J. reine angew. Math. **528** (2000), 101—148

Journal für die reine und angewandte Mathematik

On rigid-analytic Picard varieties

By *Urs Hartl* and *Werner Lütkebohmert* at Ulm[1)]

Let K be a field of any characteristic which is assumed to be complete with respect to a non-archimedean discrete valuation and let R be its valuation ring. Moreover let X_K be a *smooth* rigid space over K which is *proper and connected* and assume that there exists a K-rational point $x\colon \operatorname{Sp} K \to X_K$ of X_K. We consider the Picard functor

$$\underline{\mathrm{Pic}}_{X_K/K}\colon (\text{Smooth Rigid Spaces}) \to (\text{Sets}), \quad V_K \mapsto \underline{\mathrm{Pic}}_{X_K/K}(V_K)$$

where

$$\underline{\mathrm{Pic}}_{X_K/K}(V_K) = \left\{ \mathrm{Isoclass}(\mathscr{L},\lambda)\colon \begin{array}{l} \mathscr{L} \text{ line bundle on } X_K \times_K V_K, \\ \lambda\colon \mathscr{O}_{V_K} \xrightarrow{\sim} (x,\mathrm{id})^*\mathscr{L} \text{ isomorphism} \end{array} \right\}.$$

This is a contravariant functor. The main purpose of this paper is to show the representability of this functor under the additional assumption that X_K *admits a strict semi-stable formal model* X over the valuation ring R associated to K.

Theorem 0.1. *Under the conditions stated above, there exists a unique (up to canonical isomorphism) smooth rigid-analytic group variety* $\mathrm{Pic}_{X_K/K}$ *and a natural transformation*

$$\Theta\colon \underline{\mathrm{Pic}}_{X_K/K} \to \mathrm{Hom}_K(\_,\mathrm{Pic}_{X_K/K})$$

which is universal in the following sense: For any smooth rigid space V_K, the map

$$\Theta(V_K)\colon \underline{\mathrm{Pic}}_{X_K/K}(V_K) \xrightarrow{\sim} \mathrm{Hom}_K(V_K,\mathrm{Pic}_{X_K/K})$$

is bijective. In particular, there exists a line bundle $\mathscr{P}$ on $X_K \times \mathrm{Pic}_{X_K/K}$ and an isomorphism $\lambda_{\mathscr{P}}\colon \mathscr{O}_{\mathrm{Pic}_{X_K/K}} \xrightarrow{\sim} (x \times \mathrm{id})^*\mathscr{P}$ such that, for any smooth rigid space V_K and for any pair $(\mathscr{L},\lambda) \in \underline{\mathrm{Pic}}_{X_K/K}(V_K)$, there is a unique morphism $\varphi\colon V_K \to \mathrm{Pic}_{X_K/K}$ and a unique isomorphism $(\mathscr{L},\lambda) \xrightarrow{\sim} (\mathrm{id} \times \varphi)^*(\mathscr{P},\lambda_{\mathscr{P}})$.

[1)] The authors wish to thank the Deutsche Forschungsgemeinschaft for supporting this research.

The identity component $\mathrm{Pic}^0_{X_K/K}$ *of* $\mathrm{Pic}_{X_K/K}$ *is called the Picard variety of* X_K *and* $\mathscr{P}$ *is called the Poincaré bundle. After a finite base field extension the Picard variety is an extension of an abeloid variety by an affine torus; an abeloid variety is a smooth rigid group variety with proper underlying space.*

The Galois module $\mathrm{Pic}_{X_K/K}(\mathbb{K})/\mathrm{Pic}^0_{X_K/K}(\mathbb{K})$ *where* $\mathbb{K}$ *is the completion of the algebraic closure of* K *is called the Néron-Severi group of* X_K. *The Néron-Severi group of* X_K *is finitely generated.*

Remark 0.1.1. The proof makes use of the assumption that X_K admits a semi-stable regular model X over $\mathrm{Spf}(R)$. Presumably this hypothesis is not necessary. There is still the conjecture that any quasi-compact smooth rigid space admits such a model after a suitable base ring extension. If the residue characteristic is zero, one can apply the stable reduction theorem of Mumford; cf. [TE], p. 198 which provides the good model. So the result is valid without further conditions in this case.

Remark 0.1.2. In the complex analytic case, the Picard variety is obtained via the exponential exact sequence

$$1 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$$

as a quotient

$$\mathrm{Pic}^0_{X/\mathbb{C}} = \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \mathbb{Z}_X).$$

Due to a theorem of Blanchard [B] the group $\mathrm{H}^1(X, \mathbb{Z}_X)$ is a lattice in $\mathrm{H}^1(X, \mathcal{O}_X)$ which may happen to be of rank less than $2 \cdot \dim \mathrm{H}^1(X, \mathcal{O}_X)$. So $\mathrm{Pic}^0_{X/\mathbb{C}}$ may happen not to be a compact complex analytic torus. As reference we cite [TCGA], n^{0}16.

Remark 0.1.3. (1) If X_K is the analytification of a proper algebraic variety, the rigid-analytic Picard variety is the analytification of the classical algebraic Picard variety, due to the GAGA-principle; cf. [L1], 2.8.

(2) In the case of (1) the connected components of the Picard variety are proper as X_K is assumed to be smooth; cf. [FGA], n^{0}236, Theorem 2.1, or [BLR], 8.4/3.

(3) If X_K is a smooth connected proper group variety, the representability of $\underline{\mathrm{Pic}}_{X_K/K}$ is established in [L2] resp. [BL]. In particular, it is shown that $\mathrm{Pic}^0_{X_K/K}$ is smooth and proper and that the Néron-Severi group of X_K is finitely generated and torsion free.

(4) There are examples of (non-algebraic) proper smooth rigid varieties where the Picard variety is not proper; cf. [Ms]. Let us briefly review the example. This is an analogue of the Hopf surface. It can be defined in the following way:

$$X_K = (\mathbb{A}^2_K - \{0\})/\Gamma$$

where $\Gamma = \langle \gamma \rangle$ is generated by a single element γ which acts on $\mathbb{A}^2_K - \{0\}$ by

$$\gamma(\xi_1, \xi_2) = (\alpha_1 \xi_1 + \beta \xi_2^m, \alpha_2 \xi_2)$$

where $\alpha_1, \alpha_2 \in K^*$ with $0 < |\alpha_1| \leqq |\alpha_2| < 1$ and $\beta = 0$ if $\alpha_1 \neq \alpha_2^m$ and otherwise $\beta \in K$. One can show that X_K is smooth and proper and that it admits a strict semi-stable formal model. One computes that $\mathrm{Pic}_{X_K/K} = \mathbb{G}_{m,K}$.

Let us fix the notation for this article:

R complete discrete valuation ring,

K its field of fractions,

k its residue field,

π its uniformizing parameter,

S $= \mathrm{Spf}(R)$.

We are going to consider only rigid spaces X_K over K which in general are assumed to be smooth, quasi-compact and separated. Such a space has a flat model over $S = \mathrm{Spf}(R)$, usually denoted by X. As usual X_{rig} is the associated rigid space of X; cf. [FRG]. So, if X is a model of X_K we have an isomorphism $X_K \cong X_{\mathrm{rig}}$. If X_K is proper, then X is proper over $\mathrm{Spf}(R)$ and vice versa; cf. [L1].

We would like to thank the referee for his helpful comments. Further we would like to thank Michel Raynaud for valuable discussions on this topic.

1. Semi-stable formal schemes

In this section we will repeat the basic notions of semi-stable formal schemes and show some elementary properties like smooth fibration over polyannuli. Then we will consider the group of rig-invertible functions on such a scheme and finally we will discuss the desingularization of products of such schemes.

1.1. Definitions. Analogous to A. J. de Jong [dJ], Definition 2.16, we make the following

Definition 1.1. Let X be an admissible formal R-scheme and let X_0^σ for $\sigma = 1, \ldots, s$ be the irreducible components of the special fiber X_0 of X. For $M \subseteqq N := \{1, \ldots, s\}$ we define

$$X_0^M := \bigcap_{\sigma \in M} X_0^\sigma$$

as the scheme-theoretic intersection. X is called *strict semi-stable over R* if

(a) X_{rig} is smooth over K,

(b) X_0 is geometrically reduced,

(c) X_0^σ is a Cartier divisor on X for all $\sigma \in N$ and

(d) X_0^M is smooth over k for all $M \subseteqq N$ and equidimensional of dimension $\dim X - \# M$.

Remark 1.1.1. Condition (a) is in fact a consequence of the other three conditions; cf. Proposition 1.3.

Lemma 1.2. *Let X be a strict semi-stable admissible formal R-scheme and $x \in X_0$ a point lying on the irreducible components $X_0^{i_1}, \dots, X_0^{i_{s'}}$ and not on the other components of X_0. Then there exists an open affine neighborhood* $\mathrm{Spf}(A)$ *of x such that the completion of A with respect to the ideal I corresponding to $X_0^{\{i_1,\dots,i_{s'}\}}$ is of the form*

$$\widehat{A^I} \cong C[\![\xi_{i_1}, \dots, \xi_{i_{s'}}]\!]/(\xi_{i_1} \cdot \dots \cdot \xi_{i_{s'}} - \pi)$$

for a smooth admissible formal R-algebra C where ξ_σ are generators of the ideal associated to the Cartier divisor X_o^σ for $\sigma \in \{i_1, \dots, i_{s'}\}$.

Proof (cf. [dJ], 2.16). Let $\mathrm{Spf}(A)$ be an open neighborhood of x such that the Cartier divisor X_0^σ is principal on $\mathrm{Spf}(A)$ and generated by $\xi_\sigma \in A$ and such that $C_0 := A/I$ is integral where $I := (\xi_{i_1}, \dots, \xi_{i_{s'}})$. By (d) the subscheme C_0 is smooth over k and can therefore be lifted to a smooth admissible formal R-scheme C. Due to the smoothness we can lift the identity

$$C[\![\xi_{i_1}, \dots, \xi_{i_{s'}}]\!]/(\xi_{i_1}, \dots, \xi_{i_{s'}}, \pi) = C_0 = A/I$$

to a surjective morphism

$$C[\![\xi_{i_1}, \dots, \xi_{i_{s'}}]\!] \to \widehat{A^I}.$$

There is an equation $\xi_{i_1} \cdot \dots \cdot \xi_{i_{s'}} = u \cdot \pi$ on A. After replacing ξ_{i_1} by $u \cdot \xi_{i_1}$, we may assume $u = 1$ and hence we obtain a morphism

$$C[\![\xi_{i_1}, \dots, \xi_{i_{s'}}]\!]/(\xi_1 \cdot \dots \cdot \xi_{i_{s'}} - \pi) \xrightarrow{\sim} \widehat{A^I}$$

which is an isomorphism by reasons of dimension; use condition (d). □

This description leads to

Proposition 1.3. *Let X be an admissible formal R-scheme. The following are equivalent*:

(a) *X is strict semi-stable.*

(b) *Every closed point $x \in X_0$ of the special fiber admits an open neighborhood which for some $r \in \mathbb{N}$ is formally smooth over the formal scheme*

$$\mathrm{Spf}\, R\langle \xi_{i_1}, \dots, \xi_{i_r} \rangle/(\xi_{i_1} \cdot \dots \cdot \xi_{i_r} - \pi).$$

Proof. (a) ⇒ (b) follows from Lemma 1.2 and (b) ⇒ (a) is obvious. □

Definition 1.4. An affine strict semi-stable formal R-scheme $V = \mathrm{Spf}(A)$ is called *small* if it is formally smooth over $\mathrm{Spf}\, R\langle\zeta_1, \dots, \zeta_t\rangle/(\zeta_1 \cdot \ldots \cdot \zeta_t - \pi)$, if all strata V_0^M are geometrically irreducible for all non-empty subsets M of irreducible components of V_0 and if there is a common point on all these V_0^M.

