Langlands-Rapoport Conjecture Over Function Fields

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Abstract

In this article we prove the analogue of the Langlands-Rapoport conjecture for the moduli stacks of global $\mathcal{G}$-shtukas. Here $\mathcal{G}$ is a parahoric Bruhat-Tits group scheme over a smooth projective curve C over a finite field $\mathbb{F}_q$.

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1 Introduction

According to Langlands's philosophy it is highly desirable to understand the cohomology of Shimura varieties, as they naturally carry both Hecke and Galois actions. Thus, in particular, it is interesting to ask about the possible descriptions of the zeta functions of these varieties. Concerning this, Langlands made a conjecture about the structure of mod p points of Shimura varieties, from which the expression of the zeta function of a Shimura variety as a product of automorphic L-functions follows. The conjecture was made more precise by Kottwitz [Kot90] and consequently refined by Langlands and Rapoport [LR87]. This conjecture provides a conceptual group theoretic description of mod p points of Shimura varieties. As an evidence for the conjecture, Langlands and Rapoport proved the conjecture for simple Shimura varieties of PEL types A and C, under the assumption of Grothendieck's standard conjectures and the Tate conjecture for varieties over finite fields, and the Hodge conjecture for abelian varieties. Following work of many authors, including Morita, Milne, Pfau, Deligne, Reimann, Zink, Kottwitz [Kot92] proved the conjecture for PEL Shimura varieties of type A and C and according to technical issues arising in the general set up, Kisin, considerably later, proved the conjecture for Shimura varieties of *abelian type* with *hyperspecial* level structure [Kis13]. Recall that the Shimura varieties of abelian type are exactly those which fulfill the Deligne conception of Shimura varieties [Del79]. Indeed, concerning the issue of existence of canonical integral model for Shimura varieties, this assumption is essential even for stating the conjecture.

In this article we formulate the Langlands-Rapoport conjecture for the moduli stacks of global $\mathcal{G}$-shtukas. We further implement the local theory of global $\mathcal{G}$-shtukas and their uniformization theory, which we developed in [AH14a] and [AH14b], to establish a proof of the conjecture in the following general set up.
Let C be a smooth projective, geometrically irreducible curve over a finite field $\mathbb{F}_q$ with q elements, and let $Q = \mathbb{F}_q(C)$ be its function field. Fix an n-tuple $\underline{\nu} = (\nu_1, \ldots, \nu_n)$ of pairwise different closed points $\nu_i \in C$. In the situation of Shimura varieties, $\mathbb{Z}$ and p are the analogue of our C and $\underline{\nu}$. Consider a *parahoric* (*Bruhat-Tits*) *group scheme* $\mathcal{G}$ over C in the sense of [BT72, Définition 5.2.6] and [HR03]. According to the analogy between function fields and number fields, the moduli stacks of global $\mathcal{G}$-shtukas appear as the function field counterparts of (the canonical integral model for) Shimura varieties. In analogy with the theory of Shimura varieties, we introduce $\nabla\mathscr{H}$-data.

A $\nabla\mathscr{H}$-data is a pair $(\mathcal{G}, \widehat{Z}_{\underline{\nu}})$ where $\mathcal{G}$ is a parahoric group scheme over the curve C and $\widehat{Z}_{\underline{\nu}} = (\widehat{Z}_{\nu_i})_i$ is an n-tuple of bounds $\widehat{Z}_{\nu_i}$ which are closed subschemes of certain formal completion $\widehat{\mathcal{F}\ell}_{\mathbb{P}_{\nu_i}}$ of the twisted affine flag variety $\mathcal{F}\ell_{\mathbb{P}_{\nu_i}}$, associated with the parahoric group $\mathbb{P}_{\nu_i}$; see [PR08]. Here $\mathbb{P}_{\nu_i}$ denote the parahoric group obtained by base change of $\mathcal{G}$ to $\operatorname{Spec} \widehat{A}_{\nu_i}$, the spectrum of the completion of the stalk $\mathcal{O}_{C,\nu_i}$ at ν_i. To such a pair $(\mathcal{G}, \widehat{Z}_{\underline{\nu}})$ one may associate a tower $\nabla_n^{*,\widehat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}$ of formal algebraic stacks, which are moduli spaces for global $\mathcal{G}$-shtukas bounded by $\widehat{Z}_{\underline{\nu}}$ with fixed characteristics $\underline{\nu}$ (also called paws by V. Lafforgue [Laf12]). This tower of formal algebraic stacks is defined over a reflex ring $R := R(\mathcal{G}, \widehat{Z}_{\underline{\nu}})$ with finite residue field $\kappa := \kappa(\mathcal{G}, \widehat{Z}_{\underline{\nu}})$ and naturally carries an action of the adelic group $G(\mathbb{A}^{\underline{\nu}})$, that operates through Hecke correspondences. These formal algebraic stacks may be viewed as the function field counterparts of integral models for Shimura varieties and the parahoric case may reflect the bad reduction at the characteristic place.

On the other hand, to such $\nabla\mathscr{H}$-data, we associate a groupoid $\mathcal{L}^*(\mathcal{G}, \widehat{Z}_{\underline{\nu}}) := \coprod_{\mathcal{M}_\mathcal{G}} \mathscr{S}(\mathcal{M}_\mathcal{G})$, where the index $\mathcal{M}_\mathcal{G}$ runs over a set of equivalence classes of $\mathcal{G}$-motives and each groupoid $\mathscr{S}(\mathcal{M}_\mathcal{G})$ is naturally equipped with an action of the adelic group $G(\mathbb{A}^{\underline{\nu}})$ which again operates through Hecke correspondences, with the action of the Frobenius Φ in $Gal(\overline{\mathbb{F}}/\kappa)$, and with the action of $Z(Q)$ where $Z \subset G$ is the center; see Definition 3.9 and Remark 3.10. In the following theorem we establish an equivalence of the functors $\mathcal{L}^*(-)$ and $\nabla_n^{*,-}\mathscr{H}^1(C,-)^{\underline{\nu}}(\overline{\mathbb{F}})$.

Theorem 3.11. (Langlands-Rapoport Conjecture For $\mathcal{G}$-shtukas) There exist a canonical $G(\mathbb{A}_Q^{\underline{\nu}}) \times Z(Q)$-equivariant isomorphism of functors

$$\mathcal{L}^*(-) \xrightarrow{\sim} \nabla_n^{*,-}\mathscr{H}^1(C,-)^{\underline{\nu}}(\mathbb{F}).$$

on the category of $\nabla\mathscr{H}$-data. Moreover via this isomorphism, the operation $\underline{\Phi}$ on the left hand side of the above isomorphism corresponds to the Frobenius endomorphism σ on the right hand side. Furthermore, when G (the generic fiber of $\mathcal{G}$) is semi-simple, one may even replace $\mathcal{L}^*(-)$ by $\mathcal{L}^*_{adm}(-)$.

For the definition of the functor $\mathcal{L}^*_{adm}(-)$ see Definition 3.9.

As an advantage of working over function fields one may observe that $\mathscr{S}(\mathcal{M}_\mathcal{G})$ are not only sets, but in fact, they are special fibers of formal algebraic stacks, which are certain quotients of Rapoport-Zink spaces for local $\mathbb{P}$-shtukas. As an application of our theorem, one may use this fact together with the theory of nearby cycles to derive a decomposition for the cohomology of the moduli stack of global $\mathcal{G}$-shtukas.

The organization of the article is as follows. In section 2 we explain how one views the moduli stack of global $\mathcal{G}$-shtukas as a moduli for motives. This is done in the following way. First, we introduce the realization categories in subsection 2.2 and crystalline (resp. étale) realization functors at characteristic places (resp. away from characteristic places) in subsection 2.3 (resp. 2.4). Then in subsection 2.5 we show that the kernel of the motivic groupoid $\mathfrak{P}$ associated with a refined version of the category of Anderson t-motives is pro-reductive, see Theorem 2.29, and further we observe that the category of $\mathcal{G}$-motives, consisting of morphisms $\mathfrak{P} \to \mathfrak{G}_\mathcal{G}$

to the neutral groupoid $\mathfrak{G}_{\mathcal{G}}$ is equivalent with the category of $\mathcal{G}$-shtukas (with morphisms enlarged to quasi-isogenies); see 2.30. In section 3 we first discuss $\nabla\mathscr{H}$-data which are the function field analogues of Shimura data. Then in subsection 3.2 we state and prove the Langlands-Rapoport conjecture for the moduli stacks of global $\mathcal{G}$-shtukas. Note however that most of the way to the proof of the main theorem 3.11 has been paved according to our results in the previous articles [AH14a] and [AH14b].

1.1 Notation and Conventions

Throughout this article we denote by

$\mathbb{F}_q$	a finite field with q elements and characteristic p,
C	a smooth projective geometrically irreducible curve over $\mathbb{F}_q$,
$Q := \mathbb{F}_q(C)$	the function field of C,
ν	a closed point of C, also called a *place* of C,
$\mathbb{F}_\nu$	the residue field at the place ν on C,
$\mathbb{F}$	a finite field containing $\mathbb{F}_q$,
$\overline{\mathbb{F}}$	an algebraic closure of $\mathbb{F}$,
$\widehat{A}_\nu$	the completion of the stalk $\mathcal{O}_{C,\nu}$ at ν,
$\widehat{Q}_\nu := \mathrm{Frac}(\widehat{A}_\nu)$	its fraction field,
$\widehat{A} := \mathbb{F}[\![z]\!]$	the ring of formal power series in z with coefficients in $\mathbb{F}$,
$\widehat{Q} := \mathrm{Frac}(\widehat{A})$	its fraction field,
$\mathbb{A}^{\underline{\nu}}$	the ring of integral adeles of C outside $\underline{\nu}$,
$\mathbb{A}_Q^{\underline{\nu}} := \mathbb{A}^{\underline{\nu}} \otimes_{\mathcal{O}_C} Q$	the ring of adeles of C outside $\underline{\nu}$,
$\mathbb{D}_R := \mathrm{Spec}\, R[\![z]\!]$	the spectrum of the ring of formal power series in z with coefficients in an $\mathbb{F}$-algebra R,
$\hat{\mathbb{D}}_R := \mathrm{Spf}\, R[\![z]\!]$	the formal spectrum of $R[\![z]\!]$ with respect to the z-adic topology.

When $R = \mathbb{F}$ we drop the subscript R from the notation of $\mathbb{D}_R$ and $\hat{\mathbb{D}}_R$.

For a formal scheme $\widehat{S}$ we denote by $\mathcal{N}ilp_{\widehat{S}}$ the category of schemes over $\widehat{S}$ on which an ideal of definition of $\widehat{S}$ is locally nilpotent. We equip $\mathcal{N}ilp_{\widehat{S}}$ with the *fppf*-topology. We also denote by

$n \in \mathbb{N}_{>0}$	a positive integer,
$\underline{\nu} := (\nu_i)_{i=1\ldots n}$	an n-tuple of closed points of C,
$\widehat{A}_{\underline{\nu}}$	the completion of the local ring $\mathcal{O}_{C^n,\underline{\nu}}$ of C^n at the closed point $\underline{\nu} = (\nu_i)$,
$\mathcal{N}ilp_{\widehat{A}_{\underline{\nu}}} := \mathcal{N}ilp_{\mathrm{Spf}\,\widehat{A}_{\underline{\nu}}}$	the category of schemes over C^n on which the ideal defining the closed point $\underline{\nu} \in C^n$ is locally nilpotent,
$\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]} := \mathcal{N}ilp_{\hat{\mathbb{D}}}$	the category of $\mathbb{D}$-schemes S for which the image of z in $\mathcal{O}_S$ is locally nilpotent. We denote the image of z by ζ since we need to distinguish it from $z \in \mathcal{O}_{\mathbb{D}}$.
$\mathbb{P}$	a smooth affine group scheme of finite type over $\mathbb{D} = \mathrm{Spec}\,\mathbb{F}[\![z]\!]$,
$P := \mathbb{P} \times_{\mathbb{D}} \mathrm{Spec}\,\widehat{Q}$	the generic fiber of $\mathbb{P}$ over $\mathrm{Spec}\,\widehat{Q}$.

Let S be an $\mathbb{F}_q$-scheme. We denote by $\sigma_S : S \to S$ its $\mathbb{F}_q$-Frobenius endomorphism which acts as the identity on the points of S and as the q-power map on the structure sheaf. Likewise we let $\hat{\sigma}_S : S \to S$ be the $\mathbb{F}$-Frobenius endomorphism of an $\mathbb{F}$-scheme S. We set

$$C_S := C \times_{\mathrm{Spec}\,\mathbb{F}_q} S \quad \text{and} \quad \sigma := \mathrm{id}_C \times \sigma_S.$$

Let H be a sheaf of groups (for the *fppf*-topology) on a scheme X. In this article a (*right*) *H-torsor* (also called an *H-bundle*) on X is a sheaf $\mathcal{G}$ for the *fppf*-topology on X together with a (right) action of the sheaf H such that $\mathcal{G}$ is isomorphic to H on an *fppf*-covering of X. Here H is viewed as an H-torsor by right multiplication.

Assume that P is a connected reductive linear algebraic group over $\widehat{Q}$. Let K be the completion of the maximal unramied extension of $\widehat{Q}$. Let $\mathcal{B} = \mathcal{B}(P, \widehat{Q})$ be the (enlarged) Bruhat-Tits building. Fix a maximal $\widehat{Q}$-split torus S. Let $\mathcal{T}$ be the centralizer (a maximal torus) of S, and let N be the normalizer of S. Denote by $W_0 = N(\widehat{Q})/\mathcal{T}(\widehat{Q})$ the relative Weyl group of G with respect to S and denote by $\mathcal{A} = \mathcal{A}(P, S, \widehat{Q})$ the apartment of $\mathcal{B}$ corresponding to S. Let $\widetilde{W} := \widetilde{W}(P, S) = N(F)/\mathcal{T}^0(\mathcal{O}_K)$ denote the Iwahori-Weyl group of P with respect to S. Here $\mathcal{T}^0$ is the unique parahoric subgroup of $\mathcal{T}(\widehat{Q})$.

$\mathcal{G}$	a parahoric (Bruhat-Tits) group scheme over C; see Definition 1.1 ,
G	the generic fiber of $\mathcal{G}$ over Q, ,
$\mathbb{P}_\nu := \mathcal{G} \times_C \operatorname{Spec} \widehat{A}_\nu$	the base change of $\mathcal{G}$ to $\operatorname{Spec} \widehat{A}_\nu$,
$P_\nu := \mathcal{G} \times_C \operatorname{Spec} \widehat{Q}_\nu$	the generic fiber of $\mathbb{P}_\nu$ over $\operatorname{Spec} \widehat{Q}_\nu$,
K_ν	the completion of the maximal unramied extension of $\widehat{Q}_\nu$,
S_ν	a maximal K_ν-split torus that is defined over $\widehat{Q}_\nu$,
$\widetilde{W}_\nu := \widetilde{W}(P_\nu, S_\nu)$	the Iwahori-Weyl group of P_ν with respect to S_ν.

Definition 1.1. A smooth affine group scheme $\mathcal{G}$ over C is called a *parahoric* (*Bruhat-Tits*) *group scheme* if

(a) all geometric fibers of $\mathcal{G}$ are connected and the generic fiber of $\mathcal{G}$ is reductive over $\mathbb{F}_q(C)$,

(b) for any ramification point ν of $\mathcal{G}$ (i.e. those points ν of C, for which the fiber above ν is not reductive) the group scheme $\mathbb{P}_\nu := \mathcal{G} \times_C \operatorname{Spec} \widehat{A}_\nu$ is a parahoric group scheme over $\widehat{A}_\nu$, as defined by Bruhat and Tits [BT72, Définition 5.2.6]; see also [HR03].

