iff $X \in \mathcal{P}(\alpha) \cap \mathbf{L} \wedge \alpha \in \pi(X)$. As $0^\#$ does not exist, $\mathrm{cf}^{\mathbf{V}}(\alpha^{+\mathbf{L}}) > \omega$ as a consequence of Jensen's Covering Lemma for $\mathbf{L}$. By standard methods this implies that $\mathrm{Ult}(\mathbf{L}; U)$ is well-founded (*cf.* the proof of [Fr00, 3.13 (i)]). So $0^\#$ does exist after all. Contradiction! $\square$ (Lemma 10)

Lemma 11. *Suppose that $0^\#$ exists. Then ∇_κ holds for every regular cardinal $\kappa > \aleph_0$.*

Proof. We consider $0^\#$ as a subset of ω. Fix κ. Let $\mathfrak{M} := (\mathbf{L}_\kappa; \vec{F})$ be a model of finite type with universe $\mathbf{L}_\kappa$. Let $\theta > \kappa$ be regular, and let

$$\pi\colon (\mathbf{L}_\beta[0^\#]; \in, \mathbf{L}_\beta, 0^\#, \vec{G}) \to (\mathbf{L}_\theta[0^\#]; \in, \mathbf{L}_\theta, 0^\#, \vec{F})$$

be such that $\beta < \kappa$ and $\mathrm{ran}(\pi) \cap \kappa \in \kappa$. It is then straightforward to check that for all $\gamma < \beta$, $\gamma^{+\mathbf{L}} < \beta$. Therefore, $\beta \in \mathrm{Card}^{\mathbf{L}}$. As $\mathfrak{M} = (\mathbf{L}_\kappa; \vec{F})$ was arbitrary, this means that ∇_κ^θ holds. $\square$ (Lemma 11)

Corollary 12. *Let $\kappa \geq \aleph_3$ be a regular cardinal. Equivalent are:*

(a) ∇_κ *holds, and*
(b) $0^\#$ *exists.*

We conclude with a few remarks. Suppose that $\kappa \geq \aleph_3$ is a regular cardinal. It can be shown that $\mathbf{V} = \mathbf{L}[A] \wedge \nabla_\kappa^{\kappa^+}(A)$ implies that every element of $\mathbf{H}_\kappa$ has a sharp (but of course, $A^\#$ doesn't exist in $\mathbf{L}[A]$). Moreover, if $A \subseteq \kappa$ is such that $\mathbf{H}_\kappa^{\mathbf{L}[A^\#]} = \mathbf{H}_\kappa^{\mathbf{L}[A]}$ then $\mathbf{L}[A] \models \nabla_\kappa(A)$. In particular, if $\mathbf{V} = \mathbf{L}[E] = \mathbf{L}^\#$ (= the least extender model which is closed under sharps) then for all regular cardinals $\kappa > \aleph_0$, $\nabla_\kappa(E \cap \kappa)$ holds.

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