Remark 1.4.1. If the residue field k of R is separably closed, then every point of a strict semi-stable formal R-scheme admits an open neighborhood which is small.

1.2. rig-invertible functions on semi-stable formal schemes. Concerning the extension of line bundles from X_{rig} to the formal model X we need to know the group of rig-invertible functions on strict semi-stable models. Completely analogous to the one-dimensional case [BGR], Lemma 9.7.1.1 one shows

Lemma 1.5. *Let* $\mathrm{Sp}(A_K)$ *be a connected affinoid variety over* K *and* $\zeta_1, \dots, \zeta_{r+1}$ *be variables. Let further be* $c \in A_K^\times$ *with* $|c| \leqq 1$ *and*

$$f = \sum_{m \in \mathbb{Z}^r} a_m \cdot \zeta_1^{m_1} \cdot \ldots \cdot \zeta_r^{m_r} \in B_K := A_K\langle\zeta_1, \dots, \zeta_{r+1}\rangle/(\zeta_1 \cdot \ldots \cdot \zeta_{r+1} - c).$$

f is a unit in B_K *if and only if there is a multi-index* $n \in \mathbb{Z}^r$ *such that the term* $a_n\zeta^n$ *is dominant on* $\mathrm{Sp}\, B_K$, *i.e. if* $a_n \in A_K^\times$ *is a unit and*

$$|(a_m \cdot a_n^{-1} \cdot \zeta^{m-n})(y)| < 1$$

for all $m \neq n$ *and for all* $y \in \mathrm{Sp}(B_K)$.

Proposition 1.6. *Assume that the residue field* k *of* R *is separably closed. Let* X *be a strict semi-stable admissible formal* R*-scheme such that the ideal associated to each component* X_0^σ *is principal with generator* ξ_σ *for* $\sigma = 1, \dots, s$. *Consider a finite extension* $R \to \tilde{R}$ *of discrete valuation rings. Let* $\tilde{V}$ *be a strict semi-stable admissible formal* $\tilde{R}$*-scheme such that the ideal associated to each component* $\tilde{V}_0^\tau$ *is principal with generator* ζ_τ *for* $\tau = 1, \dots, t$. *Let* $W = \mathrm{Spf}(A)$ *be an open affine subscheme of* $X \times_R \tilde{V}$. *Assume that there is a point* $w \in W_0$ *lying on all irreducible components of the special fiber* W_0 *of* W. *Then the group of* rig-*invertible functions on* W *can be described in the following way*:

$$\mathcal{O}_{X \times \tilde{V}}(W_{\mathrm{rig}})^\times = \mathcal{O}_{X \times \tilde{V}}(W)^\times \oplus \left((\xi_{i_1}^{\mathbb{Z}} \oplus \cdots \oplus \xi_{i_{s'}}^{\mathbb{Z}}) + (\zeta_{j_1}^{\mathbb{Z}} \oplus \cdots \oplus \zeta_{j_{t'}}^{\mathbb{Z}})\right)$$

where $X_0^{i_1}, \dots, X_0^{i_{s'}}$ *are the components of* X_0 *meeting* W *resp.* $\tilde{V}_0^{j_1}, \dots, \tilde{V}_0^{j_{t'}}$ *are the components of* $\tilde{V}_0$ *meeting* W.

Remark 1.6.1. If there exists a point $x \in X_0$ resp. $v \in \tilde{V}_0$ lying on all irreducible components of X_0 resp. $\tilde{V}_0$, then there exists also a point $w \in W := (X \times_R \tilde{V})$ satisfying the analogous condition on W. Every point $x \in X_0$ resp. $v \in \tilde{V}_0$ has a neighborhood such that x resp. v lies on all irreducible components of that neighborhood.

Proof of 1.6. Since the residue field k of R is separably closed, all irreducible schemes of finite type over k are geometrically irreducible. Let $x \in X_0$ resp. $v \in \tilde{V}_0$ be the projection of the point w. We may assume that x and v satisfy the analogous condition on X resp. on V as w on W. Let $I := (\xi_{i_1}, \dots, \xi_{i_{s'}}) \subset A$ resp. $J := (\zeta_{j_1}, \dots, \zeta_{j_{t'}}) \subset A$ be the ideals generated by the functions defining the irreducible components meeting W. As in Lemma 1.2 one shows that the closed subscheme $\mathrm{Spec}(C_0) := \mathrm{V}(I,J)$ is smooth over $\tilde{k}$ and it lifts to a smooth formal $\tilde{R}$-algebra C. Furthermore we may assume that C is integral. There is an isomorphism

$$C[\![\xi_{i_1},\dots,\xi_{i_{s'}},\zeta_{j_1},\dots,\zeta_{j_{t'}}]\!]/(\xi_{i_1}\cdot\ldots\cdot\xi_{i_r}-\pi,\zeta_{j_1}\cdot\ldots\cdot\zeta_{j_{t'}}-\tilde{\pi}) \xrightarrow{\sim} \hat{A}^{(I,J)}$$

of the formal power series ring to the completion of A with respect to (I,J). Then we can look at the tubular neighborhood

$$W_{\mathrm{rig}}]I,J[:=\left\{x\in W_{\mathrm{rig}};\ \begin{matrix}|\xi_{i_1}(x)|<1,\dots,|\xi_{i_{s'}}(x)|<1,\\ |\zeta_{j_1}(x)|<1,\dots,|\zeta_{j_{t'}}(x)|<1\end{matrix}\right\}.$$

Its ring of rigid-analytic functions is given by

$$\big(C[\![\xi_{i_1},\dots,\xi_{i_{s'}},\zeta_{j_1},\dots,\zeta_{j_{t'}}]\!]/(\xi_{i_1}\cdot\ldots\cdot\xi_{i_{s'}}-\pi,\zeta_{j_1}\cdot\ldots\cdot\zeta_{j_{t'}}-\tilde{\pi})\big)\otimes_R K$$

which is isomorphic to

$$\big((C[\![\zeta_{j_1},\dots,\zeta_{j_{t'}}]\!]/(\zeta_{j_1}\cdot\ldots\cdot\zeta_{j_{t'}}-\tilde{\pi}))[\![\xi_{i_1},\dots,\xi_{i_{s'}}]\!]/(\xi_{i_1}\cdot\ldots\cdot\xi_{i_{s'}}-\pi)\big)\otimes_R K.$$

Now consider a rig-invertible function f on W. Due to Lemma 1.5 there exists a rig-invertible function $g\in\big(C[\![\zeta_{j_1},\dots,\zeta_{j_{t'}}]\!]/(\zeta_{j_1}\cdot\ldots\cdot\zeta_{j_{t'}}-\tilde{\pi})\big)\otimes_R K$ and a multi-index $m\in\mathbb{Z}^{s'}$ such that $f\cdot g\cdot\xi^m$ is invertible in the formal ring. Applying the same procedure to g we obtain a further multi-index $n\in\mathbb{Z}^{t'}$ such that $f\cdot\xi^m\cdot\zeta^n$ is a unit in the formal ring

$$f\cdot\xi^m\cdot\zeta^n\in\big(C[\![\xi_{i_1},\dots,\xi_{i_{s'}},\zeta_{j_1},\dots,\zeta_{j_{t'}}]\!]/(\xi_{i_1}\cdot\ldots\cdot\xi_{i_{s'}}-\pi,\zeta_{j_1}\cdot\ldots\cdot\zeta_{j_{t'}}-\tilde{\pi})\big)^{\times}.$$

Namely, any invertible function $a\in C_K$ is of type $a=\alpha\cdot\tilde{\pi}^{\nu}$ with $\alpha\in C$ and $\nu\in\mathbb{Z}$ as C is irreducible; then use the relation $\tilde{\pi}^{\nu}=(\zeta_{j_1}\cdot\ldots\cdot\zeta_{j_{t'}})^{\nu}$. Since w lies on all irreducible components, one shows that the rig-analytic function $f\cdot\xi^m\cdot\zeta^n$ on W_{rig} takes the sup-norm 1 on each irreducible component of W_0. Thus we see that $f\cdot\xi^m\cdot\zeta^n$ is a formal function on W. The same argument applies to its inverse. Thus $f\cdot\xi^m\cdot\zeta^n$ is an invertible formal function on W. □

This proposition implies the following description of the group of Cartier divisors with support in the special fiber

$$\mathrm{Div}_0(X\times_R\tilde{V}):=\{D\in\mathrm{Div}(X\times_R\tilde{V});\mathrm{Supp}(D)\subset(X\times_R\tilde{V})_0\}.$$

Corollary 1.7. *Assume that the residue field k of R is separably closed. Let X be a strict semi-stable admissible formal R-scheme. Consider a finite extension $R\to\tilde{R}$ of discrete valuation rings of ramification index e. Let $\tilde{V}$ be a strict semi-stable admissible formal $\tilde{R}$-scheme, which is small. Then*

$$\mathrm{Div}_0(X\times_R\tilde{V})=p_1^*\big(\mathrm{Div}_0(X)\big)+p_2^*\big(\mathrm{Div}_0(\tilde{V})\big)$$

where $p_1\colon X\times_R\tilde{V}\to X$ resp. $p_2\colon X\times_R\tilde{V}\to\tilde{V}$ are the projections. The sum is not direct; but *there is only one relation*

$$\sum_{\sigma=1}^{s} p_1^* X_0^\sigma = e \sum_{\tau=1}^{t} p_2^* \tilde{V}_0^\tau = \mathrm{V}(\pi).$$

Proof. It follows from Proposition 1.6 that any Cartier divisor D with support in the special fiber is locally of type $p_1^* D_1 + p_2^* D_2$. Let now η be the generic point of the intersection of all the irreducible components $\tilde{V}_0^\tau$ of $\tilde{V}_0$; cf. Definition 1.4. Consider the open sets meeting $X_0 \times \eta$. Using the given relation on these open sets we can normalize the local representation by requiring that the multiplicity of $\tilde{V}_0^1$ in D_2 lies between 0 and $e - 1$. So the representation becomes unique and therefore does not depend on the local situation. Since the union of the open sets just considered is dense in $X_0 \times_k \tilde{V}_0$ we see that D is globally of the form $p_1^* D_1 + p_2^* D_2$ as required. □

From this we obtain the following description of the isomorphisms between two line bundles.