We let $\mathscr{H}^1(C, \mathcal{G})$ denote the category fibered in groupoids over the category of $\mathbb{F}_q$-schemes, such that the objects over S, $\mathscr{H}^1(C, \mathcal{G})(S)$, are $\mathcal{G}$-torsors over C_S (also called $\mathcal{G}$-bundles) and morphisms are isomorphisms of $\mathcal{G}$-torsors. The resulting stack $\mathscr{H}^1(C, \mathcal{G})$ is a smooth Artin-stack locally of finite type over $\mathbb{F}_q$; see [AH14b, Theorem 2.5].

2 Moduli Stack of $\mathcal{G}$-Shtukas as a moduli for ($\mathcal{G}-$)motives

Let us consider the category $Mot(\mathbb{F})$ of motives over $\mathbb{F}$, in the sense of [Mil92]. According to the results of Janssen [Jan92] and Deligne [Del90, Théorème 7.1] it can be seen that this category is a semi-simple tannakian category. Furthermore, when one assumes Grothendieck's standard conjectures in algebraic geometry, this category admits certain fiber functors $\omega_\ell(-)$ at $\ell \neq p, \infty$ (resp. $\ell = p$) which extend the étale cohomology (resp. crystalline cohomology) functors on the category of smooth schemes over $\mathbb{F}_q$.

Recall that according to Deligne's philosophy, Shimura varieties with rational weight, may appear as moduli varieties for motives (with additional structures) [Del79]. This conception of Shimura varieties fits the case of Shimura varieties of *abelian type*, and may fail in general case, namely for Shimura varieties attached to exceptional groups. Nevertheless, as we will see in this section, one may justify that the Deligne's conception for function field counterparts of Shimura varieties, i.e. for the moduli stack of global $\mathcal{G}$-shtukas. This can be done in the following way. We consider the category of C-motives $\mathcal{M}ot_C^{\underline{\nu}}(S)$ over S; see Definition 2.1. This category is a slight modification of the category of Anderson t-motives. We prove that this category is semi-simple tannakian category. We define the category $\mathcal{G}$-$\mathcal{M}ot_C^{\underline{\nu}}$ of C-motives with $\mathcal{G}$-structure, in analogy with the category of motives with G-structure in the sense of [Mil92]. We show that this category admits crystalline (resp. étale) fiber functors at characteristic places (away from characteristic places).

2.1 G-Shtukas and G-Motives

Let $\mathbb{F}_q$ be a finite field with q elements, let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$.

In the following definition we introduce the function field analog of $Mot(\mathbb{F})$. It is a refined version of the category of Anderson t-motives [And86].

Definition 2.1. Let S be a scheme over $\mathbb{F}_q$. The category of C-motives $\mathcal{M}ot_C^{\underline{\nu}}(S)$ with characteristic $\underline{\nu}$ over S is the category whose objects are tuples $\underline{M} := (M,\tau)$ where M is a locally free sheaf of finite rank on C_S and $\tau : \sigma^*\dot{M} \xrightarrow{\sim} \dot{M}$ is an isomorphism, where $\dot{M}$ denotes the restriction of M to $\dot{C}_S := C_S \smallsetminus \bigcup_i \nu_i \times_{\mathbb{F}_q} S$. And whose morphisms are *quasi-morphism*. A quasi-morphism (resp. quasi-isogeny) from $\underline{M} := (M,\tau)$ to $\underline{M}' := \underline{M} := (M',\tau')$ is a morphism (resp. an isomorphism) $f\colon M|_{C_S\smallsetminus D_S} \to M'|_{C_S\smallsetminus D_S}$ satisfying $\tau'\circ\sigma^* f = f\circ\tau$, for some effective divisor D on C. Notice that when $S = \operatorname{Spec} L$, for some field L over $\mathbb{F}_q$, then the category $\mathcal{M}ot_C^{\underline{\nu}}(S)$ is an abelian category.

The category $\mathcal{M}ot_C^{\underline{\nu}}(\mathbb{F})$ admits an obvious fiber functor ω over $Q.\overline{\mathbb{F}} = \overline{\mathbb{F}}(C_{\mathbb{F}})$. We let $\mathfrak{P} := \mathcal{M}ot_C^{\underline{\nu}}(\mathbb{F})(\omega)$ denote the corresponding *motivic groupoid*.

Definition 2.2. Let $\operatorname{Rep}\mathcal{G}$ denote the category of representations of $\mathcal{G}$ in finite free $\mathcal{O}_C$-modules $\mathcal{V}$. By a $\mathcal{G}$-motive (resp. G-motive) over S we mean a tensor functor $\underline{M}_{\mathcal{G}} : \operatorname{Rep}\mathcal{G} \to \mathcal{M}ot_C^{\underline{\nu}}(S)$ (resp. $\underline{M}_G : \operatorname{Rep}_Q\ G \to \mathcal{M}ot_C^{\underline{\nu}}(S)$). We say that two $\mathcal{G}$-motives (resp. G-motives) are isomorphic if they are isomorphic as tensor functors. We denote the resulting category of $\mathcal{G}$-motives (resp. G-motives) over S by $\mathcal{G}$-$\mathcal{M}ot_C^{\underline{\nu}}(S)$ (resp. G-$\mathcal{M}ot_C^{\underline{\nu}}(S)$).

Remark 2.3. Let $(\mathbf{T},\otimes)$ be a Tannakian category over a field k_0. Any two fibre functors of $\mathbf{T}$ over S become isomorphic over some faithfully flat covering of S; see [Del90, §4].

Remark 2.4. Let $\mathfrak{G}_G$ denote the neutral groupoid associated to G. According to tannakian theory, instead of working with the category G-$\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}})$ one may equivalently work with the category G-$\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}})'$ whose objects are morphism $\mathfrak{P} \to \mathfrak{G}_G$ of groupoids and whose morphisms are the natural transformations.

Definition 2.5. A global $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathscr{G}, s_1,\dots,s_n,\tau)$ consisting of a $\mathcal{G}$-bundle $\mathscr{G}$ over C_S, an n-tuple of (characteristic) sections $(s_1,\dots,s_n) \in (C)^n(S)$ and an isomorphism $\tau\colon \sigma^*\mathscr{G}|_{C_S\smallsetminus\cup_i\Gamma_{s_i}} \xrightarrow{\sim} \mathscr{G}|_{C_S\smallsetminus\cup_i\Gamma_{s_i}}$. We write $\nabla_n\mathscr{H}^1(C,\mathcal{G})$ for the *stack of global $\mathcal{G}$-shtukas*. Sometimes we fix sections $(s_1,\dots,s_n) \in C^n(S)$ and we simply call $\underline{\mathscr{G}} = (\mathscr{G},\tau)$ a global $\mathcal{G}$-shtuka over S.

Definition 2.6. Fix a tuple $\underline{\nu} := (\nu_i)_{i=1\dots n}$ of places on C with $\nu_i \neq \nu_j$ for $i \neq j$. Let $\widehat{A}_{\underline{\nu}}$ be the completion of the local ring $\mathcal{O}_{C^n,\underline{\nu}}$ of C^n at the closed point $\underline{\nu}$, and let $\mathbb{F}_{\underline{\nu}}$ be the residue field of the point $\underline{\nu}$. Then $\mathbb{F}_{\underline{\nu}}$ is the compositum of the fields $\mathbb{F}_{\nu_i}$ inside $\mathbb{F}_q^{\rm alg}$, and $\widehat{A}_{\underline{\nu}} \cong \mathbb{F}_{\underline{\nu}}[\![\zeta_1,\dots,\zeta_n]\!]$ where ζ_i is a uniformizing parameter of C at ν_i. Let the stack

$$\nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} \ :=\ \nabla_n\mathscr{H}^1(C,\mathcal{G}) \,\widehat{\times}_{C^n} \operatorname{Spf}\widehat{A}_{\underline{\nu}}$$

be the formal completion of the ind-algebraic stack $\nabla_n\mathscr{H}^1(C,\mathcal{G})$ along $\underline{\nu} \in C^n$. It is an ind-algebraic stack over $\operatorname{Spf} A_{\underline{\nu}}$ which is ind-separated and locally of ind-finite type by [AH14b, Theorem 3.14].

Definition 2.7. Let $\underline{\nu}$ be as in the above definition and let $\underline{\mathscr{G}} = (\mathscr{G},\tau)$ and $\underline{\mathscr{G}}' = (\mathscr{G}',\tau')$ be two global $\mathcal{G}$-shtukas in $\nabla_n\mathscr{H}^1(C,\mathcal{G})(S)^{\underline{\nu}}$. A *quasi-isogeny* from $\underline{\mathscr{G}}$ to $\underline{\mathscr{G}}'$ is an isomorphism $f\colon \mathscr{G}|_{C_S\smallsetminus D_S} \xrightarrow{\sim} \mathscr{G}'|_{C_S\smallsetminus D_S}$ satisfying $\tau'\sigma^*(f) = f\tau$, where D is some effective divisor on C. We denote the *group of quasi-isogenies* from $\underline{\mathscr{G}}$ to itself by $\operatorname{QIsog}_S(\underline{\mathscr{G}})$. We denote by $\nabla_n\mathscr{H}^1(C,\mathcal{G})(S)_Q^{\underline{\nu}}$ the category whose objects are the same as $\nabla_n\mathscr{H}^1(C,\mathcal{G})(S)^{\underline{\nu}}$, and whose morphisms are the quasi-isogenies between them.

Remark 2.8. The assignment

$$\mathcal{G} \mapsto \nabla_n\mathscr{H}^1(C,\mathcal{G})(S)$$

is functorial. Namely, to a given morphism of algebraic groups $\rho : \mathcal{G} \to \mathcal{G}'$, we associate the following morphism

$$\begin{array}{ccc} \nabla_n \mathscr{H}^1(C, \mathcal{G})(S) & \longrightarrow & \nabla_n \mathscr{H}^1(C, \mathcal{G}')(S) \\ \underline{\mathscr{G}} := (\mathscr{G}, (s_i)_i, \tau) & \longmapsto & \rho_* \underline{\mathscr{G}} := (\rho_* \mathscr{G} := \mathscr{G} \times^{\mathcal{G}, \rho} \mathcal{G}', (s_i)_i, \tau' := \tau \times^{\mathcal{G}, \rho} \mathrm{id}_{\mathcal{G}'}) \end{array}$$

of stacks.

There is the following functor

$$\underline{\mathcal{M}}_- : \nabla_n \mathscr{H}^1(C, \mathcal{G})(S)^{\nu}_{\underline{Q}} \to \mathcal{G} - \mathcal{M}ot^{\nu}_{\underline{C}}(S) \tag{2.1}$$

that assigns to a $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$ the functor which sends the representation ρ to $\rho_* \underline{\mathscr{G}}$.

2.2 $\mathbb{P}$-Shtukas and $\mathbb{P}$-Crystals

Let $\mathbb{F}$ be a finite field and $\widehat{A} := \mathbb{F}[\![z]\!]$ be the power series ring over $\mathbb{F}$ in the variable z. We let $\mathbb{P}$ be a smooth affine group scheme over $\mathbb{D} := \operatorname{Spec} \widehat{A}$ with connected fibers, and we let $P := \mathbb{P} \times_{\mathbb{D}} \dot{\mathbb{D}}$ be the generic fiber of $\mathbb{P}$ over $\dot{\mathbb{D}} := \operatorname{Spec} \widehat{Q}$. We are mainly interested in the situation where we have an isomorphism $\mathbb{D} \cong \operatorname{Spec} \widehat{A}_\nu$ for a place ν of C and where $\mathbb{P} = \mathbb{P}_\nu := \mathcal{G} \times_C \operatorname{Spec} \widehat{A}_\nu$.

For a scheme $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ let $\mathcal{O}_S[\![z]\!]$ be the sheaf of $\mathcal{O}_S$-algebras on S for the *fpqc*-topology whose ring of sections on an S-scheme Y is the ring of power series $\mathcal{O}_S[\![z]\!](Y) := \Gamma(Y, \mathcal{O}_Y)[\![z]\!]$. Let $\mathcal{O}_S(\!(z)\!)$ be the *fpqc*-sheaf of $\mathcal{O}_S$-algebras on S associated with the presheaf $Y \mapsto \Gamma(Y, \mathcal{O}_Y)[\![z]\!][\frac{1}{z}]$. A sheaf M of $\mathcal{O}_S[\![z]\!]$-modules on S which is free *fpqc*-locally on S is already free Zariski-locally on S by [HV11, Proposition 2.3]. We call those modules *locally free sheaves of* $\mathcal{O}_S[\![z]\!]$*-modules*. We denote by $\hat{\sigma}^*$ the endomorphism of $\mathcal{O}_S[\![z]\!]$ and $\mathcal{O}_S(\!(z)\!)$ that acts as the identity on the variable z, and is the $\mathbb{F}$-Frobenius $b \mapsto (b)^{\#\mathbb{F}}$ on local sections $b \in \mathcal{O}_S$. For a sheaf $\hat{M}$ of $\mathcal{O}_S[\![z]\!]$-modules on S we set $\hat{\sigma}^* \hat{M} := \hat{M} \otimes_{\mathcal{O}_S[\![z]\!], \hat{\sigma}^*} \mathcal{O}_S[\![z]\!]$. One may define the category of $\mathbb{D}$-crystals in the following way. Note that this category also was called the category of local shtukas, see [HV11, Definition 4.1] and [Har11, Definition 2.1.1].

Definition 2.9. (a) A $\mathbb{D}$-*iso-crystal over* S (resp. $\mathbb{D}$-*crystal* over S) is a pair $\underline{\dot{\hat{M}}} := (\dot{\hat{M}}, \hat{\tau})$ (resp. $\underline{\hat{M}} := (\hat{M}, \hat{\tau})$) consisting of a locally free sheaf $\dot{\hat{M}}$ (resp. $\hat{M}$) of $\mathcal{O}_S(\!(z)\!)$-modules (resp. $\mathcal{O}_S[\![z]\!]$-modules) of finite rank on S and an isomorphism $\hat{\tau} \colon \hat{\sigma}^* \dot{\hat{M}} \xrightarrow{\sim} \dot{\hat{M}}$ (resp. $\hat{\tau} \colon \hat{\sigma}^* \hat{M} \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!) \xrightarrow{\sim} \hat{M} \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!)$).

(b) A *morphism* (resp. *quasi-morphism*) between two $\mathbb{D}$-iso-crystals $(\dot{\hat{M}}, \hat{\tau}) \to (\dot{\hat{M}}', \hat{\tau}')$ (resp. $\mathbb{D}$-crystals $(\hat{M}, \hat{\tau}) \to (\hat{M}', \hat{\tau}')$) is a morphism $f \colon \dot{\hat{M}} \to \dot{\hat{M}}'$ (resp. $f \colon \hat{M} \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!) \to \hat{M}' \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S(\!(z)\!)$) of $\mathcal{O}_S(\!(z)\!)$-modules with $\hat{\tau}' \circ \hat{\sigma}^* f = f \circ \hat{\tau}$.

(c) A $\mathbb{D}$-iso-crystal $\underline{\dot{\hat{M}}} := (\dot{\hat{M}}, \hat{\tau})$ (resp. $\mathbb{D}$-crystal $\underline{\hat{M}} := (\hat{M}, \hat{\tau})$) is called *étale* if $\dot{\hat{\tau}}$ (resp. $\hat{\tau}$) comes from an isomorphism of $\mathcal{O}_S[\![z]\!]$-modules $\hat{\sigma}^* \hat{M} \xrightarrow{\sim} \hat{M}$.

(d) We denote by $\mathbf{ICrys}_{\mathbb{D}}(S)$ (resp. $\mathbf{Crys}_{\mathbb{D}}(S)$) the category of $\mathbb{D}$-iso-crystals (resp. $\mathbb{D}$-crystals) over S with morphisms (resp. quasi-morphism) as above and by $\text{Ét}\,\mathbf{ICrys}_{\mathbb{D}}(S)$ (resp. $\text{Ét}\,\mathbf{Crys}_{\mathbb{D}}(S)$) the full sub-category of étale $\mathbb{D}$-iso-crystals (resp. étale $\mathbb{D}$-crystals) over S.

