Proving projective determinacy

Ralf Schindler

Institut für Mathematische Logik und
Grundlagenforschung
WWU Münster

and

Mathematics Department
UC Berkeley

$\boxed{\textbf{Lebesgue measurability}}$

For $A \subset [0,1]$ let $\mu^*(A)$ denote the outer measure of A.

Definition. $A \subset [0,1]$ is called Lebesgue measurable iff for all $\epsilon > 0$ there is some closed $F \subset A$ and some open $G \supset A$ such that $\mu^*(G \setminus F) < \epsilon$.

• Every open $A \subset [0,1]$ is Lebesgue measurable.

• The set of all Lebesgue measurable sets of reals is a σ-algebra, i.e., closed w.r.t. compliments, $\bigcup_{n\in\mathbb{N}}$ and $\bigcap_{n\in\mathbb{N}}$.

Definition. $A \subset [0,1]$ is a Borel set iff A is an element of the least σ-algebra which contains all open subsets of $[0,1]$.

Theorem. Every Borel set $A \subset [0,1]$ is Lebesgue measurable.

Definition. $A \subset [0,1]^n$ is called the projection of $B \subset [0,1]^{n+1}$ iff

$$\vec{x} \in A \Longleftrightarrow \exists y \ (\vec{x}, y) \in B.$$

Definition. $A \subset [0,1]^n$ is called analytic iff A is the projection of a Borel set (in $[0,1]^{n+1}$).

- There are analytic sets which are not Borel.

- $A \subset [0,1]^n$ is Borel iff both A and $[0,1] \setminus A$ are analytic.

Theorem (Luzin, 1917). Each analytic set $A \subset [0,1]$ is Lebesgue measurable.

$\boxed{\textbf{Projective sets}}$

Definition.

$$\Sigma^1_1 = \text{ analytic sets}$$

$$\Pi^1_n = \text{ complements of sets in } \Sigma^1_n$$

$$\Sigma^1_{n+1} = \text{ projections of sets in } \Pi^1_n$$

$$\text{projective sets } = \bigcup_{n \in \mathbb{N}} \Sigma^1_n = \bigcup_{n \in \mathbb{N}} \Pi^1_n.$$

Question (Luzin, 1925). Are all projective sets of reals Lebesgue measurable?

The axioms of Zermelo-Fraenkel (ZFC)

Extensionality: $\forall u(u \in x \Leftrightarrow u \in y) \Leftrightarrow x = y$

Empty set: $\emptyset$ exists

Pairing: for all x, y, $\{x, y\}$ exists

Union: for all x, $\bigcup x$ exists

Power set: for all x, $\mathcal{P}(x)$ exists

Infinity: there is an infinite set

Replacement: let F be a definable function (not necessarily a set!); for all x, $F''x$ exists

Foundation: there is no infinite descending chain $\ldots \in x_2 \in x_1 \in x_0$ (i.e., $\in$ is well-founded)

The axiom of choice: for every x, there is some (total) well-founded order (i.e., a well-order) $<_x$ on x

$[0,1]$ has a well-order $\Longrightarrow$
there is a set of reals which is not Lebesgue measurable:

For x, $y \in [0,1]$, write $x \sim y$ iff $x - y \in \mathbb{Q}$. Let $A \subset [0,1]$ be such that A has exactly one member from each equivalence class. (A is then a "Vitali set.")

Vitali sets are as complicated as a well-order of $[0,1]$.

$\boxed{\textbf{Ordinals}}$

$0 = \emptyset$

$1 = \{0\}$

$\vdots$

$n + 1 = \{0, 1, \ldots, n\}$

$\vdots$

$\omega = \{0, 1, 2, \ldots\}$

$\omega + 1 = \{0, 1, 2, \ldots, \omega\}$.

$\vdots$

$\alpha = \{\beta \mid \beta < \alpha\}$

For each well-ordered set $(x, <_x)$ there is some ordinal α such that

$$(\alpha, <) \cong (x, <_x).$$

$\boxed{\textbf{Gödel's constructible universe}}$

$J_0 = \emptyset$

$J_{\alpha+1} =$ the closure of $J_\alpha \cup \{J_\alpha\}$ under rudimentary functions (like x, $y \mapsto \{x, y\}$, $x \mapsto \cup x$,...)

$J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$ for limit ordinals λ

$L = \bigcup_\alpha J_\alpha$

Theorem (Gödel, 1938). L is an inner model of ZFC.

Theorem (Gödel, 1938). L has a well-order of $[0, 1]$ which (construed as a subset of $[0, 1]^2$) is a Σ^1_2 set. Therefore: L has a Σ^1_2 set of reals which is not Lebesgue measurable.