Corollary 1.8. *Keep the situation of Proposition* 1.6. *Consider a line bundle $\mathscr{L}$ on W. If $\lambda_K \in \mathscr{L}(W_{\mathrm{rig}})$ is a rigid-analytic section without zeros on W_{rig}, there exist multi-indices $m \in \mathbb{Z}^{s'}$ and $n \in \mathbb{Z}^{t'}$ such that $\xi^m \cdot \zeta^n \cdot \lambda_K$ extends to a generator of $\mathscr{L}$. In particular, consider line bundles $\mathscr{L}$ and $\mathscr{M}$ on W and an isomorphism φ_K: $\mathscr{L}_{\mathrm{rig}} \xrightarrow{\sim} \mathscr{M}_{\mathrm{rig}}$ over W_{rig}. Then there exist multi-indices $m \in \mathbb{Z}^{s'}$ and $n \in \mathbb{Z}^{t'}$ such that $\xi^m \cdot \zeta^n \cdot \varphi_K$ extends to an isomorphism of $\mathscr{L}$ and $\mathscr{M}$.*

Proof. Regarding λ_K as a rational section of $\mathscr{L}$ over W, it defines a Cartier divisor on W. Since there are no zeros on the rigid part, this Cartier divisor belongs to $\mathrm{Div}_0(W)$. Thus the claim follows from Corollary 1.7, since these divisors are principal. □

1.3. Desingularization of products of semi-stable formal schemes. For the extension of rigid line bundles to formal ones we need certain desingularization procedures. So let X_K be a smooth rigid analytic variety over K with strict semi-stable formal model X. Let V be an admissible formal R-scheme, smooth over

$$\mathrm{Spf}\, R\langle \zeta_1, \dots, \zeta_n \rangle / (\zeta_1 \cdot \ldots \cdot \zeta_n - \pi).$$

Assume that V is small (cf. Definition 1.4); i.e., that the strata V_0^M (cf. Definition 1.1) are geometrically irreducible for all M and that the intersection of all these strata is non-empty. Let further $R \to \tilde{R}$ be a ramified extension of discrete valuation rings of ramification index e, i.e. the uniformizers satisfy $\pi = \tilde{u} \cdot \tilde{\pi}^e$ for some unit $\tilde{u}$ in $\tilde{R}$. We want to explain how to obtain a desingularization of $X \times_R V$ resp. of $X \times_R \tilde{R}$. This leads to the desingularization of objects of the following type. We remind the reader that, in our sense, the *center Z of a blowing-up p: $Y' \to Y$* is the closed subset where the fiber of p is of dimension $\geqq 1$.

Proposition 1.9. *For $m \geqq 1$ and $n \geqq 1$ let A and B be the admissible formal R-algebras*

$$A := R\langle \xi_1, \dots, \xi_m \rangle / (\xi_1 \cdot \ldots \cdot \xi_m - \pi),$$

$$B := R\langle \zeta_1, \dots, \zeta_n \rangle / (\zeta_1 \cdot \ldots \cdot \zeta_n - \pi).$$

Then there exists a desingularization of $A \hat{\otimes}_R B$ *which is obtained as a sequence*

$$\mathrm{Spf}(A \hat{\otimes}_R B) = Y^0 \leftarrow Y^1 \leftarrow \cdots \leftarrow Y^r$$

of blowing-ups in open ideals $\mathcal{I}^\rho \subset \mathcal{O}_{Y^\rho}$ *for* $\rho = 0, \ldots, r-1$ *which are locally generated by two elements such that the following conditions hold*:

(a) Y^ρ *is normal for* $\rho = 0, \ldots, r$.

(b) Y^r *is strict semi-stable, so in particular regular.*

(c) *At each blowing-up the irreducible components of the center* Z^ρ *are of the form*

$$\mathrm{V}(\xi_i, i \in M) \times_k \mathrm{V}(\zeta_j, j \in N) \times_k (\mathbb{P}^1_k)^\alpha, \quad M \subset \{1, \ldots, m\}, \quad N \subset \{1, \ldots, n\}.$$

They all have a point above $(\xi_1, \ldots, \xi_m, \zeta_1, \ldots, \zeta_n, \pi)$ *in common. In particular* Z^ρ *is connected.*

(d) *The inverse image ideal sheaf* $\mathcal{I}^\rho \cdot \mathcal{O}_{Y^{\rho+1}}$ *induces the line bundle* $\mathcal{O}_{\mathbb{P}^1}(1)$ *on the fibers over all points of* Z^ρ.

Proof. For $1 \leqq \mu \leqq m$ and $1 \leqq \nu \leqq n$ let C be the formal R-algebra

$$R\langle \xi_1^{(r_1)}, \ldots, \xi_m^{(r_m)}, \zeta_1^{(s_1)}, \ldots, \zeta_n^{(s_n)} \rangle$$

in variables $\xi_1^{(r_1)}, \ldots, \xi_m^{(r_m)}, \zeta_1^{(s_1)}, \ldots, \zeta_n^{(s_n)}$ modulo the relations

$$\xi_\mu^{(r_\mu)} \cdot \ldots \cdot \xi_m^{(r_m)} - \zeta_\nu^{(s_\nu)} \cdot \ldots \cdot \zeta_n^{(s_n)} = 0$$

and

$$\xi_1^{(r_1)} \cdot \ldots \cdot \xi_m^{(r_m)} \cdot \zeta_1^{(s_1)} \cdot \ldots \cdot \zeta_{\nu-1}^{(s_{\nu-1})} - \pi = 0.$$

In order to control the centers of the blowing-ups which we use in the desingularization process of $A \hat{\otimes}_R B$ we call the variables $\xi_i^{(r_i)}$ and $\zeta_j^{(s_j)}$. The original variables ξ_i and ζ_j are multiples of the $\xi_i^{(r_i)}$ and $\zeta_j^{(s_j)}$. The new variables appearing in the blowing-ups are correspondingly called $\xi_i^{(1+r_i)}$ and $\zeta_j^{(1+s_j)}$. Starting with $\mu = \nu = 1$ and $r_i = s_j = 0$ for all i and j we construct step by step a desingularization of C.

For $\mu = \nu = 1$ the algebra C is isomorphic to $(A \hat{\otimes}_R B)$. The algebra C is always normal. Furthermore if $\mu = m$ or $\nu = n$ the algebra C is strict semi-stable.

We proceed by induction on $\max\{m - \mu, n - \nu\}$. For the beginning there is nothing to show. Now consider $\mu < m$ and $\nu < n$. We blow up the open ideal $(\xi_\mu^{(r_\mu)}, \zeta_\nu^{(s_\nu)})$. The center Z decomposes into irreducible components as follows:

$$\begin{aligned} Z &= \mathrm{V}(\xi_\mu^{(r_\mu)}, \xi_{\mu+1}^{(r_{\mu+1})} \cdot \ldots \cdot \xi_m^{(r_m)}, \zeta_\nu^{(s_\nu)}, \zeta_{\nu+1}^{(s_{\nu+1})} \cdot \ldots \cdot \zeta_n^{(s_n)}, \pi) \\ &= \bigcup_{\kappa=\mu+1}^{m} \bigcup_{\lambda=\nu+1}^{n} \mathrm{V}(\xi_\mu^{(r_\mu)}, \xi_\kappa^{(r_\kappa)}, \zeta_\nu^{(s_\nu)}, \zeta_\lambda^{(s_\lambda)}, \pi). \end{aligned}$$

The projection $Y^\rho \to Y^0$ maps the component $\mathrm{V}(\xi_\mu^{(r_\mu)}, \xi_\kappa^{(r_\kappa)}, \zeta_\nu^{(s_\nu)}, \zeta_\lambda^{(s_\lambda)}, \pi)$ surjectively onto

$$\mathrm{V}(\pi, \xi_i, i \in M) \times_k \mathrm{V}(\pi, \zeta_j, j \in N) \subset Y^0.$$

Thereby M and N are the sets of all i resp. j for which $\xi_i = \xi_i^{(0)}$ resp. $\zeta_j = \zeta_j^{(0)}$ belongs to the ideal $(\xi_\mu^{(r_\mu)}, \xi_\kappa^{(r_\kappa)}, \zeta_\nu^{(s_\nu)}, \zeta_\lambda^{(s_\lambda)})$. After dividing out this ideal the remaining variables $\xi_i^{(r_i)}$ and $\zeta_j^{(s_j)}$ with exponent $r_i \geqq 1$ resp. $s_j \geqq 1$ are the free coordinates of projective lines. So the fibers of the projection are isomorphic to $(\mathbb{P}_k^1)^\alpha$ and the irreducible component is isomorphic to

$$\mathrm{V}(\pi, \xi_i, i \in M) \times_k \mathrm{V}(\pi, \zeta_j, j \in N) \times_k (\mathbb{P}_k^1)^\alpha.$$

All the irreducible components intersect in

$$\mathrm{V}(\xi_\mu^{(r_\mu)}, \dots, \xi_m^{(r_m)}, \zeta_\nu^{(s_\nu)}, \dots, \zeta_n^{(s_n)}, \pi)$$

and therefore contain the point

$$\mathrm{V}(\xi_1^{(r_1)}, \dots, \xi_m^{(r_m)}, \zeta_1^{(s_1)}, \dots, \zeta_n^{(s_n)}, \pi) \cong \operatorname{Spec} k$$

above the point $\mathrm{V}(\xi_1, \dots, \xi_m, \zeta_1, \dots, \zeta_n, \pi)$. We obtain two charts of the blowing-up Y':

- $\xi_\mu^{(r_\mu)} = \zeta_\nu^{(s_\nu)} \cdot \xi_\mu^{(1+r_\mu)}$: The relations are equivalent to

$$\xi_\mu^{(1+r_\mu)} \cdot \ldots \cdot \xi_m^{(r_m)} - \zeta_{\nu+1}^{(s_{\nu+1})} \cdot \ldots \cdot \zeta_n^{(s_n)} = 0$$

and

$$\xi_1^{(r_1)} \cdot \ldots \cdot \xi_\mu^{(1+r_\mu)} \cdot \ldots \cdot \xi_m^{(r_m)} \cdot \zeta_1^{(s_1)} \cdot \ldots \cdot \zeta_\nu^{(s_\nu)} - \pi = 0.$$

Thus the number ν was increased by 1 while μ stayed constant. The number r_μ was also increased by 1.

- $\zeta_\nu^{(s_\nu)} = \xi_\mu^{(r_\mu)} \cdot \zeta_\nu^{(1+s_\nu)}$: The relations are equivalent to

$$\xi_{\mu+1}^{(r_{\mu+1})} \cdot \ldots \cdot \xi_m^{(r_m)} - \zeta_\nu^{(1+s_\nu)} \cdot \ldots \cdot \zeta_n^{(s_n)} = 0$$

and

$$\xi_1^{(r_1)} \cdot \ldots \cdot \xi_m^{(r_m)} \cdot \zeta_1^{(s_1)} \cdot \ldots \cdot \zeta_{\nu-1}^{(s_{\nu-1})} - \pi = 0.$$

Thus the number μ was increased by 1 while ν stayed constant. The number s_ν was also increased by 1.

The inverse image ideal sheaf $(\xi_\mu^{(r_\mu)}, \zeta_\nu^{(s_\nu)}) \cdot \mathcal{O}_{Y'}$ induces the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fiber over all points of the center. $\square$

Proposition 1.10. *For $m \geqq 1$ and $e \geqq 1$ let A be the admissible formal R-algebra*

$$A := R\langle \eta_1, \dots, \eta_m \rangle / (\eta_1 \cdot \ldots \cdot \eta_m - \pi^e).$$

If $e = 1$ the algebra A is strict semi-stable, in particular regular.