Remark 2.10. We say that $\underline{\hat{M}} = (\hat{M}, \hat{\tau})$ is topologically nilpotent if there is an integer n such that $\hat{\tau} \circ \hat{\sigma}^* \hat{\tau} \circ \cdots \circ \hat{\sigma}^{(n-1)*} \hat{\tau}(\hat{\sigma}^{n*} \hat{M}) \subset z\hat{M}$. If S is the spectrum of a field L, every $\mathbb{D}$-crystal $\underline{\hat{M}} = (\hat{M}, \hat{\tau})$ is canonically an extension

$$0 \to \underline{\hat{M}}^{\text{ét}} \to \underline{\hat{M}} \to \underline{\hat{M}}^{\text{nil}} \to 0,$$

where $\underline{\hat{M}}^{\text{ét}}$ (resp. $\underline{\hat{M}}^{\text{nil}}$) is étale (resp. topologically nilpotent $\mathbb{D}$-crystal). If L is perfect this extension splits canonically; see [HS15, Proposition 2.8] and also [Lau96, Lemma B.3.10].

Remark 2.11. Note that when $S = \operatorname{Spec} L$, for some field extension $\mathbb{F} \subseteq L$, the category $\mathbf{Crys}_{\mathbb{D}}(S)$ of $\mathbb{D}$-crystals over S is an abelian category. We denote by $Q\operatorname{End}(\underline{\hat{M}})$ the $\mathbb{F}((z))$-algebra of the self quasi-morphisms of $\underline{\hat{M}}$.

One can assign a Galois representation to a given étale $\mathbb{D}$-crystal as follows. Assume that S is connected. Let $\bar{s}$ be a geometric point of S and let $\pi_1 := \pi_1(S, \bar{s})$ denote the algebraic fundamental group of S at $\bar{s}$. We define the (*dual*) *Tate functor* from the category of étale crystals $\text{Ét}\,\mathbf{Crys}_{\mathbb{D}}(S)$ over S to the category $\mathfrak{FMod}_{\widehat{A}[\pi_1]}$ of $\widehat{A}[\pi_1]$-modules which are finite free over $\widehat{A}$ as follows

$$
\begin{aligned}
\check{T}_{-}\colon \text{Ét}\,\mathbf{Crys}_{\mathbb{D}}(S) &\longrightarrow \mathfrak{FMod}_{\widehat{A}[\pi_1]}, \\
\underline{\hat{M}} := (\hat{M}, \hat{\tau}) &\longmapsto \check{T}_{\underline{\hat{M}}} := (\hat{M} \otimes_{\mathcal{O}_S[\![z]\!]} \kappa(\bar{s})[\![z]\!])^{\hat{\tau}} .
\end{aligned}
\tag{2.2}
$$

Here the superscript $\hat{\tau}$ denotes $\hat{\tau}$-invariants. Inverting z we also consider the *rational* (*dual*) *Tate functor*

$$
\begin{aligned}
\check{V}_{-}\colon \text{Ét}\,\mathbf{Crys}_{\mathbb{D}}(S) &\longrightarrow \mathfrak{FMod}_{\widehat{Q}[\pi_1]}, \\
\underline{\hat{M}} := (\hat{M}, \hat{\tau}) &\longmapsto \check{V}_{\underline{\hat{M}}} := (\hat{M} \otimes_{\mathcal{O}_S[\![z]\!]} \kappa(\bar{s})[\![z]\!])^{\hat{\tau}} \otimes_{\widehat{A}} \widehat{Q}.
\end{aligned}
\tag{2.3}
$$

where $\mathfrak{FMod}_{\widehat{Q}[\pi_1]}$ denotes the category of $\widehat{Q}[\pi_1]$-modules which are finite over $\widehat{Q}$. The functor $\check{V}_{-}$ transforms quasi-isogenies into isomorphisms. Sometimes we use $\mathrm{H}^1_{\acute{e}t}(-, \widehat{A})$ (resp. $\mathrm{H}^1_{\acute{e}t}(-, \widehat{Q})$) to denote the functor $\check{T}_{-}$ (resp. $\check{V}_{-}$).

Proposition 2.12. *Let $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ be connected. We have the following statements*

(a) The functor $\check{V}_{-}$ is an equivalence between the category of étale iso-crystals $\boldsymbol{ICrys}_{\mathbb{D}}(S)$ over S and the category $\mathfrak{FMod}_{\widehat{Q}[\pi_1]}$ with isomorphisms as the only morphisms.

(b) There is a canonical isomorphism of $\kappa(\bar{s})[\![z]\!]$-modules $\check{T}_{\underline{\hat{M}}} \otimes_{\mathbb{F}[\![z]\!]} \kappa(\bar{s})[\![z]\!] \xrightarrow{\sim} \underline{\hat{M}} \otimes_{\mathcal{O}_S[\![z]\!]} \kappa(\bar{s})[\![z]\!]$ which is equivariant for the action of $\pi_1(S, \bar{s})$ and $\hat{\tau}$, where $\pi_1(S, \bar{s})$ acts trivially on $\underline{\hat{M}}$ and $\hat{\tau}$ acts trivially on $\check{T}_{\underline{\hat{M}}}$.

Proof. See [AH14a, Proposition 3.4]. □

Remark 2.13. Note that when we work over general S the category $\mathbf{Crys}_{\mathbb{D}}(S)$ may fail to be abelian. Nevertheless, the category of crystals $\mathbf{Crys}_{\mathbb{D}}(\mathbb{F})$ over a finite field $\mathbb{F}$ is indeed a tannakian category admitting a fiber functor over $\widehat{Q}.\overline{\mathbb{F}}$. We denote the corresponding motivic groupoid by $\widehat{\mathfrak{P}}_{\nu}$.

In contrast with the global version in the sense of Definition 2.2, let us now discuss the local categories which are endowed with $\mathbb{P}$-structure.

Definition 2.14. By a $\mathbb{P}$-crystal (resp. étale $\mathbb{P}$-crystal, resp. P-iso-crystal) over S we mean a tensor functor $\underline{\widehat{M}}_{\mathbb{P}} : \operatorname{Rep}_{\widehat{A}} \mathbb{P} \to \mathbf{Crys}_{\mathbb{D}}(S)$ (resp. $\underline{\widehat{M}}_{\mathbb{P}} : \operatorname{Rep}_{\widehat{A}} \mathbb{P} \to \text{Ét}\,\mathbf{Crys}_{\mathbb{D}}(S)$, resp. $\underline{\widehat{M}}_{P} : \operatorname{Rep}_{\widehat{Q}} P \to \mathbf{Crys}_{\mathbb{D}}(S)$). We say that two $\mathbb{P}$-crystals (resp. étale $\mathbb{P}$-crystals, resp. P-crystals) are isomorphic if they are isomorphic as tensor functors. We denote the resulting category of $\mathbb{P}$-crystals (resp. étale $\mathbb{P}$-crystal, resp. P-crystals) over S by $\mathbb{P}$-$\mathbf{Crys}_{\mathbb{D}}(S)$ (resp. Ét $\mathbb{P}$-$\mathbf{Crys}_{\mathbb{D}}(S)$, resp P-$\mathbf{Crys}_{\mathbb{D}}(S)$).

Remark 2.15. Over $\mathbb{F}$ the category $\mathbb{P}$-$\mathbf{Crys}_{\mathbb{D}}(\mathbb{F})$ (resp. P-$\mathbf{Crys}_{\mathbb{D}}(\mathbb{F})$) is equivalent with the category $\widehat{\mathcal{M}ot}_{\mathbb{P}}(S)'$ (resp. $\widehat{\mathcal{M}ot}_{P}(\mathbb{F})'$) whose objects are morphisms $\widehat{\mathfrak{P}}_{\nu} \to \mathfrak{G}_{\mathbb{P}}$ (resp. $\widehat{\mathfrak{P}}_{\nu} \to \mathfrak{G}_{P}$) of groupoids and whose morphisms are the natural transformations.

To construct a moduli space parametrizing such local motivic objects, that are also equipped with $\mathbb{P}$-structure, one further requires a relevant geometric modification of the notion we introduced above. For this purpose, we first recall the following

Definition 2.16. The *group of positive loops associated with* $\mathbb{P}$ is the infinite dimensional affine group scheme $L^+\mathbb{P}$ over $\mathbb{F}$ whose R-valued points for an $\mathbb{F}$-algebra R are

$$L^+\mathbb{P}(R) := \mathbb{P}(R[\![z]\!]) := \mathbb{P}(\mathbb{D}_R) := \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}_R, \mathbb{P})\,.$$

The *group of loops associated with* P is the *fpqc*-sheaf of groups LP over $\mathbb{F}$ whose R-valued points for an $\mathbb{F}$-algebra R are

$$LP(R) := P(R(\!(z)\!)) := P(\dot{\mathbb{D}}_R) := \mathrm{Hom}_{\dot{\mathbb{D}}}(\dot{\mathbb{D}}_R, P)\,,$$

where we write $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$ and $\dot{\mathbb{D}}_R := \mathrm{Spec}\, R(\!(z)\!)$. It is representable by an ind-scheme of ind-finite type over $\mathbb{F}$; see [PR08, § 1.a], or [BD, §4.5], [NP01], [Fal03] when $\mathbb{P}$ is constant. Let $\mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, L^+\mathbb{P}) := [\mathrm{Spec}\,\mathbb{F}/L^+\mathbb{P}]$ (respectively $\mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, LP) := [\mathrm{Spec}\,\mathbb{F}/LP]$) denote the classifying space of $L^+\mathbb{P}$-torsors (respectively LP-torsors). It is a stack fibered in groupoids over the category of $\mathbb{F}$-schemes S whose category $\mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, L^+\mathbb{P})(S)$ consists of all $L^+\mathbb{P}$-torsors (resp. LP-torsors) on S. The inclusion of sheaves $L^+\mathbb{P} \subset LP$ gives rise to the natural 1-morphism

$$Ł\colon \mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, L^+\mathbb{P}) \longrightarrow \mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, LP),\ \mathcal{L}_+ \mapsto \mathcal{L}\,. \tag{2.4}$$

Let us recall the definition of local $\mathbb{P}$-shtukas

Definition 2.17. (a) A local $\mathbb{P}$-shtuka over $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \hat{\tau})$ consisting of an $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and an isomorphism of the associated loop group torsors $\hat{\tau}\colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$ from (2.4).

(b) A *quasi-isogeny* $f : \underline{\mathcal{L}} \to \underline{\mathcal{L}}'$ between two local $\mathbb{P}$-shtukas $\underline{\mathcal{L}} := (\mathcal{L}_+, \hat{\tau})$ and $\underline{\mathcal{L}}' := (\mathcal{L}'_+, \hat{\tau}')$ over S is an isomorphism of the associated LP-torsors $f : \mathcal{L} \to \mathcal{L}'$ satisfying $f \circ \hat{\tau} = \hat{\tau}' \circ \hat{\sigma}^* f$. We denote by $\mathrm{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}')$ the set of quasi-isogenies between $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}'$ over S, and we write $\mathrm{QIsog}_S(\underline{\mathcal{L}}) := \mathrm{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}})$ for the quasi-isogeny group of $\underline{\mathcal{L}}$. We denote by $Sht_{\mathbb{P}}^{\mathbb{D}}(S)$ (resp. $Sht_{\mathbb{P}}^{\mathbb{D}}(S)_{\widehat{Q}}$) the category of étale local $\mathbb{P}$-shtukas over S with isomorphisms (resp. quasi-isogenies) as its morphisms.

(c) A local $\mathbb{P}$-shtuka $(\mathcal{L}_+, \hat{\tau})$ is called *étale* if $\hat{\tau}$ comes from an isomorphism of $L^+\mathbb{P}$-torsors $\hat{\sigma}^*\mathcal{L}_+ \xrightarrow{\sim} \mathcal{L}_+$. We denote by $\acute{E}tSht_{\mathbb{P}}^{\mathbb{D}}(S)$ (resp. $\acute{E}tSht_{\mathbb{P}}^{\mathbb{D}}(S)_{\widehat{Q}}$) the category of étale local $\mathbb{P}$-shtukas over S with isomorphisms (resp. quasi-isogenies) as its morphisms.

Remark 2.18. There is an equivalence of categories between the category $\mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, L^+\,\mathrm{GL}_r)(S)$ and the category of locally free sheaves of $\mathcal{O}_S[\![z]\!]$-modules of rank r; see [HV11, §4]. It induces an equivalence between the category of local GL_r-shtukas over S and the category consisting of $\mathbb{D}$-crystals over S of rank r as its objects, with isomorphisms as the only morphisms; see [HV11, Lemma 4.2].

Let $Vect_{\mathbb{D}}$ be the groupoid over $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ whose S-valued points is the category of locally free sheaves of $\mathcal{O}_S[\![z]\!]$-modules with isomorphisms as the only morphisms. Let $\mathrm{Rep}_{\widehat{A}}\,\mathbb{P}$ be the category of representations $\rho\colon \mathbb{P} \to \mathrm{GL}(V)$ of $\mathbb{P}$ in finite free $\widehat{A}$-modules V, that is, ρ is a morphism of algebraic groups over $\mathbb{F}[\![z]\!]$. Any such representation ρ gives a functor

$$\rho_*\colon \mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, L^+\mathbb{P}) \to Vect_{\mathbb{D}}$$

which sends an $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ to the sheaf of $\mathcal{O}_S[\![z]\!]$-modules associated with the following presheaf

$$Y \longmapsto \Big(\mathcal{L}_+(Y) \times \big(V \otimes_{\mathbb{F}[\![z]\!]} \mathcal{O}_S[\![z]\!](Y)\big)\Big)/L^+\mathbb{P}(Y)\,. \tag{2.5}$$

The functor $\rho_*\colon \mathscr{H}^1(\mathrm{Spec}\,\mathbb{F}, L^+\mathbb{P}) \to Vect_{\mathbb{D}}$ induces a functor from the category of local $\mathbb{P}$-shtukas over S to the category of crystals over S which we likewise denote ρ_*. Therefore we obtain the following functor

$$\mathcal{D} : Sht_{\mathbb{P}}^{\mathbb{D}}(S) \to \mathbb{P}\text{-}\mathbf{Crys}_{\mathbb{D}}(S), \tag{2.6}$$

that sends $\underline{\mathcal{L}}$ to the functor which sends a representation ρ to $\rho_*\underline{\mathcal{L}}$.
Let ${}_{A[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}}$ denote the category of tensor functors $Funct^{\otimes}(\mathrm{Rep}_{\widehat{A}}\,\mathbb{P}, \mathfrak{FMod}_{\widehat{A}[\pi_1]})$. If we restrict the above functor 2.6 to the caregory of étale local $\mathbb{P}$-shtukas then this functor induces the following functor

$$\text{Ét}\, Sht_{\mathbb{P}}^{\mathbb{D}}(S) \to \text{Ét}\,\mathbb{P}\text{-}\mathbf{Crys}_{\mathbb{D}}(S) \xrightarrow{\sim} {}_{\widehat{A}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}}. \tag{2.7}$$

The right hand side equivalence comes from 2.2; see Proposition 2.12. Note that when $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$, using tannakian theory, one may observe that the functor 2.6 is an equivalence of categories; see Remark 2.3. Also for the left hand side functor of 2.7 see Proposition 2.20 bellow. Regarding this discussion we make the following