$\boxed{\textbf{Cardinals}}$

Ordinals α for which there is no $\beta < \alpha$ of the same cardinality are called cardinals.

$0,\ 1,\ 2,\ 3,\ \ldots,\ \omega,\ \omega_1 = \aleph_1,\ \omega_2 = \aleph_2,\ \ldots,\ \omega_\omega = \aleph_\omega,\ \ldots,\ \kappa = \omega_\kappa = \aleph_\kappa,\ \ldots$

Question (Hausdorff, 1914). Is there some uncountable regular limit cardinal?

Definition. A cadinal κ is called inaccessible iff κ is uncountable, $\forall \alpha < \kappa\ 2^\alpha < \kappa$, and κ is regular.

• ZFC does not prove the existence of inaccessible cardinals. This follows from the fact that if κ is inaccessible, then

$$J_\kappa \models ZFC$$

together with Gödel's 2nd Incompleteness Theorem.

Theorem (Solovay, 1964). If κ is an inaccessible cardinal, then there is a forcing extension of the universe in which every projective $A \subset [0,1]$ is Lebesgue measurable.

So ZFC doesn't decide whether all Σ^1_2 sets are Lebesgue measurable.

Theorem (Shelah, 1979). If every projective $A \subset [0,1]$ is Lebesgue measurable, then ω_1 is an inaccessible cardinal in L.

It follows from Shelah's result that ZFC plus "there is an inaccessible cardinal" does not *prove* that every projective $A \subset [0,1]$ is Lebesgue measurable.

Is there a natural principle (which cannot be true in Gödel's L) which implies that every projective $A \subset [0,1]$ is Lebesgue measurable?

$\boxed{\textbf{Projective Determinacy}}$

Let $A \subset [0,1]$.

Gale/Stewart (1953): Consider the following game, G(A):

$$\begin{array}{c|ccccccc} I & i_0 & & i_2 & & \cdots & & \\ \hline II & & i_1 & & i_3 & & & \cdots \end{array}$$

$i_n \in \{0,1\}$. I.e. $\sum_{n<\infty} \frac{i_n}{2^{n+1}} \in [0,1]$.

I wins iff $\sum_{n<\infty} \frac{i_n}{2^{n+1}} \in A$, else II wins.

We say that G(A) (or, A itself) is determined iff either I or II has a winning strategy.

There is a well-order of $[0,1] \Longrightarrow$ there is some $A \subset [0,1]$ such that neither I nor II has a winning strategy in $\mathsf{G}(A)$. Again, such non-determined sets are roughly as complicated as the well-order of $[0,1]$.

Theorem (Martin, 1975). Let $A \subset [0,1]$ be a Borel set. Then A is determined.

Projective Determinacy (PD): Let $A \subset [0,1]$ be a projective set. Then A is determined.

Theorem (Mycielski, Swierczkowski, 1964). If PD holds, then every projective $A \subset [0,1]$ is Lebesgue measurable.

This implies that PD cannot be shown in ZFC.

$\boxed{\textbf{The property of Baire}}$

Definition. $A \subset [0,1]$ has the property of Baire iff for some open set G, $A \triangle G$ is meager (i.e., the countable union of nowhere dense sets).

Theorem (Luzin, Sierpinski, 1923). Every analytic $A \subset [0,1]$ has the property of Baire.

Theorem (Mazur, 1957). If PD holds, then every projective $A \subset [0,1]$ has the property of Baire.

Uniformization

Definition. Let $A \subset [0,1]^2$. A partial function $F\colon [0,1] \to [0,1]$ is said to uniformize A iff for all x,

$$\exists y \ (x,y) \in A \Rightarrow (x, F(x)) \in A.$$

Projective Uniformization: For each projective $A \subset [0,1]^2$ there is some uniformizing function F with a projective graph.

Theorem (Novikov, Kondo, Addison, 1959).

Theorem (Moschovakis, 1971). If PD holds, then so does Projective Uniformization.

Is PD a reasonable hypothesis? Is it true?

Question (Woodin, 1981). Suppose the following to hold true.
(1) Every projective $A \subset [0,1]$ is Lebesgue measurable.
(2) Every projective $A \subset [0,1]$ has the property of Baire
(3) Projective Uniformization holds.
Must PD hold?

This question was to become well-known as the $\mathbf{12^{th}}$**Delfino Problem**. It was open for 16 years.

$\boxed{\textbf{Large cardinals}}$

Write $V =$ the universe of all sets

Definition. $\pi: V \to M$ is called an elementary embedding iff $\pi''V \prec M$.

Definition. Let α be a cardinal. A cardinal κ is called α-strong iff for every $X \subset \alpha$ there is some inner model M and some elementary embedding $\pi: V \to M$ such that κ is the critical point of π and $X \in M$.