If $e \geqq 2$ there exists a desingularization of A, which is obtained as a sequence

$$\mathrm{Spf}(A) = Y^0 \leftarrow Y^1 \leftarrow \cdots \leftarrow Y^r$$

of blowing-ups in open ideals $\mathscr{I}^\rho \subset \mathcal{O}_{Y^\rho}$ for $\rho = 0, \ldots, r-1$ which are locally generated by two elements such that the following conditions hold:

(a) *Y^ρ is normal for $\rho = 0, \ldots, r$.*

(b) *Y^r is strict semi-stable, in particular regular.*

(c) *At each blowing-up the irreducible components of the center Z^ρ are of the form*

$$\mathrm{V}(\pi, \eta_i, i \in M) \times_k (\mathbb{P}^1_k)^\alpha, \quad M \subset \{1, \ldots, m\}.$$

They all have a point above $(\eta_1, \ldots, \eta_m, \pi)$ in common. In particular Z^ρ is connected.

(d) *The inverse image ideal sheaf $\mathscr{I}^\rho \cdot \mathcal{O}_{Y^{\rho+1}}$ induces the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers over all points of Z^ρ.*

Proof. We proceed by induction on e. Starting with $v = 0$ we stepwise desingularize the R-algebra

$$C := R\langle \eta_1^{(v)}, \eta_2, \ldots, \eta_m \rangle / (\eta_1^{(v)} \cdot \eta_2 \cdot \ldots \cdot \eta_m - \pi^{e-v})$$

with $\eta_1 = \pi^v \eta_1^{(v)}$. This algebra C is always normal.

For $e - v = 1$ the algebra C is strict semi-stable. Let now $e - v \geqq 2$. We blow up the open ideal $(\eta_1^{(v)}, \pi)$. The center Z decomposes into irreducible components as follows:

$$Z = \mathrm{V}(\eta_1^{(v)}, \eta_2 \cdot \ldots \cdot \eta_m, \pi) = \bigcup_{i=2}^{m} \mathrm{V}(\eta_1^{(v)}, \eta_i, \pi).$$

Because of $\eta_1 = \pi^v \eta_1^{(v)}$ the projection $Y^\rho \to Y^0$ maps the component $\mathrm{V}(\eta_1^{(v)}, \eta_i, \pi)$ isomorphically onto $\mathrm{V}(\eta_1, \eta_i, \pi) \subset \mathrm{Spf}(A)$. All irreducible components intersect in the point

$$\mathrm{V}(\eta_1, \eta_2, \ldots, \eta_m, \pi) \cong \mathrm{Spec}\, k.$$

We get two charts of the blowing-up Y':

- $\eta_1^{(v)} = \pi \cdot \eta_1^{(v+1)}$: The relation is equivalent to

$$\eta_1^{(v+1)} \cdot \eta_2 \cdot \ldots \cdot \eta_m - \pi^{e-(v+1)} = 0.$$

Thus the exponent of π was reduced by 1.

- $\pi = \eta_1^{(v)} \cdot \pi'$: In this case we need two equations to describe the relations:

$$\eta_2 \cdot \ldots \cdot \eta_m - \pi^{e-(v+1)}\pi' = \pi' \cdot \eta_1^{(v)} - \pi = 0.$$

This case is treated in the following Lemma 1.11 for $\mu = 2$ and $\eta_0^{(s_0)} = \pi'$.

The inverse image ideal sheaf $(\eta_1^{(v)}, \pi) \cdot \mathcal{O}_{Y'}$ induces the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers over each point of the center. □

Lemma 1.11. *Let* $1 \leqq \mu \leqq m$, $e \geqq 1$ *and* $s_j \geqq 0$ *for all* j. *For a subset*

$$J \subset \{0, \ldots, \mu - 1\}$$

let A *be the admissible formal* R*-algebra*

$$R\langle \eta_0^{(s_0)}, \ldots, \eta_\mu^{(s_\mu)}, \eta_{\mu+1}, \ldots, \eta_m \rangle$$

modulo the relations

$$\eta_\mu^{(s_\mu)} \cdot \eta_{\mu+1} \cdot \ldots \cdot \eta_m - \pi^e \cdot \prod_{j \in J} \eta_j^{(s_j)} = \eta_0^{(s_0)} \cdot \ldots \cdot \eta_{\mu-1}^{(s_{\mu-1})} - \pi = 0.$$

Then there exists a desingularization of A *as claimed in Proposition* 1.10.

Proof. The algebra A is normal and for $\mu = m$ the algebra A is strict semi-stable.

We proceed by two inductions. The inner one is done on $\min\{m - \mu, \# J\}$ and the outer one on e. We first describe how we reduce to the case $\mu = m$ or $J = \emptyset$ by induction on $\min\{m - \mu, \# J\}$. So we may assume $\mu < m$ and $J \neq \emptyset$. We blow up the open ideal $(\eta_\mu^{(s_\mu)}, \eta_j^{(s_j)})$ for some $j \in J$. The center Z decomposes into irreducible components as follows:

$$Z = \mathrm{V}(\eta_j^{(s_j)}, \eta_\mu^{(s_\mu)}, \eta_{\mu+1} \cdot \ldots \cdot \eta_m, \pi) = \bigcup_{v=\mu+1}^{m} \mathrm{V}(\eta_j^{(s_j)}, \eta_\mu^{(s_\mu)}, \eta_v, \pi).$$

We have to describe them in terms of Proposition 1.10. The projection $Y^\rho \to Y^0$ maps the component $\mathrm{V}(\eta_j^{(s_j)}, \eta_\mu^{(s_\mu)}, \eta_v, \pi)$ surjectively onto

$$\mathrm{V}(\pi, \eta_v, \eta_i, i \in M) \subset Y^0$$

where M is the set of all i for which $\eta_i = \eta_i^{(0)}$ belongs to the ideal $(\eta_\mu^{(s_\mu)}, \eta_j^{(s_j)})$. After dividing out this ideal the remaining variables $\eta_i^{(s_i)}$ with exponent $s_i \geqq 1$ are the free coordinates of projective lines. So the fibers of the projection are isomorphic to $(\mathbb{P}_k^1)^\alpha$ and the irreducible component is thus isomorphic to

$$\mathrm{V}(\pi, \eta_v, \eta_i, i \in M) \times_k (\mathbb{P}_k^1)^\alpha.$$

All the irreducible components intersect in

$$\mathrm{V}(\eta_j^{(s_j)}, \eta_\mu^{(s_\mu)}, \eta_{\mu+1}, \ldots, \eta_m, \pi)$$

and therefore contain the point

$$\mathrm{V}(\eta_0^{(s_0)}, \ldots, \eta_\mu^{(s_\mu)}, \eta_{\mu+1}, \ldots, \eta_m, \pi) \cong \operatorname{Spec} k$$

above the point $(\eta_1, \ldots, \eta_m, \pi)$. We get two charts of the blowing-up Y':

- $\eta_\mu^{(s_\mu)} = \eta_j^{(s_j)} \cdot \eta_\mu^{(1+s_\mu)}$: The relations are equivalent to

$$\eta_\mu^{(1+s_\mu)} \cdot \eta_{\mu+1} \cdot \ldots \cdot \eta_m - \pi^e \cdot \prod_{i \in J - \{j\}} \eta_i^{(s_i)} = \eta_0^{(s_0)} \cdot \ldots \cdot \eta_{\mu-1}^{(s_{\mu-1})} - \pi = 0.$$

Thus the set J was reduced by one element while μ and e stayed the same. Also the number s_μ was increased by 1.

- $\eta_j^{(s_j)} = \eta_\mu^{(s_\mu)} \cdot \eta_j^{(1+s_j)}$: The relations are equivalent to

$$\eta_{\mu+1} \cdot \ldots \cdot \eta_m - \pi^e \cdot \left(\eta_j^{(1+s_j)} \cdot \prod_{i \in J - \{j\}} \eta_i^{(s_i)} \right) = 0$$

and

$$\eta_0^{(s_0)} \cdot \ldots \cdot \eta_j^{(1+s_j)} \cdot \ldots \cdot \eta_\mu^{(s_\mu)} - \pi = 0.$$

Thus the number μ was increased by 1 while e and J stayed the same. Also the number s_j was increased by 1.

The inverse image ideal sheaf $(\eta_\mu^{(s_\mu)}, \eta_j^{(s_j)}) \cdot \mathcal{O}_{Y'}$ induces the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers over all points of the center. So we have reduced the situation for fixed e to the case $J = \emptyset$.

We now proceed by induction on e and start with $e = 1$. Using the above we reduce to the case $J = \emptyset$. Then we can apply Proposition 1.9 where we use the systems of variables $(\xi_1, \ldots, \xi_{m'}) = (\eta_\mu^{(s_\mu)}, \eta_{\mu+1}, \ldots, \eta_m)$ resp. $(\zeta_1, \ldots, \zeta_{n'}) = (\eta_0^{(s_0)}, \ldots, \eta_{\mu-1}^{(s_{\mu-1})})$. For each of the blowing-ups which appear in the process of Proposition 1.9, the irreducible components of the center are of the form

$$\mathrm{V}(\pi, \eta_j^{(s_j)}, j \in N) \times_k (\mathbb{P}_k^1)^\alpha, \quad N \subset \{0, \ldots, m\}.$$

Again we have to describe them in terms of Proposition 1.10. The projection $Y^\rho \to Y^0$ maps the component just mentioned surjectively onto

$$\mathrm{V}(\pi, \eta_i, i \in M) \subset Y^0, \quad M \subset \{1, \ldots, m\},$$

where $M \supset N - \{0\}$ is the set of all indices i for which $\eta_i = \eta_i^{(0)}$ belongs to the ideal $(\eta_j^{(s_j)}, j \in N)$. After dividing out this ideal the remaining variables $\eta_i^{(s_i)}$ with exponent $s_i \geqq 1$ are the free coordinates of additional projective lines. Together with the $(\mathbb{P}_k^1)^\alpha$ coming from

Proposition 1.9 they make the fibers of the projection being isomorphic to $(\mathbb{P}_k^1)^\beta$. Thus the irreducible component is isomorphic to

$$\mathrm{V}(\pi, \eta_i, i \in M) \times_k (\mathbb{P}_k^1)^\beta.$$

The remaining assertions also follow from Proposition 1.9.

Now consider the induction step $e \geqq 2$. Again we reduce to the case $J = \emptyset$. The relations are then equivalent to

$$\eta_\mu^{(s_\mu)} \cdot \eta_{\mu+1} \cdot \ldots \cdot \eta_m - \pi^{e-1} \cdot \eta_0^{(s_0)} \cdot \ldots \cdot \eta_{\mu-1}^{(s_{\mu-1})} = \eta_0^{(s_0)} \cdot \ldots \cdot \eta_{\mu-1}^{(s_{\mu-1})} - \pi = 0.$$

Thus the exponent e was decreased by 1 and the situation is reduced to the induction hypotheses. This settles the proof of the lemma and, moreover, its adaption to the proof of Proposition 1.10. □

Remark 1.11.1. One can give a global desingularization procedure of the product $X \times_R V$ and of the base change $X \times_R \tilde{R}$ by successively blowing up all irreducible components of the special fiber which are not yet Cartier divisors; cf. [Ha]. Propositions 1.9 & 1.10 and Lemma 1.11 are the corresponding local descriptions.

2. Extending rigid-analytic line bundles to formal models

The main purpose of this section is to study the extension of line bundles on X_{rig} to a formal model X. If X is regular, any line bundle $\mathscr{L}_K$ on X_{rig} extends to a formal line bundle $\mathscr{L}$ on X. This assertion remains true for the base change $X \times_R V$ for any smooth formal scheme V over R because regularity is preserved under smooth base change, but already for a ramified base ring extension $R \to \tilde{R}$ one will loose the regularity in general. For later use let us formulate the well-known extension property in the regular case.