Definition 2.19. Let $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ be a connected scheme, $\bar{s}$ a geometric point of S and set $\pi_1 := \pi_1(S, \bar{s})$. We define ${}_{\widehat{A}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}} := Funct^{\otimes}(\mathrm{Rep}_{\widehat{A}}\,\mathbb{P}, \mathfrak{FMod}_{\widehat{A}[\pi_1]})$ (resp. ${}_{\widehat{Q}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}} := Funct^{\otimes}(\mathrm{Rep}_{A}\,\mathbb{P}, \mathfrak{FMod}_{\widehat{Q}[\pi_1]})$, as the category whose objects are tensor functors from $\mathrm{Rep}_{\widehat{A}}\,\mathbb{P}$ to $\mathfrak{FMod}_{\widehat{A}[\pi_1]}$ (resp. to $\mathfrak{FMod}_{\widehat{Q}[\pi_1]}$), and whose morphisms are isomorphisms of functors. We define the (*dual*) *Tate functor* $\check{\mathcal{T}}_-$, respectively the *rational* (*dual*) *Tate functor* $\check{\mathcal{V}}_-$ as the functors

$$\begin{array}{rrcl} \check{\mathcal{T}}_-\colon & \acute{E}tSht_{\mathbb{P}}^{\mathbb{D}}(S) & \longrightarrow & {}_{\widehat{A}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}}, \\ & \underline{\mathcal{L}} & \longmapsto & \check{\mathcal{T}}_{\underline{\mathcal{L}}}\colon\ \rho \mapsto \check{T}_{\rho_*\underline{\mathcal{L}}}, \\ \check{\mathcal{V}}_-\colon & \acute{E}tSht_{\mathbb{P}}^{\mathbb{D}}(S) & \longrightarrow & {}_{\widehat{Q}[\pi_1]}\widehat{\mathcal{M}od}_{P} \\ & \underline{\mathcal{L}} & \longmapsto & \check{\mathcal{V}}_{\underline{\mathcal{L}}}\colon\ \rho \mapsto \check{V}_{\rho_*\underline{\mathcal{L}}}. \end{array}$$

That $\check{\mathcal{T}}_-$ and $\check{\mathcal{V}}_-$ are indeed tensor functors, follows from the fact that $\underline{\mathcal{L}} \mapsto \rho_*\underline{\mathcal{L}}$ is a tensor functor and from the equivariant isomorphism $\check{T}_{\rho_*\underline{\mathcal{L}}} \otimes_{\mathbb{F}[\![z]\!]} \kappa(\bar{s})[\![z]\!] \xrightarrow{\sim} \rho_*\underline{\mathcal{L}} \otimes_{\mathcal{O}_S[\![z]\!]} \kappa(\bar{s})[\![z]\!]$ from Proposition 2.12. If $\underline{\mathcal{L}}$ is an étale local $\mathbb{P}$-shtuka then the composition of the tensor functor $\check{\mathcal{T}}_{\underline{\mathcal{L}}}$ followed by the forgetful functor $F\colon \mathfrak{FMod}_{\widehat{A}[\pi_1]} \to \mathfrak{FMod}_{\widehat{A}}$ is isomorphic to the forgetful fiber functor $\omega^\circ\colon \mathrm{Rep}_{\widehat{A}}\,\mathbb{P} \to \mathfrak{FMod}_{\widehat{A}}$ according to [AH14a, Corollary 2.9]. Indeed, the base change $\underline{\mathcal{L}}_{\bar{s}}$ of $\underline{\mathcal{L}}$ to $\bar{s} = \operatorname{Spec}\kappa(\bar{s})$ is isomorphic to $\underline{\mathbb{L}}_0 := \big((L^+\mathbb{P})_{\bar{s}}, 1\cdot\hat{\sigma}^*\big)$ and the functor $F \circ \check{\mathcal{T}}_{\underline{\mathbb{L}}_0}$ is isomorphic to ω°. This yields a conjugacy class of isomorphisms $\mathrm{Aut}^{\otimes}(F \circ \check{\mathcal{T}}_{\underline{\mathcal{L}}}) \cong \mathrm{Aut}^{\otimes}(\omega^\circ) = \mathbb{P}$. Since every $\gamma \in \pi_1(S, \bar{s})$ acts as a tensor automorphism of $\check{\mathcal{T}}_{\underline{\mathcal{L}}}$, the tensor functor $\check{\mathcal{T}}_{\underline{\mathcal{L}}}$ corresponds to a conjugacy class of Galois representations $\pi\colon \pi_1(S, \bar{s}) \to \mathbb{P}(\widehat{A})$. Now consider the full subcategory ${}_{\widehat{A}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P},0}$ of ${}_{\widehat{A}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}}$ consisting of those tensor functors $\mathcal{F}$ for which $F \circ \mathcal{F} \cong \omega^\circ$. Then, as we proved in [AH14a, Proposition 3.6], Proposition 2.12 generalizes as follows.

Proposition 2.20. *Let $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ be a connected scheme. Then the functor $\check{\mathcal{T}}_-$ (resp. $\check{\mathcal{V}}_-$) between the category $\acute{E}tSht_{\mathbb{P}}^{\mathbb{D}}(S)$ (resp. $\acute{E}tSht_{\mathbb{P}}^{\mathbb{D}}(S)_{\widehat{Q}}$) and the category ${}_{\widehat{A}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P},0}$ (resp. ${}_{\widehat{Q}[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}}$) is equivalence (resp. fully faithful).*

Remark 2.21. Let $\nu \in C$ be a closed point of C and set $C' := C \smallsetminus \{\nu\}$. We let $\mathscr{H}_e^1(C', \mathcal{G})$ denote the category fibered in groupoids over the category of $\mathbb{F}_q$-schemes, such that $\mathscr{H}_e^1(C', \mathcal{G})(S)$ is the full subcategory of $[C'_S/\mathcal{G}](C'_S)$ consisting of those $\mathcal{G}$-torsors over C'_S that can be extended to a $\mathcal{G}$-torsor over the whole relative curve C_S. We denote by $(\dot{\ })$ the restriction morphism

$$(\dot{\ })\colon \mathscr{H}^1(C, \mathcal{G}) \longrightarrow \mathscr{H}_e^1(C', \mathcal{G})$$

which assigns to a $\mathcal{G}$-torsor $\mathscr{G}$ over C_S the $\mathcal{G}$-torsor $\dot{\mathscr{G}} := \mathscr{G} \times_{C_S} C'_S$ over C'_S. Let $\widetilde{\mathbb{P}}_\nu := \mathrm{Res}_{\mathbb{F}_\nu/\mathbb{F}_q}\mathbb{P}_\nu$ and $\widetilde{P}_\nu := \mathrm{Res}_{\mathbb{F}_\nu/\mathbb{F}_q} P_\nu$ be the Weil restrictions. Then $\widetilde{\mathbb{P}}_\nu$ is a smooth affine group scheme over $\operatorname{Spec}\mathbb{F}_q[\![z]\!]$. Let $\mathbb{F} = \mathbb{F}_q$ and let $\hat{\widetilde{\mathbb{P}}}_\nu := \widetilde{\mathbb{P}}_\nu \widehat{\times}_{\operatorname{Spec}\mathbb{F}_q[\![z]\!]} \operatorname{Spf}\mathbb{F}_q[\![z]\!] = \mathrm{Res}_{\mathbb{F}_\nu/\mathbb{F}_q}\hat{\mathbb{P}}_\nu$ be the ν-adic completion. We write $A_\nu \cong \mathbb{F}_\nu[\![z]\!]$ for a uniformizer $z \in \mathbb{F}_q(C)$. Then for every $\mathbb{F}_q$-algebra R we have

$$\begin{array}{rclcl} A_\nu \widehat{\otimes}_{\mathbb{F}_q} R & \cong & (R \otimes_{\mathbb{F}_q} \mathbb{F}_\nu)[\![z]\!] & = & R[\![z]\!] \otimes_{\mathbb{F}_q} \mathbb{F}_\nu \quad \text{and} \\ Q_\nu \widehat{\otimes}_{\mathbb{F}_q} R & \cong & (R \otimes_{\mathbb{F}_q} \mathbb{F}_\nu)(\!(z)\!) & = & R(\!(z)\!) \otimes_{\mathbb{F}_q} \mathbb{F}_\nu\,. \end{array}$$

This implies that

$$\begin{aligned} L^+\widetilde{\mathbb{P}}_\nu(R) &= \widetilde{\mathbb{P}}_\nu(R[\![z]\!]) &= \mathbb{P}_\nu(A_\nu\widehat{\otimes}_{\mathbb{F}_q}R) \quad \text{and} \\ L\widetilde{P}_\nu(R) &= \widetilde{P}_\nu(R(\!(z)\!)) &= P_\nu(Q_\nu\widehat{\otimes}_{\mathbb{F}_q}R). \end{aligned}$$

If $\mathscr{G} \in \mathscr{H}^1(C,\mathcal{G})(S)$, its completion $\widehat{\mathscr{G}_\nu} := \mathscr{G} \widehat{\times}_{C_S}(\operatorname{Spf} A_\nu \widehat{\times}_{\mathbb{F}_q} S)$ is a formal $\hat{\mathbb{P}}_\nu$-torsor (see [AH14a, Definition 2.2] for the definition of formal torsor) over $\operatorname{Spf} A_\nu \widehat{\times}_{\mathbb{F}_q} S$. The Weil restriction $\operatorname{Res}_{\mathbb{F}_\nu/\mathbb{F}_q} \widehat{\mathscr{G}_\nu}$ is a formal $\hat{\widetilde{\mathbb{P}}}_\nu$-torsor over $\operatorname{Spf} \mathbb{F}_q[\![z]\!] \widehat{\times}_{\mathbb{F}_q} S$ and may be viewed as an $L^+\widetilde{\mathbb{P}}_\nu$-torsor over S which we denote $Ł_\nu^+(\mathscr{G})$; see [AH14a, Proposition 2.4]. We obtain the functor

$$Ł_\nu^+ \colon \mathscr{H}^1(C,\mathcal{G})(S) \longrightarrow \mathscr{H}^1(\operatorname{Spec}\mathbb{F}_q, L^+\widetilde{\mathbb{P}}_\nu)(S)\,, \quad \mathscr{G} \mapsto Ł_\nu^+(\mathscr{G})\,.$$

Finally there is a funcor

$$Ł_\nu \colon \mathscr{H}_e^1(C',\mathcal{G})(S) \longrightarrow \mathscr{H}^1(\operatorname{Spec}\mathbb{F}_q, L\widetilde{P}_\nu)(S)\,, \quad \dot{\mathscr{G}} \mapsto Ł_\nu(\dot{\mathscr{G}})$$

which sends the $\mathcal{G}$-torsor $\dot{\mathscr{G}}$ over C'_S, having some extension $\mathscr{G}$ over C_S, to the $L\widetilde{P}_\nu$-torsor $Ł(Ł_\nu^+(\mathscr{G}))$ associated with $Ł_\nu^+\mathscr{G}$ under (2.4). Notice that $Ł_\nu(\mathscr{G})$ is independent of the extension $\mathscr{G}$, and that we hence may write $Ł_\nu(\dot{\mathscr{G}}) := Ł(Ł_\nu^+(\mathscr{G}))$. The above maps assign to each $\mathcal{G}$-torsor $\mathscr{G}$ over C_S a triple $(\dot{\mathscr{G}}, Ł_\nu^+(\mathscr{G}), \varphi)$ where $\varphi \colon Ł_\nu(\dot{\mathscr{G}}) \xrightarrow{\sim} Ł(Ł_\nu^+(\mathscr{G}))$ is the canonical isomorphism of $L\widetilde{P}_\nu$-torsors. The groupoid $\mathscr{H}^1(C,\mathcal{G})(S)$ is equivalent to the category of such triples; see [AH14a, Lemma 5.1]. In other words, the following diagram of groupoids is cartesian

$$\begin{array}{ccc} \mathscr{H}^1(C,\mathcal{G}) & \xrightarrow{\;\dot{(\;)}\;} & \mathscr{H}_e^1(C',\mathcal{G}) \\ {\scriptstyle Ł_\nu^+}\big\downarrow & & \big\downarrow{\scriptstyle Ł_\nu} \\ \mathscr{H}^1(\operatorname{Spec}\mathbb{F}_q, L^+\widetilde{\mathbb{P}}_\nu) & \xrightarrow{\;Ł\;} & \mathscr{H}^1(\operatorname{Spec}\mathbb{F}_q, L\widetilde{P}_\nu)\,. \end{array}$$

2.3 Crystalline Realization Functors

Let ν be a place on C and let $\mathbb{D}_\nu := \operatorname{Spec} \widehat{A}_\nu$ and $\hat{\mathbb{D}}_\nu := \operatorname{Spf} \widehat{A}_\nu$. Let $\deg \nu := [\mathbb{F}_\nu : \mathbb{F}_q]$ and fix an inclusion $\mathbb{F}_\nu \subset \widehat{A}_\nu$. Assume that we have a section $s \colon S \to C$ which factors through $\operatorname{Spf} \widehat{A}_\nu$, that is, the image in $\mathcal{O}_S$ of a uniformizer of $\widehat{A}_\nu$ is locally nilpotent. In this case we have

$$\hat{\mathbb{D}}_\nu \widehat{\times}_{\mathbb{F}_q} S \cong \coprod_{\ell \in \mathbb{Z}/(\deg \nu)} \mathrm{V}(\mathfrak{a}_{\nu,\ell}) \cong \coprod_{\ell \in \mathbb{Z}/(\deg \nu)} \hat{\mathbb{D}}_{\nu,S}, \tag{2.8}$$

where $\hat{\mathbb{D}}_{\nu,S} := \hat{\mathbb{D}}_\nu \widehat{\times}_{\mathbb{F}_\nu} S$ and where $\mathrm{V}(\mathfrak{a}_{\nu,\ell})$ denotes the component identified by the ideal $\mathfrak{a}_{\nu,\ell} = \langle a \otimes 1 - 1 \otimes s^*(a)^{q^\ell} : a \in \mathbb{F}_\nu \rangle$. Note that σ cyclically permutes these components and thus the $\mathbb{F}_\nu$-Frobenius $\sigma^{\deg \nu} =: \hat{\sigma}$ leaves each of the components $\mathrm{V}(\mathfrak{a}_{\nu,\ell})$ stable. Also note that there are canonical isomorphisms $\mathrm{V}(\mathfrak{a}_{\nu,\ell}) \cong \hat{\mathbb{D}}_{\nu,S}$ for all ℓ.

Remark 2.22. Let us recall the following interpretation of the component $\mathrm{V}(\mathfrak{a}_{\nu,0})$. The section $s \colon S \to C$ induces an isomorphism of the component $\mathrm{V}(\mathfrak{a}_{\nu,0})$ with the formal completion $\widehat{C_S}^{\Gamma_s}$ of C_S along the graph Γ_s of s. In particular $\widehat{C_S}^{\Gamma_s}$ is canonically isomorphic to $\hat{\mathbb{D}}_{\nu,S}$. For a proof we refer the reader to [AH14a, Lemma 5.3].

Definition 2.23. We set $\mathbb{P}_{\nu_i} := \mathcal{G} \times_C \operatorname{Spec} A_{\nu_i}$ and $\hat{\mathbb{P}}_{\nu_i} := \mathcal{G} \times_C \operatorname{Spf} A_{\nu_i}$. Let $(\mathscr{G}, s_1, \ldots, s_n, \tau) \in \nabla_n \mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)$, that is, $s_i \colon S \to C$ factors through $\operatorname{Spf} \widehat{A}_{\nu_i}$. By 2.8 we may decompose

$$\mathscr{G} \widehat{\times}_{C_S} (\operatorname{Spf} A_{\nu_i} \widehat{\times}_{\mathbb{F}_q} S) \cong \coprod_{\ell \in \mathbb{Z}/(\deg \nu_i)} \mathscr{G} \widehat{\times}_{C_S} \mathrm{V}(\mathfrak{a}_{\nu_i,\ell})$$

into a finite product with components $\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})\in\mathscr{H}^1(\hat{\mathbb{D}}_{\nu_i},\hat{\mathbb{P}}_{\nu_i})$. Using [AH14a, Poposition 2.4], we view $\big(\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,0}),\tau^{\deg\nu_i}\big)$ as a local $\mathbb{P}_{\nu_i}$-shtuka over S, where $\tau^{\deg\nu_i}\colon(\sigma^{\deg\nu_i})^*\mathcal{L}_i\xrightarrow{\sim}\mathcal{L}_i$ is the $\mathbb{F}_{\nu_i}$-Frobenius on the loop group torsor $\mathcal{L}_i$ associated with $\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,0})$. We define the *crystalline realization functor* $\omega_{\nu_i}(-)$ at ν_i by

$$\begin{aligned}\omega_{\nu_i}(-)\colon\ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)&\longrightarrow\ \mathbb{P}_{\nu_i}\text{-}\mathbf{Crys}_{\mathbb{D}}(S)\,,\\ (\mathscr{G},\tau)&\longmapsto\ \mathcal{D}\big(\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,0}),\tau^{\deg\nu_i}\big)\,,\\ \omega_{\underline{\nu}}:=\prod_i\omega_{\nu_i}\colon\ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)&\longrightarrow\ \prod_i\mathbb{P}_{\nu_i}\text{-}\mathbf{Crys}_{\mathbb{D}}(S)\,;\end{aligned}\tag{2.9}$$

see 2.7.