Definition. κ is a measurable cardinal iff κ is 0-strong (iff κ is κ-strong).

There is a measurable cardinal iff for some set X, there is some σ-complete 2-valued nontrivial measure on X. (There is no such measure on $\mathbb{R}$.)

Theorem (Steel, 1997; Schindler, 2000) The following theories are equiconsistent.

(1) ZFC + every projective $A \subset [0,1]$ is Lebesgue measurable and has the property of Baire + Projective Uniformization holds

(2) ZFC + there is an infinite sequence $\kappa_0 < \kappa_1 < \kappa_2 < \ldots$ of cardinals with supremum λ such that each κ_n is λ-strong.

Corollary. The answer to Woodin's question is "no."

$\boxed{\text{More large cardinals}}$

Definition. Let α be a cardinal, and let Y be a set of ordinals. A cardinal κ is called α,Y-strong iff for every $X \subset \alpha$ there is some inner model M and some elementary embedding $\pi: V \to M$ such that κ is the critical point of π, $X \in M$, and $Y \cap \alpha = \pi(Y) \cap \alpha$.

Definition. A cardinal λ is a Woodin cardinal iff for every $Y \subset \lambda$ there is some $\kappa < \lambda$ which is α,Y-strong for every $\alpha < \lambda$.

Theorem (Martin, Steel, 1985). Suppose that there are infinitely many Woodin cardinals. Then PD holds.

There are not many "natural" statements known to be equivalent to PD.

Theorem (Martin, Steel, Woodin, Neeman, 199?). The following statements are equivalent.

(1) PD.

(2) Turing determinacy for projective sets.

(3) For every $n \in \mathbb{N}$ and for every real x there is a (countable) ω_1-iterable mouse which contains x and has n Woodin cardinals.

$\boxed{\textbf{Mice}}$

Let E be any set.

$J_0[E] = \emptyset$

$J_{\alpha+1}[E] =$ the closure of $J_\alpha[E] \cup \{J_\alpha[E]\}$ under functions which are rudimentary in E (i.e., rudimentary or $x \mapsto x \cap E$)

$J_\lambda[E] = \bigcup_{\alpha<\lambda} J_\alpha[E]$ for limit ordinals λ

Definition. A premouse $\mathcal{M}$ is a model of the form $J_\alpha[E]$, where E codes a sequence of elementary embeddings whose domains are initial segments of $\mathcal{M}$.

Definition. A mouse is a premouse which is sufficiently iterable.

Hjorth has shown that Π^1_2 Wadge determinacy is equivalent with Π^1_2 determinacy.

(1) Does Wadge determinacy for projective sets imply PD?

Steel has shown that if all projective sets are Lebesgue measurable and have the Baire property, and if all Π^1_3 relations can be uniformized by Π^1_3 functions, then Π^1_2 determinacy holds.

(2) If all projective sets are Lebesgue measurable and have the Baire property, and if all Π^1_{2n+1} relations can be uniformized by Π^1_{2n+1} functions, does PD hold?

Whereas there are not many statements known to be equivalent with PD, there are in fact a lot of statements which *imply* PD.

Many such statements come from areas of set theory with apparently no connection to the study of projective sets of reals.

The proof that a given statement implies PD always factors through the production of mice with Woodin cardinals.

Any such proof uses its own variant of the **core model induction**. This is a method which inductively produces mice which yield PD (or stronger forms of determinacy).

At any given stage of the induction, the inductive hypothesis allows us to build a preliminary core model K^c which is closed under all mice so far and which satisfies the K-**existence dichotomy**:

Either (1) K^c is fully iterable, hence the true core model K exists,

or else (2) the "next" mouse with Woodin cardinals exists.

The hypothesis at hand will allow us to rule out (1).

$\boxed{\textbf{PFA}}$

The Proper Forcing Axiom, PFA, is a generalization of Martin's Axiom, MA. Both MA and PFA have applications in topology and abelian group theory.

Theorem (Schimmerling, Steel, Woodin, 199?) If PFA holds, then PD holds.

In fact:

Theorem (Jensen, Schimmerling, Schindler, Steel, 2006). If PFA holds, then $\mathsf{AD}_{\mathbb{R}}$ holds in an inner model.

$\boxed{\textbf{Trees}}$

A cardinal δ is said to have the tree property iff every δ-tree has a branch of length δ through it.

- ω has the tree property, but ω_1 does not.

- Consistenty, all $\aleph_n$'s for $n \geq 2$ may simultaneously have the tree property.

Theorem (Foreman, Magidor, Schindler, 1997) Suppose that $(\delta_n : n < \omega)$ is a strictly increasing sequence of cardinals with supremum δ, a strong limit cardinal, such that for every $n < \omega$, both δ_n and δ_n^+ have the tree property. Then PD holds.