Lemma 2.1. *Let X be a regular formal scheme and let V be a smooth formal scheme over R. Then any line bundle $\mathscr{L}_K$ on $(X \times_R V)_{\mathrm{rig}}$ extends to a line bundle $\mathscr{L}$ on $X \times_R V$.*

Proof. Due to [L1], Lemma 2.2 there exists a coherent formal $\mathcal{O}_X$-module $\mathscr{F}$ extending $\mathscr{L}_K$. Then the double dual $\mathscr{L} := \mathscr{H}om_X(\mathscr{H}om_X(\mathscr{F}, \mathcal{O}_X), \mathcal{O}_X)$ is a line bundle on X since X is regular and $\mathscr{L}$ extends $\mathscr{L}_K$. □

The main topic in the following is to discuss the obstructions to such an extension problem. We will start with a strict semi-stable formal scheme X over R which is regular by definition. For technical reasons, we have to perform base ring extension $\tilde{X} := X \times_R \tilde{R}$ where $R \to \tilde{R}$ can be ramified. Moreover we have to study the case where we perform a further base change by another semi-stable formal scheme $\tilde{V}$ over $\tilde{R}$. This will be the situation we are mainly concerned with when proving the representability of the Picard functor.

2.1. Line bundles on the reduction. Let us start with the algebraic situation X_0 where X_0 is the reduction of the strict semi-stable formal scheme X we started with. We will explain the behavior of the Picard group under the base change by the 1-dimensional vector group $\mathbb{G}_{a,k}$ resp. by the multiplicative group $\mathbb{G}_{m,k}$. The result is known and mainly contained in the paper [BM] of Bass and Murthy. This result will later be used to show that the Picard variety we construct has semi-abelian reduction.

Proposition 2.2. *Let X be a strict semi-stable formal R-scheme and consider its special fiber $X_0 := X \otimes_R k$.*

(1) *In the case of the additive vector group $\mathbb{G}_{a,k}$, the morphisms*

$$\mathrm{Pic}(X_0) \xrightarrow{\sim} \mathrm{Pic}(X_0 \times \mathbb{G}_{a,k}), \quad \mathcal{L} \mapsto p^*\mathcal{L};\ \sigma^*\mathcal{E} \leftarrow\!\shortmid \mathcal{E}$$

are bijective and inverse to each other where $\sigma: X_0 \to X_0 \times \mathbb{G}_{a,k}$ is the zero section.

(2) *Locally, in the case of the multiplicative group $\mathbb{G}_{m,k}$, there exists for each closed point $x \in X_0$ a neighborhood U_0 of x in X_0 such that the morphisms*

$$\mathrm{Pic}(U_0) \xrightarrow{\sim} \mathrm{Pic}(U_0 \times \mathbb{G}_{m,k}), \quad \mathcal{L} \mapsto p^*\mathcal{L};\ \sigma^*\mathcal{E} \leftarrow\!\shortmid \mathcal{E}$$

are isomorphisms where $\sigma: X_0 \to X_0 \times \mathbb{G}_{m,k}$ is the unit section.

(3) *Globally, in the case of the multiplicative group $\mathbb{G}_{m,k}$, the morphism*

$$\mathrm{Pic}(X_0) \oplus \mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Pic}(X_0 \times \mathbb{G}_{m,k}), \quad (\mathcal{L}, n) \mapsto p^*\mathcal{L} \otimes (\xi^n);$$

is an isomorphism where ξ is a coordinate on $\mathbb{G}_{m,k}$.

(4) *The assertions (1)–(3) remain valid if one replaces X_0 by $\tilde{X}_0 := X_0 \times_k \tilde{k}$ for any finite field extension $k \to \tilde{k}$.*

(5) *If the irreducible components of each intersection $X_0^i \cap X_0^j$ are geometrically irreducible, the canonical map $\mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(X_0 \otimes_k \tilde{k}, \mathbb{Z})$ is bijective for any field extension $k \to \tilde{k}$.*

Proof. Since these statements are buried in a mass of a general situation in [BM], we will give a short proof. First of all we mention that the statements are well-known in the case where X_0 is normal. So we are only concerned with the singular case which will be reduced to the normal case. The map σ^* is a section of the map p^*, so it suffices to proof the surjectivity of p^*. Since the group of units $A[\xi]^\times$ of a polynomial ring in one variable ξ over a reduced ring A is equal to the group of units $A^\times$ of A, the problem is local on X_0 in the first case. So we may assume that $X_0 = \mathrm{Spec}(A)$ for an affine k-algebra A. Now consider both cases (1) and (2).

In the case where $A = k[\xi_1, \dots, \xi_r]/(\xi_1 \cdot \ldots \cdot \xi_r)$, we have the following exact sequence

$$0 \to A \to A/(\xi_1) \times \cdots \times A/(\xi_r) \to \prod_{i<j} A/(\xi_i, \xi_j),$$

$$f \mapsto (\bar{f}, \dots, \bar{f}), (\overline{f_1}, \dots, \overline{f_r}) \mapsto (\dots, \bar{f}_i - \bar{f}_j, \dots).$$

In the general case, any point $x \in X_0$ has an open neighborhood which is smooth over such a ring so that we have such an exact sequence in the general case as well; cf. Proposition

1.3. The situation of line bundles can be translated into invertible modules. So we have to start with an invertible module E over $A[\xi]$ resp. over $A[\xi, \xi^{-1}]$ where ξ is a variable and have to show the existence of an isomorphism $\sigma^* E \otimes_A A[\xi] \xrightarrow{\sim} E$ resp. $\sigma^* E \otimes_A A[\xi, \xi^{-1}] \xrightarrow{\sim} E$ over a suitable neighborhood of a given point x. Since E is flat over A, we have the exact sequence

$$0 \to E \to E/(\xi_1) \times \cdots \times E/(\xi_r) \to \prod_{i<j} E/(\xi_i, \xi_j),$$

$$f \mapsto (\bar{f}, \ldots, \bar{f}), (\overline{f_1}, \ldots, \overline{f_r}) \mapsto (\ldots, \bar{f}_i - \bar{f}_j, \ldots).$$

Thus E is the kernel of the last map. Now consider $L := \sigma^* E$. Then we obtain the exact sequence

$$0 \to L \to L/(\xi_1) \times \cdots \times L/(\xi_r) \to \prod_{i<j} L/(\xi_i, \xi_j)$$

and hence the exact sequence

$$0 \to L \otimes_A B \to L/(\xi_1) \otimes_A B \times \cdots \times L/(\xi_r) \otimes_A B \to \prod_{i<j} L/(\xi_i, \xi_j) \otimes_A B$$

where $B := A[\xi]$ in case (1) resp. $B := A[\xi, \xi^{-1}]$ in case (2). Then $L \times_A B$ is the kernel of the last map. Thus we have to show that there are isomorphisms from the exact sequence with kernel $L \otimes_A B$ to the exact sequence with kernel E. Since $A_i := A/(\xi_i)$ is smooth over k and hence normal, there exist isomorphisms

$$\varphi_i \colon L \otimes_A B \otimes_A A_i \xrightarrow{\sim} E \otimes_A A_i$$

which are compatible with the section σ^*. For the last term of the exact sequence we obtain two isomorphisms

$$\varphi_i \otimes_A A_j, \quad \varphi_j \otimes_A A_i \colon L \otimes_A B \otimes_A A_i \otimes_A A_j \xrightarrow{\sim} E \otimes_A A_i \otimes_A A_j.$$

Using the canonical isomorphism $A_{ij} := A_i \otimes_A A_j \cong A/(\xi_i, \xi_j)$ as an identity, we can compare the restrictions of the isomorphisms φ_i on the intersections.

(1) In the case of the vector group $\mathbb{G}_{a,k}$, the isomorphism φ_i is uniquely determined and, hence, $\varphi_i \otimes_A A_j = \varphi_j \otimes_A A_i$ since they coincide after pull-back by the section σ^*. So the diagram is commutative, and we obtain the desired isomorphism $\sigma^* E \otimes_A A[\xi] \xrightarrow{\sim} E$.

(2) In the case of the multiplicative group $\mathbb{G}_{m,k}$, the isomorphism φ_i is uniquely determined only up to a power $\xi^{n(i)}$ of the variable because the group of units of the ring of Laurent polynomials is $A[\xi, \xi^{-1}]^\times = A^\times \cdot \xi^{\mathbb{Z}}$ for a reduced connected ring A. So we obtain an equation

$$\varphi_i \otimes_{A_i} A_{ij} = \xi^{n(ij)} \cdot \varphi_j \otimes_{A_j} A_{ij}$$

where $n(ij) \in \mathbb{Z}(\mathrm{V}(\xi_i, \xi_j))$. If we choose the neighborhood of the given point $x \in X_0$ so small that all the subschemes defined by (ξ_i, ξ_j) are connected, the elements $n(ij) \in \mathbb{Z}$ are true integers. These numbers satisfy a cocycle condition over the one-point space $\{x\}$. Therefore they can be written as $n(ij) = n(i) - n(j)$. Thus, after multiplying the isomorphisms φ_i by $\xi^{-n(i)}$ we obtain a coherent system of morphisms and, hence, we obtain an isomorphism between the kernels $L \otimes_A B$ and E.

(3) Due to (2) there exists an open covering $\mathfrak{U} := \{U_0^1, \dots, U_0^N\}$ of X_0 such that for each line bundle $\mathscr{E}$ on $X_0 \times \mathbb{G}_{m,k}$ there exist isomorphisms

$$\varphi_i \colon p^*\sigma^*\mathscr{E}|U^i \times \mathbb{G}_{m,k} \xrightarrow{\sim} \mathscr{E}|U^i \times \mathbb{G}_{m,k}$$

which are compatible with the unit section σ. Using the canonical identifications on the overlaps we can compare these isomorphisms. They can differ only by a power of the coordinate, so we obtain

$$\varphi_i|(U^i \cap U^j) \times \mathbb{G}_{m,k} = \xi^{n(ij)} \cdot \varphi_j|(U^i \cap U^j) \times \mathbb{G}_{m,k}$$

for a unique $n(ij) \in \mathbb{Z}(U^i \cap U^j)$. The numbers $\big(n(ij)\big)$ give rise to a cocycle in $\mathrm{H}^1(\mathfrak{U}, \mathbb{Z})$ and, hence to a line bundle $\mathscr{M} := (\xi^{n(ij)})$ on $X_0 \times \mathbb{G}_{m,k}$. Then we can write $\mathscr{E} = \mathscr{M} \otimes \bar{\mathscr{E}}$ where $\bar{\mathscr{E}}$ satisfies $\bar{\mathscr{E}} \cong p^*\sigma^*\bar{\mathscr{E}}$. Thus we see that the map is surjective. Since we have the section σ, for the injectivity it suffices to show that a line bundle (ξ^n) for a cocycle $n \in \mathrm{H}^1(X_0, \mathbb{Z})$ is trivial if and only if $n = 0$. This follows from the unique decomposition of the group of units $A[\xi, \xi^{-1}]^\times = A^\times \cdot \xi^{\mathbb{Z}}$ in a ring of Laurent polynomials over a connected reduced ring A.

(4) The exact sequences we used remain exact and also the base change of smooth algebras are still smooth over $\tilde{k}$ and, hence normal; the fact we really need.