Remark 2.24. Consider the preimages in $\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})$ of the graphs $\Gamma_{s_j}\subset C_S$ of s_j. Since $\nu_i\neq\nu_j$ for $i\neq j$ the preimage of Γ_{s_j} is empty for $j\neq i$. Also the preimage of Γ_{s_i} equals $\mathrm{V}(\mathfrak{a}_{\nu_i,0})$ and does not meet $\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})$ for $\ell\neq0$. Thus for $\ell\neq0$ the restriction of τ to $\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})$ is an isomorphism

$$\tau\times\mathrm{id}\colon\sigma^*\big(\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,\ell-1})\big)=(\sigma^*\mathscr{G})\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})\xrightarrow{\sim}\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,\ell})\,.\tag{2.10}$$

This allows to recover $(\mathscr{G},\tau)\,\widehat{\times}_{C_S}(\operatorname{Spf}A_{\nu_i}\,\widehat{\times}_{\mathbb{F}_q}\,S)$ from $\big(\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,0}),\tau^{\deg\nu_i}\big)$ via the isomorphism

$$\begin{aligned}\textstyle\prod_\ell(\tau^\ell\,\widehat{\times}\,\mathrm{id})\colon\ &\left(\textstyle\prod_\ell\sigma^{\ell*}\big(\mathscr{G}\,\widehat{\times}_{C_S}\,\mathrm{V}(\mathfrak{a}_{\nu_i,0})\big),\begin{pmatrix}0&&\tau^{\deg\nu_i}\\1&\ddots&\\&\ddots&\\&1&0\end{pmatrix}\right)\\ &\xrightarrow{\sim}(\mathscr{G},\tau)\,\widehat{\times}_{C_S}(\operatorname{Spf}A_{\nu_i}\,\widehat{\times}_{\mathbb{F}_q}\,S).\end{aligned}\tag{2.11}$$

Recall from Remark 2.21 that the Weil restriction $\operatorname{Res}_{\mathbb{F}_{\nu_i}/\mathbb{F}_q}\widehat{\mathscr{G}_\nu}$ of the torsor $\widehat{\mathscr{G}_\nu}:=\mathscr{G}\,\widehat{\times}_{C_S}(\operatorname{Spf}A_{\nu_i}\,\widehat{\times}_{\mathbb{F}_q}\,S)$ corresponds by to an $L^+\widetilde{\mathbb{P}}_{\nu_i}$-torsor $Ł^+_{\nu_i}(\mathscr{G})$ according to [AH14a, Poposition 2.4]. Then $(Ł^+_{\nu_i}(\mathscr{G}),\tau\,\widehat{\times}\,\mathrm{id})$ is a local $\widetilde{\mathbb{P}}_{\nu_i}$-shtuka over S. We call it the *local $\widetilde{\mathbb{P}}_{\nu_i}$-shtuka associated with $\underline{\mathscr{G}}$ at the place ν_i*. By equation (2.11) there is an equivalence between the category of local $\mathbb{P}_{\nu_i}$-shtukas over schemes $S\in\mathcal{N}ilp_{A_{\nu_i}}$ and the category of local $\widetilde{\mathbb{P}}_{\nu_i}$-shtukas over S for which the Frobenius τ is an isomorphism outside $\mathrm{V}(\mathfrak{a}_{\nu_i,0})$.

2.4 Etale Realization Functors

Definition 2.25. Let $\underline{\mathscr{G}}=(\mathscr{G},\tau)$ be a global $\mathcal{G}$-shtuka and let ν be a place on C outside the characteristic places ν_i. Let $Ł^+_\nu(\underline{\mathscr{G}})$ be the local $\widetilde{\mathbb{P}}_\nu$-shtuka associated with $\operatorname{Res}_{\mathbb{F}_\nu/\mathbb{F}_q}\big(\underline{\mathscr{G}}\,\widehat{\times}_{C_S}(\operatorname{Spf}A_\nu\,\widehat{\times}_{\mathbb{F}_q}\,S)\big)$ by [AH14a, Poposition 2.4]. Hence we obtain a functor

$$Ł^+_\nu(-)\colon\ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)\ \longrightarrow\ \acute{E}tSht^{\operatorname{Spec}A_\nu}_{\widetilde{\mathbb{P}}_\nu}(S)\,,\tag{2.12}$$

note that $Ł^+_\nu(\underline{\mathscr{G}})$ is étale because τ is an isomorphism at ν. We call $Ł^+_\nu(\underline{\mathscr{G}})$ the *étale local $\widetilde{\mathbb{P}}_\nu$-shtuka associated with $\underline{\mathscr{G}}$ at the place $\nu\notin\underline{\nu}$*.

Set $\widehat{A}_\nu\cong\mathbb{F}_\nu[\![z]\!]$. For every representation $\rho\colon\mathbb{P}_\nu\to\mathrm{GL}_{r,\widehat{A}_\nu}$ in $\operatorname{Rep}_{\widehat{A}_\nu}\mathbb{P}_\nu$ we consider the representation $\tilde{\rho}\in\operatorname{Rep}_{\mathbb{F}_q[\![z]\!]}\widetilde{\mathbb{P}}_\nu$ which is the composition of $\operatorname{Res}_{\mathbb{F}_\nu/\mathbb{F}_q}(\rho)\colon\widetilde{\mathbb{P}}_\nu\to\operatorname{Res}_{\mathbb{F}_\nu/\mathbb{F}_q}\mathrm{GL}_{r,\widehat{A}_\nu}$ followed by the natural inclusion $\operatorname{Res}_{\mathbb{F}_\nu/\mathbb{F}_q}\mathrm{GL}_{r,\widehat{A}_\nu}\subset\mathrm{GL}_{r\cdot[\mathbb{F}_\nu:\mathbb{F}_q],\mathbb{F}_q[\![z]\!]}$. We set $\underline{\tilde{\mathcal{L}}}=Ł^+_\nu(\underline{\mathscr{G}})$ and define $\omega^\nu(\underline{\mathscr{G}}):=\check{\mathcal{T}}_{\underline{\tilde{\mathcal{L}}}}(\tilde{\rho}):=\check{T}_{\tilde{\rho}_*\underline{\tilde{\mathcal{L}}}}$ (resp. $\omega^\nu_Q(\underline{\mathscr{G}}):=\check{\mathcal{V}}_{\underline{\tilde{\mathcal{L}}}}(\tilde{\rho}):=\check{V}_{\tilde{\rho}_*\underline{\tilde{\mathcal{L}}}}$). According to this procedure we obtain the *étale realization functor* with integral (resp. rational) coefficients

$$\begin{aligned} \omega^\nu(-)\colon\ \ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S) \ &\longrightarrow\ {}_{\widehat{A}_\nu[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}_\nu}\,, \\ (resp. \qquad \omega^\nu_Q(-)\colon\ \ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S) \ &\longrightarrow\ {}_{\widehat{Q}_\nu[\pi_1]}\widehat{\mathcal{M}od}_{\mathbb{P}_\nu}\,. \end{aligned} \tag{2.13}$$

Note that there is a canonical isomorphism of A_ν-modules $\omega^\nu(\underline{\mathscr{G}})(\rho)\cong\lim\limits_{\longleftarrow\, n}\rho_*\big(\underline{\mathscr{G}}\times_C \operatorname{Spec} A_\nu/(\nu^n)\big)^\tau$. Putting all the étale realization functors together in the adelic way, we obtain

$$\begin{aligned} \omega^{\underline{\nu}}(-)\colon\ \ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S) \ &\longrightarrow\ {}_{\mathbb{A}^{\underline{\nu}}[\pi_1]}\mathcal{M}od_{\mathcal{G}}\,, \\ (resp. \qquad \omega^{\underline{\nu}}_Q(-)\colon\ \ \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S) \ &\longrightarrow\ {}_{\mathbb{A}^{\underline{\nu}}_Q[\pi_1]}\mathcal{M}od_{\mathcal{G}}\,. \end{aligned} \tag{2.14}$$

Here ${}_{\mathbb{A}^{\underline{\nu}}[\pi_1]}\mathcal{M}od_{\mathcal{G}}$ (resp. ${}_{\mathbb{A}^{\underline{\nu}}_Q[\pi_1]}\mathcal{M}od_{\mathcal{G}}$) denote the category of tensor functors $Funct^{\otimes}(\operatorname{Rep}_{\mathbb{A}^{\underline{\nu}}}\mathcal{G},\ \mathfrak{FMod}_{\mathbb{A}^{\underline{\nu}}[\pi_1(S,\bar{s})]})$ (resp. $Funct^{\otimes}(\operatorname{Rep}_{\mathbb{A}^{\underline{\nu}}}\mathcal{G},\ \mathfrak{FMod}_{\mathbb{A}^{\underline{\nu}}_Q[\pi_1(S,\bar{s})]})$). Note that there is a canonical isomorphism of $\mathbb{A}^{\underline{\nu}}$-modules $\omega^{\underline{\nu}}(\underline{\mathscr{G}})(\rho)\cong\lim\limits_{\longleftarrow\, D}\rho_*\big(\underline{\mathscr{G}}|_{D_{\bar{s}}}\big)^\tau$, where D runs over all divisors on $C\smallsetminus\{\nu_1,\dots,\nu_n\}$.

Remark 2.26. If $\mathbb{F}_\nu\subset\mathcal{O}_S$ there also exists the decomposition (2.8) and we can associate a local $\mathbb{P}_\nu$-shtuka $\underline{\mathcal{L}}$ with $Ł^+_\nu(\underline{\mathscr{G}})$. The main difference to Definition 2.23 and Remark 2.24 is that there is no distinguished component of $\mathscr{G}\,\widehat{\times}_{C_S}(\operatorname{Spf}A_\nu\,\widehat{\times}_{\mathbb{F}_q}S)$, like the one given by the characteristic section at ν_i. But τ induces isomorphisms between all components as in (2.10). Therefore we may take any component and the associated local $\mathbb{P}_\nu$-shtuka $\underline{\widetilde{\mathcal{L}}}$. Equation (2.11) shows that over $\mathbb{F}_\nu$-schemes S we obtain an equivalence between the category of étale local $\widetilde{\mathbb{P}}_\nu$-shtukas and the category of étale local $\mathbb{P}_\nu$-shtukas.

Let us recall the following theorem from [Tam94, § 2].

Theorem 2.27. *(Tate Conjecture for $\mathcal{M}ot_C^{\underline{\nu}}(L)$) Let ν be a place on C different from the characteristic places ν_i. Let $\underline{M}$ and $\underline{M}'$ be in $\mathcal{M}ot_C^{\underline{\nu}}(L)$, where L is a finitely generated field over $\mathbb{F}_q$. Then*

$$\operatorname{Hom}(\underline{M},\underline{M}')\otimes\widehat{Q}_\nu\cong\operatorname{Hom}_{Gal(L^{sep}/L)}(\check{V}_{\underline{\hat{M}}_\nu},\check{V}_{\underline{\hat{M}}'_\nu}).$$

Theorem 2.28. *(Tate Conjecture for $\mathcal{G}$-C-Motives) Let $\underline{M}_{\mathcal{G}}$ and $\underline{M}'_{\mathcal{G}}$ be in $\mathcal{G}$-$\mathcal{M}ot_C^{\underline{\nu}}(L)$, where L is a finitely generated field. Then*

$$\operatorname{Hom}(\underline{M}_{\mathcal{G}},\underline{M}'_{\mathcal{G}})\otimes\widehat{Q}_\nu\cong\operatorname{Hom}(\omega^\nu(\underline{M}_{\mathcal{G}}),\omega^\nu(\underline{M}'_{\mathcal{G}})).$$

Proof. This follows from Theorem 2.27 and tannakian theory. □

2.5 Poincare-Weil Theorem for C-Motives

Theorem 2.29. *The category $\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}})$ with the fiber functor ω, is a semi-simple tannakian category. In particular the kernel $P:=\mathfrak{P}^\Delta$ of the corresponding motivic groupoid $\mathfrak{P}:=\mathcal{M}ot_C^{\underline{\nu}}(\omega)$ is a pro-reductive group.*

Proof. According to [AH14b, Theorem 3.14], we may suppose that a given motive $\underline{M}\in\mathcal{M}ot^{\underline{\nu}}(\overline{\mathbb{F}})$ comes from a motive over a finite extension $L/\mathbb{F}_q$, which we again denote by $\underline{M}$.
It is enough to show that after a finite extension $L\subset L'\subset\mathbb{F}$, the image $\underline{M}'$ of $\underline{M}$ under the obvious functor $\mathcal{M}ot^{\underline{\nu}}(L)\to\mathcal{M}ot^{\underline{\nu}}(L')$ is semi-simple, or equivalently the endomorphism algebra $E:=QEnd(\underline{M}')$ is a semi-simple algebra over Q.
Let ν be a place on C which is distinct from the characteristic places ν_i. Let $\underline{\hat{M}}'_\nu$ denote the crystal associated to $\omega^\nu(\underline{M}')=(\hat{M}'_\nu,\tau'_\nu)$, see Remark 2.18 and Proposition 2.12. By [Bou58, Corollaire 7.6/4] it is enough to show that after such a field extension the endomorphism algebra $E_\nu:=E\otimes\widehat{Q}_\nu=QEnd(\underline{\hat{M}}'_\nu)$ is semi-simple. Note that the last equality follows from Theorem 2.27.

By [Bou58, Corollaire the proposition 6.4/9] one can equivalently show that $F = \widehat{Q}_\nu(\pi'_\nu)$ is semi-simple, where $\pi'_\nu := \tau' \circ \cdots \circ (\hat{\sigma}^*)^{[L':\mathbb{F}_q]-1}\tau'$ is the Frobenius endomorphism of $\hat{M}'_\nu$. Take a representative matrix B_{π_ν} for $\pi_\nu \otimes 1 : End(\hat{[\underline{M}]}_\nu \otimes_{A_\nu} \widehat{Q}_\nu^{alg})$ and write B_{π_ν} in the Jordan normal form. We let L'/L to be a field extension such that $[L' : L]$ is a power of the characteristic of $\mathbb{F}_q$ and $[L' : L] \geq \operatorname{rank} M$. Clearly $B_{\pi_\nu}^{[L':L]}$ is diagonal. This represents the Frobenius endomorphism $\pi' \otimes \widehat{Q}_\nu^{alg}$ of $\hat{M}'_\nu \otimes \widehat{Q}_\nu^{alg}$. Since $\widehat{Q}_\nu^{alg}/Q$ is perfect we may argue by [Bou58, Proposition 9.2/4] that π'_ν is semi-simple, and as we mentioned above, this suffices to argue that the image of $\underline{M}'$ is semi-simple.
Now the second statement follows from [DM82, Proposition 2.23]. □

Consider the following functor

$$\underline{\mathcal{M}}_- : \nabla_n \mathscr{H}^1(C, \mathcal{G})(S)^{\underline{\nu}} \to \mathcal{M}ot_{\mathcal{G}}^{\underline{\nu}}(S) \tag{2.15}$$

that assigns to a $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$ the functor which sends the representation ρ to $\rho_*\underline{\mathscr{G}}$.