$\boxed{\textbf{No choice}}$

In the absence of the axiom of choice, ω_1 need not be regular or non-measurable. In fact, successor cardinals may be singular or Ramsey.

Theorem (Busche, 2007) Suppose that all uncountable cardinals are singular. Then PD holds.

Theorem (Busche, 2007) Suppose that all uncountable successor cardinals are weakly compact and all uncountable limit cardinals are singular. Then PD holds.

In fact, Busche derives AD to hold in $L(\mathbb{R})$.

$\boxed{\textbf{Ideals}}$

An ideal I on ω_1 is called ω_1 dense iff the structure $(\mathcal{P}(\omega_1)/I, \supset)$ has a dense subset of size $\aleph_1$.

Theorem (Woodin, 199?) Suppose that there is an ω_1-dense ideal on ω_1. Then PD holds.

In fact, Woodin derives AD to hold in $L(\mathbb{R})$.

$\boxed{\textbf{Cardinal arithmetic}}$

Cardinal arithmetic is the study of the (possible) behavior of the continuum function, $\kappa \mapsto 2^\kappa =$ the cardinality of $\mathcal{P}(\kappa)$.

More generally, it is the study of the (possible) behavior of the function λ, $\kappa \mapsto \lambda^\kappa$.

"Possible" because of the independence results of Gödel, Cohen, Solovay, Easton, and others; for instance:

CH = continuum hypothesis = $2^{\aleph_0} = \aleph_1$.

Theorem (Gödel, 1938; Cohen 1963). ZFC does not decide CH.

We say that GCH holds at κ if $2^\kappa = \kappa^+$.

Theorem (Solovay, 196?). GCH can first fail at any regular cardinal κ.

Of particular interest in cardinal arithmetic are therefore *singular* cardinals.

Typical question: Let κ be singular; given $\lambda \mapsto 2^\lambda$ for $\lambda < \kappa$, what are the possible values of 2^κ ?

For instance, what can $2^{\aleph_\omega}$ be if $\aleph_\omega$ is a strong limit cardinal?

Or: what can $2^{\aleph_{\omega_1}}$ be if $\aleph_{\omega_1}$ is a strong limit cardinal?

Theorem (Silver; 1974). GCH cannot fail for the first time at a singular cardinal of *uncountable* cofinality.

Hence if GCH holds below $\aleph_{\omega_1}$ then it holds *at* $\aleph_{\omega_1}$, i.e., $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

One of the key questions, given Silver's result:

Can GCH fail for the first time at $\aleph_\omega$? More generally, can it fail for the first time at a cardinal of *countable* cofinality?

More generally, what are possible values for $2^{\aleph_\omega}$ if GCH holds below $\aleph_\omega$, say? or if $\aleph_\omega$ is a strong limit cardinal?

Theorem (Magidor; 1977). If there is a supercompact cardinal, then there is a transitive model of ZFC in which GCH first fails at $\aleph_\omega$.

Theorem (Gitik; 1989). If there is a measurable cardinal κ of Mitchell order $o(\kappa) = (2^\kappa)^+$ then there is a generic extension of V in which GCH first fails at $\aleph_\omega$.

Gitik has also shown that his theorem gives the optimal upper bound.

Theorem (Schindler 2001). Suppose that $\aleph_\omega$ is a strong limit cardinal, and $2^{\aleph_\omega} > \aleph_{\omega_1}$. Then PD holds.

It is open whether the hypothesis of this theorem is consistent. This question is related to one of the key open problems of Shelah's pcf-theory.

We do have:

Theorem (Shelah, 199?). If $\aleph_\omega$ is a strong limit cardinal then

$$2^{\aleph_\omega} < \aleph_{\omega_4}.$$

Theorem (Gitik, Schindler; 2001). Suppose that κ is a singular (strong limit) cardinal of uncountable cofinality such that

$$\{\mu < \kappa \colon 2^\mu = \mu^+\}$$

is stationary and costationary. Then PD holds.

Theorem (Gitik; 2004). Suppose that there is a supercompact cardinal. Then there is a forcing extension in which there is a singular (strong limit) cardinal κ of uncountable cofinality such that

$$\{\mu < \kappa \colon 2^\mu = \mu^+\}$$

is stationary and costationary.

$\boxed{\textbf{Challenges}}$

• Find a version of the 12^{th} Delfino Problem which has a positive solution, i.e., find a set of "regularity properties" about the projective sets of reals which is equivalent with PD!

• Verify that stronger forms of determinacy follow from the hypotheses of the previous theorems, i.e., push the core model induction and its application further!

• Construct larger core models!!