(5) The open covering $\mathfrak{U} := \{U_0^1, \dots, U_0^N\}$ of X_0 used in the proof of (3) had only to satisfy that U is affine and that all intersections $U \cap (X_0^i \cap X_0^j)$ are connected for all $U \in \mathfrak{U}$. If the irreducible components of the intersections $X_0^i \cap X_0^j$ are geometrically irreducible, this condition remains valid after base change. So we can use the Cech cohomology group $\mathrm{H}^1(\mathfrak{U}, \mathbb{Z})$ to describe the isomorphism in (3) for X_0 as well as for $X_0 \otimes_k \tilde{k}$ which are equal. Thus we see that $\mathrm{H}^1(X_0, \mathbb{Z})$ is not altered by base change. □

2.2. Local models of line bundles. Coming back to our main task we want to show here the existence of local extensions of rigid line bundles to semi-stable models. Let us start with a well-known result on existence of global sections on fibers. We need only the following simple case which can easily be proved by looking at a suitable exact cohomology sequence.

Lemma 2.3. *Let S be a Noetherian scheme and let $p \colon Y \to S$ be the blowing-up of an ideal $\mathscr{I}$ which is locally generated by two elements. Assume $p_*\mathcal{O}_Y = \mathcal{O}_S$. Let s be a closed point of the center Z of p and let $Y_s := Y \times_S k(s)$ be the fiber above s. Let $\mathscr{F}$ be a coherent sheaf on Y with $\mathrm{H}^1(Y_s, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_s}) = (0)$. Then there exists an open neighborhood S' of s in S such that $\mathrm{H}^1(Y', \mathscr{F}|_{Y'}) = (0)$ on $Y' := Y \times_S S'$ and the canonical map*

$$\mathrm{H}^0(Y', \mathscr{F}|_{Y'}) \to \mathrm{H}^0(Y_s, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_s})$$

is surjective.

Lemma 2.4. *Let S be an admissible formal R-scheme and let $p: Y \to S$ be the formal blowing-up of an open ideal $\mathscr{I}$ which is locally generated by two elements. Assume that $p_*\mathcal{O}_Y = \mathcal{O}_S$. Further assume that the center Z of p is connected and that the pull-back $\mathscr{J} := \mathscr{I} \cdot \mathcal{O}_Y$ induces the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers above all points of Z. If $\mathscr{L}$ is a line bundle on Y, there exists an integer $n \in \mathbb{Z}$ such that $p_*(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n})$ is a line bundle on S and the canonical map*

$$p^*p_*(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n}) \xrightarrow{\sim} \mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n}$$

is an isomorphism.

Proof. Due to the GAGA-principle [EGA], III, Corollaire 5.1.3, we may assume that we deal with a projective situation $p: Y \to S$. For any point $s \in Z \subset \mathrm{V}(\pi)$ the fiber $Y_s \hookrightarrow \mathbb{P}^1_{k(s)}$ is a closed subset since the ideal $\mathscr{I}$ is locally generated by two elements. Since $p_*\mathcal{O}_Y = \mathcal{O}_S$, the fibers of p are connected due to Zariski's main theorem so that p is an isomorphism outside the center Z. For $s \in Z$, we have $\dim Y_s = 1$ due to the definition of the center and, hence, $Y_s = \mathbb{P}^1_{k(s)}$. Thus we have $\mathscr{L}_s := \mathscr{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_s} \cong \mathcal{O}_{\mathbb{P}^1_{k(s)}}(n(s))$ for a uniquely determined $n(s) \in \mathbb{Z}$. For a fixed point $s \in Z$ set $n = n(s)$ and replace $\mathscr{L}$ by $\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes -n}$. So we may assume $n(s) = 0$ so that $\mathscr{L}_s \cong \mathcal{O}_{Y_s}$ and $\mathrm{H}^1(Y_s, \mathscr{L}_s) = (0)$. Now let f_s be a global generator of $\mathrm{H}^0(Y_s, \mathscr{L}_s)$. Due to Lemma 2.3 there exists an open neighborhood S' of s in S and a section $f \in \mathrm{H}^0(Y \times_S S', \mathscr{L})$ inducing f_s. For any $y \in Y_s$ the stalk $\mathscr{L}_y$ is generated by f due to the lemma of Nakayama. Thus the closed subset A of Y where f does not generate $\mathscr{L}$ is disjoint from Y_s. Since $Y \to S$ is proper, there exists an open neighborhood U of s such that its inverse image in Y is disjoint from A. Thus we see that the canonical map

$$\mathcal{O}_Y | Y \times_S U \xrightarrow{\sim} \mathscr{L} | Y \times_S U, \quad 1 \mapsto f$$

is an isomorphism. In particular we see that the number $n(s)$ is locally constant on S. Due to the connectedness of Z the function $n(s) = n$ is constant. Since $p_*\mathcal{O}_Y = \mathcal{O}_S$ it is clear that $p_*(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n})$ is a line bundle and that the canonical map

$$p^*p_*(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n}) \xrightarrow{\sim} \mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{J}^{\otimes n}$$

is an isomorphism. □

Proposition 2.5. *Let X_K be a smooth rigid analytic variety over K which admits a strict semi-stable formal model X over* $\mathrm{Spf}(R)$. *Furthermore let $R \to \tilde{R}$ be a finite extension of discrete valuation rings and set $\tilde{X} := X \otimes_R \tilde{R}$. Then there exists an open covering $\{U^1, \ldots, U^n\}$ of X satisfying the following conditions*:

(1) *Let V_K be a smooth rigid analytic variety over K which admits a strict semi-stable formal model $V \to \mathrm{Spf}(R)$ over $\mathrm{Spf}(R)$. Assume that V is small in the sense of* 1.4. *Consider a rigid analytic line bundle $\mathscr{L}_K$ on $(X \times_R V)_{\mathrm{rig}}$. Then there exist line bundles $\mathscr{L}^i$ on $(U^i \times_R V)$ extending the restriction $\mathscr{L}_K|(U^i \times_R V)_{\mathrm{rig}}$ for $i = 1, \ldots, n$.*

(2) *Consider a smooth formal scheme $\tilde{V} \to \mathrm{Spf}(\tilde{R})$ over $\tilde{R}$ which is geometrically connected and a rigid analytic line bundle $\tilde{\mathscr{L}}_K$ on $(\tilde{X} \times_{\tilde{R}} \tilde{V})_{\mathrm{rig}}$. Then there exist line bundles $\tilde{\mathscr{L}}^i$ on $(U^i \times_R \tilde{V})$ extending the restriction $\tilde{\mathscr{L}}_K|(U^i \times_R \tilde{V})_{\mathrm{rig}}$ for $i = 1, \ldots, n$.*

Proof. For any point $x \in X_0$ we have to find an open neighborhood U of x satisfying the assertion. Due to Proposition 1.3 there exists an open neighborhood U of x which is formally smooth over

$$\mathrm{Spf}\big(R\langle\xi_1,\dots,\xi_s\rangle/(\xi_1\cdot\ldots\cdot\xi_t-\pi)\big).$$

In case (1) the product $Y := U \times_R V$ is formally smooth over

$$S := \mathrm{Spf}\big(R\langle\xi_1,\dots,\xi_s,\zeta_1,\dots,\zeta_t\rangle/(\xi_1\cdot\ldots\cdot\xi_s-\pi,\zeta_1\cdot\ldots\cdot\zeta_t-\pi)\big).$$

In case (2) the base change $Y := U \times_R \tilde{V}$ is formally smooth over

$$S := \mathrm{Spf}\big(\tilde{R}\langle\xi_1,\dots,\xi_s\rangle/(\xi_1\cdot\ldots\cdot\xi_s-\tilde{\pi}^e)\big)$$

where e is the ramification index of $\tilde{R}$ over R. Now we choose a desingularization S' of S as constructed in Proposition 1.9 resp. Proposition 1.10. After base change with Y we obtain a Cartesian diagram

$$\begin{array}{ccc} Y & \longleftarrow & Y' = S' \times_S Y \\ \downarrow & & \downarrow \\ S & \longleftarrow & S'. \end{array}$$

Since Y' is formally smooth over the regular formal scheme S', the product Y' is regular. Due to Lemma 2.1 the line bundle $\mathscr{L}_K$ on $Y'_{\mathrm{rig}} = Y_{\mathrm{rig}}$ extends to a line bundle $\mathscr{L}'$ on Y'. In the following we want to show that this line bundle descends to a line bundle $\mathscr{L}$ on Y. In Proposition 1.9 resp. Proposition 1.10, the desingularization $S' \to S$ is constructed stepwise $p\colon S^{\nu+1} \to S^{\nu}$ by blowing-up open ideals $\mathscr{I}^{\nu} \subset \mathcal{O}_{S^{\nu}}$ which are locally generated by two elements. Now consider the diagram obtained by the base change $Y \to S$

$$\begin{array}{ccc} Y^{\nu} & \overset{q}{\longleftarrow} & Y^{\nu+1} = S^{\nu+1} \times_S Y \\ \downarrow & & \downarrow \\ S^{\nu} & \overset{p}{\longleftarrow} & S^{\nu+1}. \end{array}$$

Since Y is flat over S, the map $q\colon Y^{\nu+1} \to Y^{\nu}$ is the blowing up of Y^{ν} in $\mathscr{J} := \mathscr{I}^{\nu}\cdot\mathcal{O}_{Y^{\nu}}$. Proceeding by induction, we have an extension $\mathscr{L}^{\nu+1}$ on $Y^{\nu+1}$ of the line bundle $\mathscr{L}_K$. Then we have to construct an extension $\mathscr{L}^{\nu}$ on Y^{ν}. This will be done by looking at the direct image of a twist of $\mathscr{L}^{\nu+1}$ by $\mathscr{J}^{n(\nu)}$. Let W^{ν} be the center of the blowing-up $p\colon S^{\nu+1} \to S^{\nu}$. Due to Proposition 1.9 resp. Proposition 1.10, S^{ν} is normal and, hence, $p_*\mathcal{O}_{S^{\nu+1}} = \mathcal{O}_{S^{\nu}}$. Thus we get

$$W^{\nu} = \{s \in S^{\nu}; p^{-1}(s) = \mathbb{P}^1_{k(s)}\},$$
$$p\colon S^{\nu+1} - p^{-1}(W^{\nu}) \xrightarrow{\sim} S^{\nu} - W^{\nu}.$$

Since $Y \to S$ is flat, the fibers of the blowing-up are compatible with this base change. Set $Z^{\nu} := W^{\nu} \times_S Y$, so Z^{ν} is the center of the blowing-up q. In particular, we have

$$Z^{\nu} = \{y \in Y^{\nu}; q^{-1}(y) = \mathbb{P}^1_{k(y)}\},$$

$$q\colon Y^{\nu+1} - q^{-1}(Z^{\nu}) \xrightarrow{\sim} Y^{\nu} - Z^{\nu}.$$

Due to Proposition 1.9 resp. Proposition 1.10 the set W^{ν} and, hence, Z^{ν} is the union of subschemes of the following type in case (1)

$$\mathrm{V}(\pi, \xi_i, i \in M) \times_k \mathrm{V}(\pi, \zeta_j, j \in N) \times_k (\mathbb{P}^1_k)^{\alpha} = X_0^M \times_k V_0^N \times_k (\mathbb{P}^1_k)^{\alpha},$$

resp. in case (2)

$$\mathrm{V}(\tilde{\pi}, \xi_i, i \in M) \times_k (\mathbb{P}^1_k)^{\alpha} = X_0^M \times_k V_0 \times_k (\mathbb{P}^1_k)^{\alpha}.$$