Proposition 2.30. *Let $\mathbb{F}_q \subset L$ be a field extension. Then the functor 2.15 induces an equivalence between the following categories*

(a) The category $\nabla_n \mathscr{H}^1(C, \mathcal{G})(L)_Q^{\underline{\nu}}$ of global $\mathcal{G}$-shtukas over L,

(b) the category G-$\mathcal{M}ot_C^{\underline{\nu}}(L)$.

Proof. (a)⇒(b) Fix a global $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$. Now for every representation in $\operatorname{Rep}_Q G$ we choose an extension to a representation $\rho : \mathcal{G} \to GL(\mathcal{V})$ where $\mathcal{V}$ is a vector bundle on the whole curve C and we consider $\rho_*\underline{\mathscr{G}} \in \mathcal{M}ot_C^{\underline{\nu}}(\mathbb{F})_Q$. Note that if we choose a different representation $\rho : \mathcal{G} \to GL(\mathcal{V})$, since ρ and ρ' are generically equal, therefore there is a unique quasi-isogeny $\rho_*\underline{\mathscr{G}} \to \rho'_*\underline{\mathscr{G}}$.

(b)⇒(a) We choose a lift $\operatorname{Rep} \mathcal{G} \to \mathcal{M}ot^{\underline{\nu}}(\mathbb{F})$, by tannakian formalism this yields a global $\mathcal{G}$-shtuka. One can check that different choices for the lift differ by a unique quasi-isogeny.

□

Remark 2.31. According to the Proposition 2.30 above and Remark 2.4, the étale (resp. crystalline) realization functors $\omega^\nu(-)$ (resp. $\omega_{\nu_i}(-)$) induce étale (resp. crystalline) realization $\omega^\nu(-)$ (resp. $\omega_{\nu_i}(-)$) on the category G-$\mathcal{M}ot_C^{\underline{\nu}}(L)'$.

3 Langlands-Rapoport Conjecture

Let (G, X) be a Shimura data which roughly consists of a reductive group G over $\mathbb{Z}_{(p)}$ with center Z, and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $\mathbb{S} \to G_\mathbb{R}$ for the Deligne torus $\mathbb{S}$, that satisfies certain conditions. Let E be the reflex field of (G, X), choose a p-adic completion E_ν of E, and let p^m be the cardinality of the residue field of E_ν. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$, let $B(\mathbb{F})$ be the fraction field of the ring $W(\mathbb{F})$ of Witt vectors with values in $\mathbb{F}$, and let σ^* be the p-Frobenius lift on $B(\mathbb{F})$ and $W(\mathbb{F})$. Langlands and Rapoport [LR87] construct a "pseudomotivic groupoid" $\mathscr{P}$ over $\overline{\mathbb{Q}}/\mathbb{Q}$, together with homomorphisms $\zeta_\ell : \mathfrak{G}_\ell \to \mathscr{P}$ for all primes ℓ including ∞. Here $\mathscr{P}$ is a transitive $\overline{\mathbb{Q}}/\mathbb{Q}$-groupoid that can equivalently be described as a gerb. Moreover, $\mathfrak{G}_\ell$ for $\ell \neq p, \infty$ is the trivial $\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell$-groupoid, and $\mathfrak{G}_p$ is the $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$-groupoid which is the fundamental groupoid of the category of isocrysttals over $\mathbb{F}$. Finally $\mathfrak{G}_\infty$ is the $\mathbb{C}/\mathbb{R}$-groupoid whose group of complex points is the real Weil group; see [Mil92, 3.18-3.26]. However, the existence of ζ_∞ relies on Grothendiecks standard conjectures and the remaining ζ_ℓ on the Tate conjecture (this issue has been avoided in [LR87, Kis13] by purposing a quasi-motivic Galois gerb $\mathfrak{Q}$ together with morphisms $\zeta_\ell : \mathfrak{G}_\ell \to \mathfrak{Q}(\ell)$ instead of working with the pseudomotivic groupoid $(\mathscr{P}, \zeta_\ell)$). Furthermore, to a given homomorphism $\varphi : \mathscr{P} \to \mathfrak{G}_G$

(here $\mathfrak{G}_G = G \times_{\mathbb{Q}} (Spec\overline{\mathbb{Q}} \times_{\mathbb{Q}} Spec\overline{\mathbb{Q}})$ denotes the neutral groupoid over $\overline{\mathbb{Q}}/\mathbb{Q}$) they associate a set $\mathscr{S}(\varphi)$, which carries a $G(\mathbb{A}_f^p) \times \mathbb{Z}(\mathbb{Q}_p) \times \langle \mathrm{Frob}_{p^m} \rangle$-action. The Langlands-Rapoport conjecture predicts a canonical $G(\mathbb{A}_f^p) \times \mathbb{Z}(\mathbb{Q}_p) \times \langle \mathrm{Frob}_{p^m} \rangle$-equivariant bijection

$$Sh_p(X,G)(\mathbb{F}) = \bigsqcup_{\varphi} \mathscr{S}(\varphi)$$

which is functorial in (G,X). Here φ runs over the set of equivalence classes of "admissible" morphisms $\varphi : \mathscr{P} \to \mathfrak{G}_G$ and $\mathrm{Sh}_p(G,X)$ denotes the canonical integral model of the Shimura variety

$$\varprojlim_K G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$$

in the sense of Milne [Mil92].
Let us briefly recall the construction of the set $\mathscr{S}(\varphi)$. By Tannakian theory φ corresponds to a unique tensor functor $M : (\mathrm{Rep}_Q G, \omega^\circ) \to (Mot(\mathbb{F}), \omega)$, where ω° is the forgetful fiber functor. Such M are called "G-motives. Then

$$X^p(\varphi) := X^p(M) := \mathrm{Isom}^{\otimes}(\mathbb{A}_f^p \otimes_{\mathbb{Q}} \omega^\circ, \prod_{\ell \neq p,\infty} \omega_\ell \circ M)$$

is the set of tensor isomorphisms where $\mathbb{A}_f^p$ is the finite adeles of $\mathbb{Q}$ away from the place p. At the place p, the morphism φ defines an element $b = b(\varphi) \in G(B(\mathbb{F}))$ unique up to $\hat{\sigma}^*$-conjugation. Since G is unramied at p, the Hermitian symmetric domain X in the Shimura p-datum (G,X) gives rise to a double coset $C_p \in G(W(\mathbb{F}))\backslash G(B(\mathbb{F}))/G(W(\mathbb{F}))$. Then $X_p(\varphi)$ is defined as follows

$$X_p(\varphi) := X_{C_p}(b) := \{g \in G(B(F))/G(W(F)) : g^{-1} b \hat{\sigma}^* g \in C_p\};$$

see [Mil92, Chapter 4]. One lets Z^p be the closure of $Z(\mathbb{Z}_{(p)})$ in $Z(\mathbb{A}_f^p)$, set $I_\varphi(\mathbb{Q}) := \mathrm{Aut}(\varphi)(\mathbb{Q}) := \{g \in G(\overline{\mathbb{Q}}); ad_g \circ \varphi = \varphi\} = \mathrm{Aut}^{\otimes}(M)(\mathbb{Q})$ and defines $\mathscr{S}(\varphi) := I_\varphi(\mathbb{Q}) \backslash (X^p(\varphi)/Z^p \times X_p(\varphi))$. This set is equipped with an action of $G(\mathbb{A}_f^p) \times Z(\mathbb{Q}_p) \times \langle \mathrm{Frob}_{p^m} \rangle$ where the first factor acts on $X^p(\varphi)$ and the other two on $X_p(\varphi)$.

We now return to the arithmetic function fields. Recall that (formal completion of) the moduli stack of global $\mathcal{G}$-shtukas (at certain characteristic places) may be viewed as the function field counterpart of (the canonical integral model for) Shimura varieties. In this section we state and prove this conjecture for the moduli stack of global $\mathcal{G}$-shtukas. To this purpose we first introduce the function field analogue of Shimura data.

3.1 The $\nabla\mathscr{H}$-data

First let us introduce boundedness conditions for local $\mathbb{P}$-shtukas. For this purpose we fix an algebraic closure $\mathbb{F}((\zeta))^{\mathrm{alg}}$ of $\mathbb{F}((\zeta))$. Since its ring of integers is not complete we prefer to work with finite extensions of discrete valuation rings $R/\mathbb{F}[\![\zeta]\!]$ such that $R \subset \mathbb{F}((\zeta))^{\mathrm{alg}}$. For such a ring R we denote by κ_R its residue field, and we let $\mathcal{N}ilp_R$ be the category of R-schemes on which ζ is locally nilpotent. We also set $\widehat{\mathcal{F}\ell}_{\mathbb{P},R} := \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \mathrm{Spf}\, R$ and $\widehat{\mathcal{F}\ell}_{\mathbb{P}} := \widehat{\mathcal{F}\ell}_{\mathbb{P},\mathbb{F}[\![\zeta]\!]}$. Before we can define "bounds" we need to make the following observations.

Definition 3.1. (a) For a finite extension $\mathbb{F}[\![\zeta]\!] \subset R \subset \mathbb{F}((\zeta))^{\mathrm{alg}}$ of discrete valuation rings we consider closed ind-subschemes $\hat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$. We call two closed ind-subschemes $\hat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$ and $\hat{Z}'_{R'} \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R'}$ *equivalent* if there is a finite extension of discrete valuation rings $\mathbb{F}[\![\zeta]\!] \subset \widetilde{R} \subset \mathbb{F}((\zeta))^{\mathrm{alg}}$ containing R and R' such that $\hat{Z}_R \widehat{\times}_{\mathrm{Spf}\, R} \mathrm{Spf}\, \widetilde{R} = \hat{Z}'_{R'} \widehat{\times}_{\mathrm{Spf}\, R'} \mathrm{Spf}\, \widetilde{R}$ as closed ind-subschemes of $\widehat{\mathcal{F}\ell}_{\mathbb{P},\widetilde{R}}$.

(b) Let $\hat{Z} = [\hat{Z}_R]$ be an equivalence class of closed ind-subschemes $\hat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$ and let $G_{\hat{Z}} := \{\gamma \in \mathrm{Aut}_{\mathbb{F}[\![\zeta]\!]}(\mathbb{F}((\zeta))^{\mathrm{alg}}) \colon \gamma(\hat{Z}) = \hat{Z}\}$. We define the *ring of definition* $R_{\hat{Z}}$ *of* $\hat{Z}$ as the intersection of the fixed field of $G_{\hat{Z}}$ in $\mathbb{F}((\zeta))^{\mathrm{alg}}$ with all the finite extensions $R \subset \mathbb{F}((\zeta))^{\mathrm{alg}}$ of $\mathbb{F}[\![\zeta]\!]$ over which a representative $\hat{Z}_R$ of $\hat{Z}$ exists.

For further explanation about the above definition see [AH14a, Remark 4.6 and Remark 4.7]

Definition 3.2. (a) We define a *bound* to be an equivalence class $\hat{Z} := [\hat{Z}_R]$ of closed ind-subschemes $\hat{Z}_R \subset \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$, such that for all R the ind-subscheme $\hat{Z}_R$ is stable under the left $L^+\mathbb{P}$-action on $\mathcal{F}\ell_{\mathbb{P}}$, and the special fiber $Z_R := \hat{Z}_R \widehat{\times}_{\operatorname{Spf} R} \operatorname{Spec} \kappa_R$ is a quasi-compact subscheme of $\mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spec} \kappa_R$. The ring of definition $R_{\hat{Z}}$ of $\hat{Z}$ is called the *reflex ring* of $\hat{Z}$. Since the Galois descent for closed ind-subschemes of $\mathcal{F}\ell_{\mathbb{P}}$ is effective, the Z_R arise by base change from a unique closed subscheme $Z \subset \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \kappa_{R_{\hat{Z}}}$. We call Z the *special fiber* of the bound $\hat{Z}$. It is a projective scheme over $\kappa_{R_{\hat{Z}}}$ by Remark [AH14a, Remark 4.3] and [HV11, Lemma 5.4], which implies that every morphism from a quasi-compact scheme to an ind-projective ind-scheme factors through a projective subscheme.

(b) Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}}$. Let $\mathscr{L}_+$ and $\mathscr{L}'_+$ be $L^+\mathbb{P}$-torsors over a scheme S in $\mathcal{N}ilp_{R_{\hat{Z}}}$ and let $\delta\colon \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ be an isomorphism of the associated LP-torsors. We consider an étale covering $S' \to S$ over which trivializations $\alpha\colon \mathscr{L}_+ \xrightarrow{\sim} (L^+\mathbb{P})_{S'}$ and $\alpha'\colon \mathscr{L}'_+ \xrightarrow{\sim} (L^+\mathbb{P})_{S'}$ exist. Then the automorphism $\alpha' \circ \delta \circ \alpha^{-1}$ of $(LP)_{S'}$ corresponds to a morphism $S' \to LP \widehat{\times}_{\mathbb{F}} \operatorname{Spf} R_{\hat{Z}}$. We say that δ is *bounded by* $\hat{Z}$ if for any such trivialization and for all finite extensions R of $\mathbb{F}[\![\zeta]\!]$ over which a representative $\hat{Z}_R$ of $\hat{Z}$ exists the induced morphism

$$S' \widehat{\times}_{R_{\hat{Z}}} \operatorname{Spf} R \to LP \widehat{\times}_{\mathbb{F}} \operatorname{Spf} R \to \widehat{\mathcal{F}\ell}_{\mathbb{P},R}$$

factors through $\hat{Z}_R$. Furthermore we say that a local $\mathbb{P}$-shtuka $(\mathscr{L}_+, \hat{\tau})$ is *bounded by* $\hat{Z}$ if the isomorphism $\hat{\tau}$ is bounded by $\hat{Z}$.

Remark 3.3. Note that the condition of Definition 3.2(b) is satisfied for *all* trivializations and for *all* such finite extensions R of $\mathbb{F}[\![\zeta]\!]$ if and only if it is satisfied for *one* trivialization and for *one* such finite extension [AH14a, Remark 4.9].

Definition 3.4. (a) Fix an n-tuple $\underline{\nu} := (\nu_i)$ of closed points of C. A $\nabla\mathscr{H}^{\underline{\nu}}$*-data* is a tuple $(\mathcal{G}, (\hat{Z}_{\nu_i})_i, H)$ consisting of

(i) a parahoric group scheme $\mathcal{G}$ over C,

(ii) an n-tuple $(\hat{Z}_{\nu_i})_i$ of bounds with reflex rings $R_{\hat{Z}_{\nu_i}}$,

(iii) a compact open subgroup $H \subseteq \mathcal{G}(\mathbb{A}_Q^{\underline{\nu}}) \cong Aut^{\otimes}(\omega^{\circ}_{\mathbb{A}^{\underline{\nu}}})$.

We use the shorthand $\nabla\mathscr{H}$-data when $\underline{\nu}$ is clear from the context. A morphism between two $\nabla\mathscr{H}$-data $(\mathcal{G}, (\hat{Z}_{\nu_i}), H)$ and $(\mathcal{G}', (\hat{Z}'_{\nu_i}), H')$ is a morphism $\varphi : \mathcal{G} \to \mathcal{G}'$ such that the induced morphism $\widehat{\mathcal{F}\ell}_{\mathbb{P}_{\nu_i}} \to \widehat{\mathcal{F}\ell}_{\mathbb{P}'_{\nu_i}}$ send $\hat{Z}_{\nu_i}$ to $\hat{Z}'_{\nu_i}$, for every i, and $\varphi(H) \subseteq H'$.