Each stratum X_0^M decomposes into irreducible components. Since they are smooth over k there exists a unique component containing x. So we can replace the open subscheme U by an open neighborhood of x such that it meets only the irreducible component of X_0^M which contains x. The subschemes V_0^N resp. V_0 are geometrically irreducible due to the assumption, therefore the subscheme Z^{ν} decomposes into the irreducible components $X_0^M \times_k V_0^N \times_k (\mathbb{P}^1_k)^{\alpha}$ resp. into $X_0^M \times_k V_0 \times_k (\mathbb{P}^1_k)^{\alpha}$ which meet in a common point above $\{x\}$; cf. Proposition 1.9 resp. Proposition 1.10. This means that Z^{ν} is connected. The pullback $\mathscr{J} \cdot \mathcal{O}_{Y^{\nu+1}}$ induces the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibers of all points of the center Z^{ν} of q as this is true for the blowing-up $p\colon S^{\nu+1} \to S^{\nu}$. Due to Proposition 1.9 resp. Proposition 1.10, S^{ν} is normal so that $p_*\mathcal{O}_{S^{\nu+1}} = \mathcal{O}_{S^{\nu}}$ and, hence, $q_*\mathcal{O}_{Y^{\nu+1}} = \mathcal{O}_{Y^{\nu}}$ due to the flatness of Y over S. Due to Lemma 2.4 the line bundle $\mathscr{L}^{\nu+1}$ on $Y^{\nu+1}$ can be twisted by a suitable power of $\mathscr{J}$ so that the modified line bundle $\mathscr{L}^{\nu+1}$ descends to a line bundle $\mathscr{L}^{\nu}$ on Y^{ν}. Since $\mathscr{J}_{\mathrm{rig}} \cong \mathcal{O}_{Y_{\mathrm{rig}}}$ the modification of $\mathscr{L}^{\nu+1}$ does not change $\mathscr{L}_K$. So we end up with a line bundle $\mathscr{L}^{\nu}$ on Y^{ν} which extends $\mathscr{L}_K$ on Y^{ν}. By induction we finally obtain a line bundle $\mathscr{L}$ on Y which extends $\mathscr{L}_K$. □

2.3. Global models of line bundles. In this section we study the obstructions of gluing the local extensions of rigid line bundles obtained in Proposition 2.5. The following proposition is a generalization of a result in [Ge].

Proposition 2.6. *Assume that the residue field k of R is separably closed. Let X_K be a smooth rigid analytic variety over K which admits a strict semi-stable formal model X over* $\mathrm{Spf}(R)$. *Let $R \to \tilde{R}$ be a finite extension of discrete valuation rings. Let $\tilde{V} \to \mathrm{Spf}(\tilde{R})$ be a strict semi-stable formal scheme. Assume that $\tilde{V}$ is small in the sense of* 1.4; *so let $\tilde{V}$ be smooth over $\tilde{R}\langle\zeta_1, \dots, \zeta_{t+1}\rangle/(\zeta_1 \cdot \ldots \cdot \zeta_{t+1} - \tilde{\pi})$. Let $\tilde{v} \in \tilde{V}(\tilde{R})$ be a point above $\{\zeta_1 = \cdots = \zeta_t = 1\}$. Consider a rigid analytic line bundle $\tilde{\mathscr{L}}_K$ on $(X \times_R \tilde{V})_{\mathrm{rig}}$ which is trivialized along $\tilde{v}$.*

Then $\tilde{\mathscr{L}}_K$ is a tensor product $\tilde{\mathscr{L}}_K \cong \tilde{\mathscr{M}}_K \otimes \tilde{\mathscr{N}}_{\mathrm{rig}}$ of two line bundles where

(1) *$\tilde{\mathscr{M}}_K \cong (\zeta_1^{n_1} \otimes \cdots \otimes \zeta_t^{n_t})$ is the line bundle on $(X \times_R \tilde{V})_{\mathrm{rig}}$ associated to suitable elements $n_1, \dots, n_t \in \mathrm{H}^1(X_0, \mathbb{Z})$. Such line bundles are called multiplicative.*

(2) *$\tilde{\mathscr{N}}$ is a formal line bundle on $X \times_R \tilde{V}$.*

The decomposition $\tilde{\mathscr{L}}_K \cong \tilde{\mathscr{M}}_K \otimes \tilde{\mathscr{N}}_{\mathrm{rig}}$ *is unique. In particular, the Picard group of line bundles which are trivial along* $\tilde{v}$ *decomposes into*

$$\mathrm{Pic}^0\big((X \times_R \tilde{V})_{\mathrm{rig}}\big) = \mathrm{H}^1(X_0, \mathbb{Z})^t \oplus \mathrm{Pic}^0(X \times_R \tilde{V}).$$

Remark 2.6.1. By applying the theory of ideals of coefficients as used in [L1], Proposition 2.9, one can show the following assertion in a more general situation.

If $V = \mathrm{Spf}(A)$ is small in the sense of 1.4 and sufficiently small in the topological sense, the ring A has a topological basis over the valuation ring R; cf. [FRG], II, 2.9. Namely, the ring $R\langle\zeta_1, \dots, \zeta_t\rangle/(\zeta_1 \cdot \dots \cdot \zeta_t - \pi)$ has such a basis over the valuation ring R and A has one over the latter ring. Lowering the assumptions on X, assume only that X is normal. Then for a given line bundle $\mathscr{L}_K$ on $(X \times_R V)_{\mathrm{rig}}$ there exists an admissible formal blowing-up $X' \to X$ such that $\mathscr{L}_K$ extends to a formal line bundle $\mathscr{L}'$ on $(U' \times_R V)$ where U' is a Zariski open covering of X'. Then one gets a similar global result for the given $\mathscr{L}_K$ as proposed in Proposition 2.6. However one thereby has to change the formal model X to X'. Whereas we are not allowed to do that here; cf. Proposition 3.3.

Proof of 2.6. The proof will be done in several steps:

(1) Let us first assume $R = \tilde{R}$ and let us first prove the assertion in this case. In the following we drop the tilde ˜. Due to Proposition 2.5 there exists an open covering $\{U^1, \dots, U^n\}$ of X and a line bundle $\mathscr{L}'$ on the disjoint union

$$Y' := X' \times_R V := \coprod_{i=1}^{n} U^i \times_R V$$

which extends the rigid analytic line bundle $\mathscr{L}'_K := p^*\mathscr{L}_K$ given on

$$Y'_K := Y'_{\mathrm{rig}} = X'_{\mathrm{rig}} \times_K V_{\mathrm{rig}} = \coprod_{i=1}^{n} U^i_{\mathrm{rig}} \times_K V_{\mathrm{rig}}$$

where p: $Y' \to X \times_R V$ is the projection. Now consider the descent situations

$$Y'' := Y' \times_Y Y' \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y' \xrightarrow{p} Y := X \times_R V,$$

$$Y''_{\mathrm{rig}} = Y'_{\mathrm{rig}} \times_{Y_{\mathrm{rig}}} Y'_{\mathrm{rig}} \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y'_{\mathrm{rig}} \xrightarrow{p} Y_{\mathrm{rig}} = X_{\mathrm{rig}} \times_K V_{\mathrm{rig}}$$

where p_i is the projection onto the i-th factor and p is the canonical projection. Since $\mathscr{L}_K$ is trivialized along v_K, we can choose the extension $\mathscr{L}'$ so that it is trivialized along v. The line bundle $\mathscr{L}'_{\mathrm{rig}}$ is isomorphic to $p^*\mathscr{L}_K$. So it gives rise to a descent datum on $Y''_K := Y''_{\mathrm{rig}}$

$$\varphi_K\colon (p_1^*\mathscr{L}')_{\mathrm{rig}} \xrightarrow{\sim} (p_2^*\mathscr{L}')_{\mathrm{rig}}.$$

Now we want to extend φ_K to a descent datum on the formal line bundle $\mathscr{L}'$. Therefore consider the formal line bundle

$$\mathscr{H} := \mathscr{H}om_{Y''}(p_1^*\mathscr{L}', p_2^*\mathscr{L}')$$

on Y'' as well as its associated rigid analytic line bundle on Y''_K

$$\mathscr{H}_K := \mathscr{H} \otimes_R K = \mathscr{H}om_{Y''_K}\big((p_1^*\mathscr{L}')_{\mathrm{rig}}, (p_2^*\mathscr{L}')_{\mathrm{rig}}\big).$$

φ_K is a global section of $\mathrm{Isom}_{Y''_K}\big((p_1^*\mathscr{L}')_{\mathrm{rig}}, (p_2^*\mathscr{L}')_{\mathrm{rig}}\big) \subset \Gamma(Y''_K, \mathscr{H}_K)$. Since $\mathscr{L}'$ is trivialized along v, so is $\mathscr{H}$, say by an isomorphism θ: $(\mathrm{id}_{X''} \times v)^*\mathscr{H} \xrightarrow{\sim} \mathscr{O}_{X''}$. Let U be a connected component of $X'' := X' \times_X X'$. So U is a union of open subschemes U^ν of X which satisfy the assumptions of Corollary 1.8; cf. Remark 1.6.1. Due to Corollary 1.8 we can write

$$\varphi_K | U_K^\nu \times_K V_K = a_\nu \psi^\nu \zeta_1^{n_1} \cdot \ldots \cdot \zeta_t^{n_t}$$

where the $n_\tau \in \mathbb{Z}(U^\nu)$ are uniquely determined, the $\psi^\nu \in \mathrm{Isom}_{U^\nu \times_R V}(p_1^*\mathscr{L}', p_2^*\mathscr{L}')$ and the $a_\nu \in \mathscr{O}_{X''_K}(U_K^\nu)^\times$. Next we use the trivialization $\theta(v^*\psi^\nu) \in \mathscr{O}_{X''}(U^\nu)^\times$ along v. After replacing ψ^ν by $\theta(v^*\psi^\nu)^{-1}\psi^\nu$ and a_ν by $a_\nu\theta(v^*\psi^\nu) \in \mathscr{O}_{X''_K}(U_K^\nu)^\times$, we have $\theta(v^*\psi^\nu) = 1$ and, hence, $a_\nu = v^*a_\nu$ is uniquely determined. This shows that $a = a_\nu$ does not depend on ν and, hence, that $a \in \mathscr{O}_{X''_K}(X''_K)^\times$ is a cocycle. Since $\mathscr{L}$ is trivialized along v_K, the cocycle a is a boundary. Thus we can transform the isomorphism between $\mathscr{L}'_{\mathrm{rig}}$ and $p^*\mathscr{L}_K$ so that the descent datum of $\mathscr{L}'_{\mathrm{rig}}$ obtains the form

$$\varphi'_K | U_K^\nu \times_K V_K := a^{-1} \cdot \varphi_K | U_K^\nu \times_K V_K = \psi^\nu \zeta_1^{n_1} \cdot \ldots \cdot \zeta_t^{n_t}.$$

The number n_τ is the order of φ_K on the irreducible component V_0^τ of V_0. So it is constant on U. Thus they define an element $n_\tau \in \mathrm{H}^1(X_0, \mathbb{Z})$. Then we define the multiplicative line bundle $\mathscr{M}_K := (\zeta_1^{n_1} \otimes \cdots \otimes \zeta_t^{n_t})$ on $X_K \times_K V_K$ via the cocycle

$$(\zeta_1^{n_1} \otimes \cdots \otimes \zeta_t^{n_t}) \in Z^1\big((X'' \times V)_{\mathrm{rig}}, \mathscr{O}^\times_{(X'' \times V)_{\mathrm{rig}}}\big).$$

In particular $p^*\mathscr{M}_K$ is trivial. Now define

$$\mathscr{N}_K := \mathscr{L}_K \otimes \mathscr{M}_K^\vee.$$

Then $p^*\mathscr{L}_K \cong p^*\mathscr{N}_K$ on Y'_K and the descent datum on $p^*\mathscr{N}_K$ is given by

$$\varphi'_K := \varphi_K \otimes \zeta_1^{-n_1} \otimes \cdots \otimes \zeta_t^{-n_t} = (\psi^\nu).$$

Therefore φ'_K extends to a formal descent datum

$$\varphi' := (\psi^\nu) : p_1^*\mathscr{L}' \xrightarrow{\sim} p_2^*\mathscr{L}'.$$

φ' satisfies the cocycle condition since φ'_K does. So $(\mathscr{L}', \varphi')$ descends to a formal line bundle $\mathscr{N}$ on $X \times_R V$. This yields the decomposition $\mathscr{L}_K \cong \mathscr{M}_K \otimes \mathscr{N}_{\mathrm{rig}}$ we are looking for. The decomposition is unique since the decomposition of the isomorphism was unique. This settles the first step.