(b) For a compact open subgroup $H \subseteq \mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$ we define a *rational* H*-level structure* $\bar{\gamma}$ on a global $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$ over $S \in \mathcal{N}ilp_{A_{\underline{\nu}}}$ as a $\pi_1(S, \bar{s})$-invariant H-orbit $\bar{\gamma} = \gamma H$ in $\operatorname{Isom}^{\otimes}(\omega^{\circ}, \omega_{\underline{\nu}}(\underline{\mathscr{G}}))$.

(c) We denote by $\nabla_n^H\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}$ the category fibered in groupoids whose S-valued points $\nabla_n^H\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}(S)$ is the category whose objects are tuples $(\underline{\mathscr{G}}, \bar{\gamma})$, consisting of a global $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$ in $\nabla_n\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}(S)$ together with a rational H-level structure $\bar{\gamma}$, and whose morphisms are quasi-isogenies of global $\mathcal{G}$-shtukas (see Definition 2.7) that are isomorphisms at the characteristic places ν_i and are compatible with the H-level structures.

(d) Fix an n-tuple $\underline{\nu} = (\nu_i)$ of places on the curve C with $\nu_i \neq \nu_j$ for $i \neq j$ and let $\nabla\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}$ denote the formal completion of the stack $\nabla\mathscr{H}^1(C, \mathcal{G})$ at $\underline{\nu}$. Let $\hat{Z}_{\underline{\nu}} := (\hat{Z}_{\nu_i})_i$ be a tuple of closed ind-subschemes $\hat{Z}_i$

of $\widehat{\mathcal{F}\ell}_{\mathbb{P}_{\nu_i}}$ which are bounds in the sense of Definition 3.2. Let $\underline{\mathscr{G}}$ be a global $\mathcal{G}$-shtuka in $\nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)$. We say that $\underline{\mathscr{G}}$ *is bounded by* $\hat{Z}_{\underline{\nu}} := (\hat{Z}_{\nu_i})_i$ if for every i the associated local $\mathbb{P}_{\nu_i}-$shtuka $\omega_{\nu_i}(\underline{\mathscr{G}})$ is bounded by $\hat{Z}_{\nu_i}$.

(e) To a $\nabla\mathscr{H}^{\underline{\nu}}$-data $(\mathcal{G},(\hat{Z}_{\nu_i})_i,H)$ we associate a moduli stack $\nabla_n^{H,\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}$ parametrizing $\mathcal{G}$-shtukas bounded by $\hat{Z}_{\underline{\nu}}$ at place ν which are additionally equipped with a level H-structure. Furthermore this correspondence is functorial, i.e. a morphism $\varphi : (\mathcal{G},(\hat{Z}_i),H) \to (\mathcal{G}',(\hat{Z}_i'),H')$ between $\nabla\mathscr{H}$-data induces the following morphism

$$\varphi_* : \nabla_n^{H,\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} \to \nabla_n^{H',\hat{Z}'_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G}')^{\underline{\nu}}$$

(f) The group $\mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$ acts on $\nabla_n^{H,\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}$ by means of Hecke correspondences which is explicitly given as follows. Let $H,H' \subset \mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$ be compact open subgroups. Then the Hecke correspondences $\pi(g)_{H,H'}$ are given by the diagrams

$$\begin{array}{ccc} & \nabla_n^{(H'\cap g^{-1}Hg),\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} & \\ \swarrow & & \searrow \\ \nabla_n^{H',\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} & \dashleftarrow & \nabla_n^{H,\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} \end{array} \tag{3.16}$$

$$\begin{array}{ccc} & (\underline{\mathcal{G}},(H'\cap g^{-1}Hg)\gamma) & \\ \swarrow & & \searrow \\ (\underline{\mathcal{G}},H'\gamma) & & (\underline{\mathcal{G}},Hg\gamma). \end{array}$$

(g) As we may desire to think of Hecke operation as a true operation on the moduli stack of global $\mathcal{G}$-shtukas, hence we must take care of all possible level structures. Accordingly, we sometimes don't specify level structure H in the above notation for $\nabla\mathscr{H}$-data and simply write $(\mathcal{G},(\hat{Z}_{\nu_i})_i)$. We define the following functorial assignment

$$(\mathcal{G},(\hat{Z}_{\nu_i})_i) \mapsto \nabla_n^{*,\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} := \varprojlim_H \nabla_n^{H,\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}},$$

where H runs over all compact open subgroups of $\mathcal{G}(\mathbb{A}_Q^{\underline{\nu}}) \cong Aut^{\otimes}(\omega^{\circ}_{\mathbb{A}^{\underline{\nu}}})$.

Definition 3.5. A $\mathcal{G}$-shtuka $\underline{\mathscr{G}}$ is called *special $\mathcal{G}$-shtuka* if it comes from a $\mathcal{T}$-shtuka $\underline{\mathscr{T}}$, i.e. $\underline{\mathscr{G}} = i_*\underline{\mathscr{T}}$, where $\mathcal{T}$ is a torus in $\mathcal{G}$ and $i : \mathcal{T} \to \mathcal{G}$ denotes the inclusion.

Definition 3.6. Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}} = \kappa[\![\xi]\!]$ and special fiber $Z \subset \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spec}\kappa$; see Definition 3.2. Let $\underline{\mathbb{L}} = (L^+\mathbb{P}, b\hat{\sigma}^*)$ be a trivialized local $\mathbb{P}$-shtuka over a field k in $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$. Assume that b is decent with integer s and let $\ell \subset k^{\mathrm{alg}}$ be the compositum of the residue field κ of $R_{\hat{Z}}$ and the finite field extension of $\mathbb{F}$ of degree s. Then $b \in LP(\ell)$ by Remark 3.13. So $\underline{\mathbb{L}}_0$ is defined over ℓ and we may replace k by ℓ. Note that $\ell[\![\xi]\!]$ is the unramified extension of $R_{\hat{Z}}$ with residue field ℓ.
We define the associated *affine Deligne-Lusztig variety* $X_Z(\underline{\mathbb{L}})$ as the reduced closed ind-subscheme $X_Z(b) \subset \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spec} k$ whose K-valued points (for any field extension K of k) are given by

$$X_Z(\underline{\mathbb{L}})(K) := X_Z(b)(K) := \big\{g \in \mathcal{F}\ell_{\mathbb{P}}(K) \colon g^{-1}\, b\,\hat{\sigma}^*(g) \in Z(K)\big\}.$$

We set $X_{\preceq\omega}(b) := X_{\mathcal{S}(\omega)}(b)$ if Z is the affine Schubert variety $\mathcal{S}(\omega)$ with $\omega \in \widetilde{W}$, see [PR08].

Remark 3.7. Notice that the following operator

$$\Phi_b.g = \mathcal{N}b.\hat{\sigma}^m g$$

acts on $X_Z(b)$. Here $\mathcal{N}b := b.\hat{\sigma}b \ldots \hat{\sigma}^{m-1}b$ and $m = [R_{\hat{Z}} : \mathbb{F}_q[\![\zeta]\!]]$.

Let $B(P_\nu)$ be the set of $\hat{\sigma}$-conjugacy classes of elements in $P(\widehat{Q}_\nu)$. Let $B(\widetilde{W}_\nu)$ be the set of $\hat{\sigma}$-conjugacy classes in the Iwahori-Weyl group $\widetilde{W}_\nu := \widetilde{W}(P_\nu, S_\nu)$.

Definition 3.8. (a) Let $\omega \in \widetilde{W}^{\mathbb{P}} \backslash \widetilde{W} / \widetilde{W}^{\mathbb{P}}$ and let $\mathbb{F}_\omega$ be the fixed field in $\mathbb{F}^{\text{alg}}$ of $\{\gamma \in \text{Gal}(\mathbb{F}^{\text{alg}}/\mathbb{F})\colon \gamma(\omega) = \omega\}$. There is a representative $g_\omega \in LP(\mathbb{F}_\omega)$ of ω; see [AH14a, Example 4.12]. The *Schubert variety* $\mathcal{S}(\omega)$ associated with ω is the ind-scheme theoretic closure of the $L^+\mathbb{P}$-orbit of g_ω in $\mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \mathbb{F}_\omega$. It is a reduced projective variety over $\mathbb{F}_\omega$. For further details see [PR08] and [Ric10].

(b) Let $\Psi_\nu : B(\widetilde{W}_\nu) \to B(P_\nu)$ denote the natural morphism induced from the natural inclusion $N(K_\nu) \to G(K_\nu)$. We say that a global $\mathcal{G}$-shtuka $\underline{\mathscr{G}} \in \nabla_n^{\underline{w}}\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}(\mathbb{F})$ is admissible if the $\hat{\sigma}$-conjugacy class $[b_i] := [\omega_{\nu_i}(\tau_{\underline{\mathscr{G}}})] \in B(P_{\nu_i})$ lies in $\bigcup_{w' \preceq w_i} \Psi(w')$, for all $1 \leq i \leq n$. Here $\nabla_n^{\underline{w}}\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}$ denotes the stack $\nabla_n^{\hat{Z}_{\underline{\nu}}}\mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}}$ with $\hat{Z}_{\underline{\nu}} := (\mathcal{S}(w_i) \widehat{\times}_{\mathbb{F}_{\omega_i}} \text{Spf}\, \mathbb{F}_{\omega_i}[\![\zeta]\!])_i$.

Definition 3.9. Keep the above notation. Define the functor $\mathcal{L}$ (resp. $\mathcal{L}_{spe}$, resp. $\mathcal{L}_{adm}$) which sends a $\nabla\mathscr{H}-$data $(\mathcal{G}, (\hat{Z}_{\nu_i})_i, H)$ (resp. $(\mathcal{G}, (\hat{Z}_{\nu_i})_i)$) to the disjoint union $\coprod_{\underline{\mathscr{G}}} \mathscr{S}(\underline{\mathscr{G}})$ of the quotient stack

$$\mathscr{S}(\underline{\mathscr{G}}) := I(Q)(\underline{\mathscr{G}}) \backslash \prod_{\nu_i} X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}})) \times X^{\underline{\nu}}/H, \tag{3.17}$$

where $\underline{\mathscr{G}}$ runs over the quasi-isogeny classes of (resp. quasi-isogeny classes of special, resp. quasi-isogeny classes of admissible) $\mathcal{G}$-shtukas and $X^{\underline{\nu}} = \text{Isom}^{\otimes}(\omega^\circ, \omega^{\underline{\nu}}(\underline{\mathscr{G}}))$. We define the functor $\mathcal{L}^*$ (resp. $\mathcal{L}^*_{spe}$, resp. $\mathcal{L}^*_{adm}$) as the functor which sends $(\mathcal{G}, (\hat{Z}_i)_i)$ to $\varprojlim_H \mathcal{L}(\mathcal{G}, (\hat{Z}_{\nu_i})_i, H)$ (resp. $\varprojlim_H \mathcal{L}_{spe}(\mathcal{G}, (\hat{Z}_{\nu_i})_i, H)$, resp. $\varprojlim_H \mathcal{L}_{adm}(\mathcal{G}, (\hat{Z}_{\nu_i})_i, H)$).

Remark 3.10. Notice that the above stacks $\mathscr{S}(\underline{\mathscr{G}})$ are equipped with the following actions

(a) $\underline{\Phi} = (\Phi_i)$ operates on $\mathcal{L}(\mathcal{G}, (\hat{Z}_i)_i, H)$ via the action of Φ_i on $X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}}))$. Here $\Phi_i := \Phi_{b_i}$, where $b_i := \omega_{\nu_i}(\tau_{\underline{\mathscr{G}}})$; see Remark 3.7.

(b) $\mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$ operates via the Hecke correspondence. More explicitly let $H, H' \subset \mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$ be compact open subgroups. Then the Hecke correspondences $\pi(g)_{H,H'}$ are given by the diagrams

$$\begin{array}{ccc} & \prod_i X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}})) \times X^{\underline{\nu}}/(H' \cap g^{-1}Hg) & \\ \swarrow & & \searrow \\ \prod_i X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}})) \times X^{\underline{\nu}}/H' & \dashleftarrow & \prod_i X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}})) \times X^{\underline{\nu}}/H \end{array} \tag{3.18}$$

$$\begin{array}{ccc} & (x_i)_i \times h(H' \cap g^{-1}Hg) & \\ \swarrow & & \searrow \\ (x_i)_i \times hH' & & (x_i)_i \times hg^{-1}H \end{array}$$

This defines a true operation of $\mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$ on $\varprojlim_H \mathcal{L}(\mathcal{G}, (\hat{Z}_i)_i, H)$.

(c) Let Z denote the center of G. Then $Z(Q)$ operates on $\prod_{\nu_i} X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}})) \times X^{\underline{\nu}}/H$ via the factor $X_{Z_{\nu_i}}(\omega_{\nu_i}(\underline{\mathscr{G}}))$, according to the inclusion $Z(Q) \subseteq I(\underline{\mathscr{G}})_{\nu_i} := \text{QIsog}(\omega_{\nu_i}(\underline{\mathscr{G}}))$.

3.2 Langlands-Rapoport Conjecture for the Moduli Stack of Global $\mathcal{G}$-Shtukas

Theorem 3.11. *(Langlands-Rapoport Conjecture For $\mathcal{G}$-shtukas) There exist a canonical $\mathcal{G}(\mathbb{A}_Q^{\underline{\nu}}) \times Z(Q)$-equivariant isomorphism of functors*

$$\mathcal{L}^*(-) \xrightarrow{\sim} \nabla_n^{*,-}\mathscr{H}^1(C,-)^{\underline{\nu}}(\overline{\mathbb{F}}).$$

Moreover via this isomorphism, the operation $\underline{\Phi}$ on the left hand side of the above isomorphism corresponds to the Frobenius endomorphism σ on the right hand side. Furthermore, when G is semi-simple, one may replace $\mathcal{L}^*(-)$ by $\mathcal{L}^*_{adm}(-)$.

Lemma 3.12. *Let $\underline{\mathscr{G}} \in \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)$ be a global $\mathfrak{G}$-shtuka over S. Let $\underline{\mathcal{L}}_{\nu_i}(\underline{\mathscr{G}})$ be the local $\mathbb{P}_{\nu_i}$-shtuka associated with $\underline{\mathscr{G}}$ via the crystalline realization functor $\omega_{\nu_i}(-)$ at the characteristic place ν_i. Let $\tilde{f}\colon \tilde{\underline{\mathcal{L}}}'_{\nu_i} \to \underline{\mathcal{L}}_{\nu_i}(\underline{\mathscr{G}})$ be a quasi-isogeny of local $\mathbb{P}_{\nu_i}$-shtukas over S. Then there exists a unique global $\mathfrak{G}$-shtuka $\underline{\mathscr{G}}' \in \nabla_n\mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(S)$ and a unique quasi-isogeny $g = g_{\tilde{f}}\colon \underline{\mathscr{G}}' \to \underline{\mathscr{G}}$ which is the identity outside ν_i, such that the local $\mathbb{P}_{\nu_i}$-shtuka associated with $\underline{\mathscr{G}}'$ is $\tilde{\underline{\mathcal{L}}}'_\nu$ and the quasi-isogeny of local $\mathbb{P}_\nu$-shtukas induced by g is $\tilde{f}$. We denote $\underline{\mathscr{G}}'$ by $\tilde{f}^*\underline{\mathcal{G}}$.*

Proof. It is a particular case of [AH14a, Proposition 5.7]. □

Recall that a formal scheme over $k[\![\xi]\!]$ in the sense of [EGA, I_{new}, 10] is called *locally formally of finite type* if it is locally noetherian and adic and its reduced subscheme is locally of finite type over k. It is called *formally of finite type* if in addition it is quasi-compact.

Let P denote the generic fiber of $\mathbb{P}$ and let $b \in LP(k)$ for some field $k \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$. With b Kottwitz associates a slope homomorphism

$$\nu_b\colon D_{k(\!(z)\!)} \to P_{k(\!(z)\!)},$$

called Newton polygon of b; see [Kot85, 4.2]. Here D is the diagonalizable pro-algebraic group over $k(\!(z)\!)$ with character group $\mathbb{Q}$. The slope homomorphism is characterized by assigning the slope filtration of $(V \otimes_{\mathbb{F}(\!(z)\!)} k(\!(z)\!), \rho(b)\cdot(\mathrm{id}\otimes\hat{\sigma}))$ to any $\mathbb{F}(\!(z)\!)$-rational representation (V,ρ) of P; see [Kot85, Section 3]. We assume that $b \in LP(k)$ satisfies a *decency equation for a positive integer s*, that is,

$$(b\hat{\sigma})^s \;=\; s\nu_b(z)\,\hat{\sigma}^s \qquad \text{in} \quad LP(k) \ltimes \langle\hat{\sigma}\rangle. \tag{3.19}$$

Remark 3.13. Assume that $b \in LP(k)$ is decent with the integer s and let $\ell \subset k^{\mathrm{alg}}$ be the finite field extension of $\mathbb{F}$ of degree s. Then $b \in LP(\ell)$ because by (3.19) the element b has values in the fixed field of $\hat{\sigma}^s$ which is ℓ. Note that if k is algebraically closed, any $\hat{\sigma}$-conjugacy class in $LP(k)$ contains an element satisfying a decency equation; see [Kot85, Section 4].

Remark 3.14. With the element $b \in LP(k)$ one can associate a connected algebraic group J_b over $\mathbb{F}(\!(z)\!)$ which is defined by its functor of points that assigns to an $\mathbb{F}(\!(z)\!)$-algebra R the group

$$J_b(R) := \big\{ g \in P(R \otimes_{\mathbb{F}(\!(z)\!)} k(\!(z)\!))\colon g^{-1}b\hat{\sigma}(g) = b \big\}.$$

Let b satisfy a decency equation for the integer s and let F_s be the fixed field of $\hat{\sigma}^s$ in $k(\!(z)\!)$. Then ν_b is defined over F_s and $J_b \times_{\mathbb{F}(\!(z)\!)} F_s$ is the centralizer of the 1-parameter subgroup $s\nu_b$ of P and hence a Levi subgroup of P_{F_s}; see [RZ96, Corollary 1.9]. In particular $J_b(\mathbb{F}(\!(z)\!)) \subset P(F_s) \subset LP(\ell)$ where ℓ is the finite field extension of $\mathbb{F}$ of degree s.

Let $\hat{Z}$ be a bound with reflex ring $R_{\hat{Z}} = \kappa[\![\xi]\!]$ and special fiber $Z \subset \mathcal{F}\ell_{\mathbb{P}} \,\widehat{\times}_{\mathbb{F}}\, \mathrm{Spec}\,\kappa$; see Definition 3.2. Let $\underline{\mathbb{L}}_0 = (L^+\mathbb{P}, b\hat{\sigma}^*)$ be a trivialized local $\mathbb{P}$-shtuka over a field k in $\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$. Assume that b is decent with integer s and let $\ell \subset k^{\mathrm{alg}}$ be the compositum of the residue field κ of $R_{\hat{Z}}$ and the finite field extension of $\mathbb{F}$ of degree s. Then $b \in LP(\ell)$ by Remark 3.13. So $\underline{\mathbb{L}}_0$ is defined over ℓ and we may replace k by ℓ. Note that $\ell[\![\xi]\!]$ is the unramified extension of $R_{\hat{Z}}$ with residue field ℓ.

Definition 3.15. Keep the notation from above and set $\underline{\mathcal{L}}_0 := \underline{\mathbb{L}}_0$.

(a) Define the functor

$$\begin{aligned} \underline{\mathcal{M}}_{\underline{\mathbb{L}}_0}^{\hat{Z}} \colon (\mathcal{N}ilp_{\ell[\![\xi]\!]})^o &\longrightarrow \mathcal{S}ets \\ S &\longmapsto \Big\{ \text{Isomorphism classes of } (\underline{\mathscr{L}}, \bar{\delta}) \text{where} \\ &\qquad \underline{\mathscr{L}} \text{ is a local } \mathbb{P}\text{-shtuka bounded by } \hat{Z} \text{and} \\ &\qquad \bar{\delta} \colon \mathscr{L}_{\overline{S}} \to \mathbb{L}_{\nu_i, \overline{S}} \text{ is a quasi-isogeny} \Big\}. \end{aligned} \tag{3.20}$$

Note that this functor is represented by a closed ind-subscheme of $\widehat{\mathcal{F}\ell}_{\mathbb{P},\ell[\![\xi]\!]} := \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spf} \ell[\![\xi]\!]$ by [AH14a, Proposition 4.11].

(b) We define the associated *affine Deligne-Lusztig variety* as the reduced closed ind-subscheme $X_Z(b) \subset \mathcal{F}\ell_{\mathbb{P}} \widehat{\times}_{\mathbb{F}} \operatorname{Spec} \ell$ whose K-valued points (for any field extension K of ℓ) are given by

$$X_Z(b)(K) := \big\{ g \in \mathcal{F}\ell_{\mathbb{P}}(K) \colon g^{-1}\, b\, \hat{\sigma}^*(g) \in Z(K) \big\}.$$

If $\omega \in \widetilde{W}$ and $Z = \mathcal{S}(\omega)$ is the Schubert variety, we set $X_{\preceq \omega}(b) := X_{\mathcal{S}(\omega)}(b)$.

The representablity of the above functor 3.20 was studied in [HV11] and [AH14a]. Let us recall the following theorem from [AH14a, Theorem 4.18].

Theorem 3.16. *The functor $\underline{\mathcal{M}}_{\underline{\mathbb{L}}_0}^{\hat{Z}} \colon (\mathcal{N}ilp_{\ell[\![\xi]\!]})^o \to \mathcal{S}ets$ is pro-representable by a formal scheme over* $\operatorname{Spf} \ell[\![\xi]\!]$ *which is locally formally of finite type. Its underlying reduced subscheme equals $X_Z(b)$. In particular $X_Z(b)$ is a scheme locally of finite type over ℓ. The formal scheme representing $\underline{\mathcal{M}}_{\underline{\mathbb{L}}_0}^{\hat{Z}}$ is called a* bounded Rapoport-Zink space *for local $\mathbb{P}$-shtukas.*

Remark 3.17. By our assumptions $\operatorname{QIsog}_{\ell}(\underline{\mathbb{L}}_0)$ equals the group $J_b(\mathbb{F}(\!(z)\!))$ from Remark 3.14. This group acts on the functor $\underline{\mathcal{M}}_{\underline{\mathbb{L}}_0}^{\hat{Z}}$ via $g \colon (\underline{\mathcal{L}}, \bar{\delta}) \mapsto (\underline{\mathcal{L}}, g \circ \bar{\delta})$ for $g \in \operatorname{QIsog}_{\ell}(\underline{\mathbb{L}}_0)$.

Proof. of Theorem 3.11. Consider a compact open subgroup $H \subset \mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})$. Let $\underline{\mathscr{G}}_0$ be a global $\mathcal{G}$-shtuka over an algebraically closed field k with characteristic $\underline{\nu}$, bounded by $\hat{Z}_{\underline{\nu}}$. We let $I(Q)$ denote the group $\operatorname{QIsog}_k(\underline{\mathscr{G}}_0)$ of quasi-isogenies of $\underline{\mathscr{G}}_0$; see Definition 2.7. Let $(\underline{\mathbb{L}}_i)_{i=1\ldots n} := \omega_{\underline{\nu}}(\underline{\mathscr{G}}_0)$ denote the associated tuple of local $\mathbb{P}_{\nu_i}$-shtukas over k, where $\omega_{\underline{\nu}}$ is the crystalline realization functor from Definition 2.23. Let $\mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_i}$ denote the Rapoport-Zink space of $\underline{\mathbb{L}}_i$; see Definition 3.15 and Theorem 3.16. Since k is algebraically closed we may assume that all $\underline{\mathbb{L}}_i$ are trivialized and decent by [AH14a, Remark 4.10]. By Theorem 3.16 the product $\prod_i \mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_i} := \mathcal{M}_{\underline{\mathbb{L}}_1}^{\hat{Z}_1} \widehat{\times}_k \ldots \widehat{\times}_k \mathcal{M}_{\underline{\mathbb{L}}_n}^{\hat{Z}_n}$ is a formal scheme locally formally of finite type over $\operatorname{Spec} k \widehat{\times}_{\mathbb{F}_{\underline{\nu}}} \operatorname{Spf} A_{\underline{\nu}} = \operatorname{Spf} k[\![\zeta_1, \ldots, \zeta_n]\!] =: \operatorname{Spf} k[\![\underline{\zeta}]\!]$. Recall that the group $J_{\underline{\mathbb{L}}_i}(Q_{\nu_i}) = \operatorname{QIsog}_k(\underline{\mathbb{L}}_i)$ of quasi-isogenies of $\underline{\mathbb{L}}_i$ over k acts naturally on $\mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_i}$; see Remark 3.17. Especially we see that the group $I(Q)$ acts on $\prod_i \mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_i}$ via the natural morphism

$$I(Q) \longrightarrow \prod_i J_{\underline{\mathbb{L}}_i}(Q_{\nu_i}), \;\; \alpha \mapsto \big(\omega_{\nu_i}(\alpha)\big)_i =: (\alpha_i)_i. \tag{3.21}$$

Consider the following morphism

$$\prod_i \mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_{\nu_i}} \times X^{\underline{\nu}}(\underline{\mathscr{G}}_0)/H \quad \longrightarrow \quad \nabla_n^{H,\hat{Z}_{\underline{\nu}}} \mathscr{H}^1(C, \mathcal{G})^{\underline{\nu}} \widehat{\times}_{\mathbb{F}_{\underline{\nu}}} \operatorname{Spec} k,$$

which sends $(\underline{\mathcal{L}}_{\nu_i}, \delta_i)_i \times \alpha H$ to $(\delta_n^* \dots \delta_1^* \underline{\mathcal{G}}_0, \omega^{\underline{\nu}}(g_{\delta_n})^{-1} \circ \dots \circ \omega^{\underline{\nu}}(g_{\delta_1})^{-1} \circ \alpha)$; see Lemma 3.12. This induces the following morphism

$$\Theta = \Theta(\underline{\mathscr{G}}_0) \colon I(Q) \backslash \prod_i \mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_{\nu_i}} \times X^{\underline{\nu}}(\underline{\mathscr{G}}_0)/H \longrightarrow \nabla_n^{H,\hat{Z}_{\underline{\nu}}} \mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} \widehat{\times}_{\mathbb{F}_{\underline{\nu}}} \operatorname{Spec} k$$

of ind-algebraic stacks over $\operatorname{Spf} k[\![\underline{\zeta}]\!]$ which is ind-proper and formally étale. Now let $\{T_j\}$ be a set of representatives of $I(Q)$-orbits of the irreducible components of $\prod_i \mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_i} \times \mathcal{G}(\mathbb{A}_Q^{\underline{\nu}})/H$. Then the image $\Theta(T_j)$ of T_j under Θ' is closed and each $\Theta(T_j)$ intersects only finitely many others. Let $\mathcal{Z}$ denote the union of the $\Theta(T_j)$ and let $\nabla_n^{H,\hat{Z}_{\underline{\nu}}}, \mathscr{H}^1(C,\mathcal{G})_{/\mathcal{Z}}^{\underline{\nu}}$ be the formal completion of $\nabla_n^{H,\hat{Z}_{\underline{\nu}}}, \mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} \widehat{\times}_{\mathbb{F}_{\underline{\nu}}} \operatorname{Spec} k$ along $\mathcal{Z}$. According to [AH14b,], Θ induces an isomorphism of formal algebraic stacks over $\operatorname{Spf} k[\![\underline{\zeta}]\!]$

$$\Theta_{\mathcal{Z}} \colon I(Q) \backslash \prod_i \mathcal{M}_{\underline{\mathbb{L}}_i}^{\hat{Z}_{\nu_i}} \times X^{\underline{\nu}}(\underline{\mathscr{G}})/H \xrightarrow{\ \sim\ } \nabla_n^{H,\hat{Z}_{\underline{\nu}}} \mathscr{H}^1(C,\mathcal{G})_{/\mathcal{Z}}^{\underline{\nu}}.$$

This isomorphism is compatible with the action of $\mathcal{G}(\mathbb{A}_Q^{\underline{\nu}}) \times Z(Q) \times \underline{\Phi}$. To see this notice that the first factor acts through Hecke-correspondences on source (3.18) and target (3.16). Moreover since $\Phi_i(g).b_i.\Phi_i(g)^{-1} = \widehat{\sigma}^{m_i}\left(g^{-1}.b.\widehat{\sigma}(g)\right)$, where $m_i := [R_{\hat{Z}_i} : \mathbb{F}_q[\![\zeta_\nu]\!]]$, using Remark 2.21 we see that the operation $\underline{\Phi}$ on the left hand side of the above isomorphism corresponds to the Frobenius endomorphism σ on the right hand side. Now, after passing to the special fiber, the theorem follows from Theorem 3.16, Lemma 3.18 bellow and [He14, Theorem 2.1]

□

The following lemma shows that the map Θ covers the Newton stratum.

Lemma 3.18. *Let $\underline{\mathscr{G}}_0$, $\underline{\mathscr{G}}_0'$ be $\mathcal{G}$-shtukas in $\nabla_n \mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}}(\mathbb{F})$. Then the following are equivalent*

(a) $\operatorname{im} \Theta_{\underline{\mathscr{G}}_0} \cap \operatorname{im} \Theta_{\underline{\mathscr{G}}_0'} \neq \emptyset$

(b) $\operatorname{im} \Theta_{\underline{\mathscr{G}}_0} = \operatorname{im} \Theta_{\underline{\mathscr{G}}_0'}$

(c) There is a quasi-isogeny $\underline{\mathscr{G}}_0 \to \underline{\mathscr{G}}_0'$ over $\mathbb{F}$.

Proof. (a) ⇒ (b) obvious, (b)⇒ (c) follows from the fact that the image of the morphism $\Theta(\underline{\mathscr{G}}_0)$ lies in the quasi-isogeny locus of $\underline{\mathscr{G}}_0$. It remains to show that (c) implies (a).
Let $f : \underline{\mathscr{G}}_0 \to \underline{\mathscr{G}}_0'$ be a quasi-isogeny over $\mathbb{F}$. The quasi-isogeny f induces an n-tuple of quasi-isogenies $\omega_{\nu_i}(f) : \underline{\mathcal{L}}_{\nu_i} \to \underline{\mathcal{L}}_{\nu_i}'$ of local $\mathbb{P}_{\nu_i}$-shtukas and also a tensor isomorphism $\check{\mathcal{V}}_f : \check{\mathcal{V}}_{\underline{\mathscr{G}}_0} \xrightarrow{\sim} \check{\mathcal{V}}_{\underline{\mathscr{G}}_0'}$. These data in turn define the vertical arrows which make the following

$$\begin{array}{ccc}
\prod_i X_{\nu_i}(\underline{\mathcal{L}}_{\nu_i}) \times X^{\underline{\nu}}(\underline{\mathscr{G}}_0)/H & \xrightarrow{\Theta_{\underline{\mathscr{G}}_0}} & \nabla_n^{H,\hat{Z}_{\underline{\nu}}} \mathscr{H}^1(C,\mathcal{G})^{\underline{\nu}} \widehat{\times}_{\mathbb{F}_{\underline{\nu}}} \operatorname{Spec} k \\
\downarrow \uparrow & \nearrow_{\Theta_{\underline{\mathscr{G}}_0'}} & \\
\prod_i X_{\nu_i}(\underline{\mathcal{L}}_{\nu_i}') \times X^{\underline{\nu}}(\underline{\mathscr{G}}_0'.)/H & &
\end{array}$$

commutative.

□

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