Now we want to discuss the problem after base change by a ring extension $R \to \tilde{R}$. So consider the formal scheme $\tilde{X} := X \times_R \tilde{R}$.

(2) Let us first consider the case where $\tilde{V} \to \mathrm{Spf}(\tilde{R})$ is formally smooth over $\tilde{R}$ with geometrically irreducible special fiber and let $\tilde{v} \in \tilde{V}(\tilde{R})$ be a point. Then any line bundle $\tilde{\mathscr{L}}_K$ on $\tilde{X} \times \tilde{V}$ which is trivial on $\tilde{X}_K \times \tilde{v}$ extends to a line bundle $\tilde{\mathscr{L}}$ on $\tilde{X} \times_{\tilde{R}} \tilde{V}$. Namely, we can proceed similarly as in step (1). Due to Proposition 2.5 we have the local extension on $U^i \times_R \tilde{V}$. Then we can do the descent as explained in (1). All the $\zeta_1, \dots, \zeta_t$ disappear so that no multiplicative line bundle is necessary to achieve the situation where we can apply the formal descent.

Now we start with the general situation.

(3) Let $U := \coprod_{i=1}^{n} U^i$ be the open covering of X constructed in Proposition 2.5. Assume that all the U^i are so small that $H^1(U_0^i, \mathbb{Z}) = 0$ for $i = 1, \dots, n$ and that each U^i admits a semi-stable desingularization $\tilde{U}' \to \tilde{U} := U \otimes_R \tilde{R}$ as constructed in Proposition 1.10. Due to (1) there exists a line bundle $\tilde{\mathscr{M}}_K'$ on $(\tilde{U}' \times_{\tilde{R}} \tilde{V})_{\mathrm{rig}}$ and a line bundle $\tilde{\mathscr{N}}^*$ on $\tilde{U}' \times_{\tilde{R}} \tilde{V}$ such that

$$\tilde{\mathscr{L}}_K | (\tilde{U}' \times_{\tilde{R}} \tilde{V})_{\mathrm{rig}} \cong \tilde{\mathscr{M}}_K' \otimes \tilde{\mathscr{N}}_{\mathrm{rig}}^*$$

where $\tilde{\mathscr{M}}_K'$ is given by a cocycle $(\zeta_1^{n_1} \otimes \cdots \otimes \zeta_t^{n_t})$ for elements $n_1, \dots, n_t \in \mathrm{H}^1(\tilde{U}_0', \mathbb{Z})$. Due to Lemma 2.7 below, the elements n_τ are defined over U_0. Since $H^1(U_0, \mathbb{Z}) = 0$ they are trivial. Thus we see that $\tilde{\mathscr{L}}_K | (\tilde{U}' \times_{\tilde{R}} \tilde{V})_{\mathrm{rig}}$ extends to a formal line bundle $\tilde{\mathscr{N}}^*$ on $\tilde{U}' \times \tilde{V}$. Due to the general procedure in the proof of Proposition 2.5, in particular of its part in the case (2), the line bundle $\tilde{\mathscr{N}}^*$ descends to a line bundle $\tilde{\mathscr{N}}'$ on $\tilde{U} \times_{\tilde{R}} \tilde{V}$, after suitable modifications which take place in the special fiber only. Now we proceed as in step (1) by looking at the descent datum. Namely, one shows that there is a multiplicative line bundle $\tilde{\mathscr{M}}_K$ of the desired type such that $\tilde{\mathscr{L}}_K \otimes \tilde{\mathscr{M}}_K^{-1}$ extends to a line bundle $\tilde{\mathscr{N}}$ on $\tilde{X} \times_{\tilde{R}} \tilde{V}$.

Thus we obtained the desired result. □

Lemma 2.7. *Consider the situation of Proposition* 2.6. *Let $R \to \tilde{R}$ be a finite extension of discrete valuation rings. Let $\tilde{X} := X \times_R \tilde{R}$ and let $\tilde{X}' \to \tilde{X}$ be the desingularization of* 1.10.

(1) *Let $n \in \mathrm{H}^1(X_{\mathrm{rig}}, \mathbb{Z})$ be a cohomology class. Then the line bundle (ζ^n) on $X_{\mathrm{rig}} \times \overline{\mathbb{G}}_{m,K}$ is trivial if and only if $n = 0$.*

(2) $\mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(X_{\mathrm{rig}}, \mathbb{Z})$ *is bijective.*

(3) $\mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\tilde{X}_0, \mathbb{Z})$ *is bijective.*

(4) $\mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\tilde{X}_0', \mathbb{Z})$ *is bijective.*

Proof. (1) follows from the uniqueness of the decomposition of the group of invertible rigid analytic functions on $X_K \times_K \overline{\mathbb{G}}_{m,K}$; cf. [BGR], Lemma 9.7.1/1.

(2) The map is bijective due to (1) and step (1) in the proof of Proposition 2.6.

(3) follows because k is separably closed.

(4) Due to step (1) in the proof of Proposition 2.6 the canonical map

$$\mathrm{H}^1(\tilde{X}_0', \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\tilde{X}_{\mathrm{rig}}', \mathbb{Z})$$

is bijective. Since $\tilde{X}_{\mathrm{rig}}' \xrightarrow{\sim} \tilde{X}_{\mathrm{rig}}$ is an isomorphism, the map $\mathrm{H}^1(\tilde{X}_{\mathrm{rig}}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(\tilde{X}_{\mathrm{rig}}', \mathbb{Z})$ is bijective. For a cohomology class $\tilde{n}_K \in \mathrm{H}^1(\tilde{X}_{\mathrm{rig}}, \mathbb{Z})$ consider the associated line bundle $\tilde{\mathscr{M}}_K = (\zeta^{\tilde{n}_K})$ on $\tilde{X}_K \times_{\tilde{K}} \overline{\mathbb{G}}_{m,\tilde{K}}$. Due to step (2) in the proof of Proposition 2.6 this line bundle has a model $\tilde{\mathscr{M}}$ on $\tilde{X} \times_{\tilde{R}} \overline{\mathbb{G}}_{m,\tilde{R}}$ since $\overline{\mathbb{G}}_{m,\tilde{R}} := \operatorname{Spf} \tilde{R}\langle\zeta, \zeta^{-1}\rangle$ is formally smooth over $\operatorname{Spf}(\tilde{R})$. Due to Proposition 2.2/3, there exists an element $\tilde{n} \in \mathrm{H}^1(\tilde{X}_0, \mathbb{Z})$ such that the reduction $\tilde{\mathscr{M}}_0$ on $\tilde{X}_0 \times \mathbb{G}_{m,k}$ is associated to the cocycle $(\zeta^{\tilde{n}})$. Then $\tilde{\mathscr{M}}$ is isomorphic to the line bundle on $\tilde{X} \times \overline{\mathbb{G}}_{m,R}$ associated to the cocycle $(\zeta^{\tilde{n}})$. Namely, by the Nakayama Lemma one shows that $\tilde{\mathscr{M}}$ trivializes on an open covering $\{\tilde{U}^i \times_R \overline{\mathbb{G}}_{m,R}\}$ where $\tilde{U}^i$ are open subschemes of $\tilde{X}$. Then $\tilde{n}_K$ must be equivalent to $\tilde{n}$. Due to (3) we can replace $\tilde{n}$ by some $n \in \mathrm{H}^1(X_0, \mathbb{Z})$. □

3. Proof of the main theorem

In this section we want to prove our main theorem announced in the introduction. Let us fix the notations for this section.

Let X_K be a proper smooth connected rigid space over K with a rational point x_K. Let X be a formal model of X_K over the formal spectrum $\operatorname{Spf}(R)$ of the complete discrete valuation ring R of K. The given point x_K extends to an R-point x of X in a unique way. Assume that X is strict semi-stable; cf. Definition 1.1. For technical reasons which will be explained later, we consider also $\tilde{X} := X \times_R \tilde{R}$ after base change by a finite extension $R \to \tilde{R}$, so we may loose the regularity. We remind the reader that we do not make use of the regularity of the given X; we can prove the main result also for schemes like $\tilde{X}$ which are obtained by base change from a semi-stable one. For a fixed field K, by abuse of notations it can already be an extension of the field we started with, we consider the following functor

$$\mathrm{Pic}_{X_K/K}\colon (\text{smooth rigid } K\text{-spaces}) \to (\text{sets}), \quad V_K \mapsto \mathrm{Pic}_{X_K/K}(V_K)$$

where

$$\mathrm{Pic}_{X_K/K}(V_K) = \left\{ \mathrm{Isoclass}(\mathscr{L}, \lambda)\colon \begin{array}{l} \mathscr{L} \text{ line bundle on } X_K \times_K V_K, \\ \lambda\colon \mathcal{O}_{V_K} \xrightarrow{\sim} (x \times \mathrm{id}_{V_K})^* \mathscr{L} \text{ isomorphism} \end{array} \right\}.$$

This is a contravariant functor. The main purpose of this section is to show the representability of this functor. For technical reasons we are first interested only in the representability of the neutral component of the representing space. Therefore we introduce the following categories:

$\mathfrak{C}_K$ the category of pointed rigid spaces (V_K, v_K) where V_K is smooth and connected over K and where $v_K \in V_K(K)$ is a K-rational point. The morphisms in this category are the rigid morphisms respecting the points.

$\overline{\mathfrak{C}}_K$ the full subcategory of $\mathfrak{C}_K$ consisting of such V_K which admit smooth formal models V over $\operatorname{Spf}(R)$.

$\hat{\mathfrak{C}}_K$ the full subcategory of $\mathfrak{C}_K$ consisting of such V_K which satisfy

$$\mathrm{H}^1(V_K \hat{\otimes}_K \mathbb{K}, \mathbb{Z}) = 0$$

where $\mathbb{K}$ is the topological algebraic closure of K.

If $K \to \tilde{K}$ is a ramified resp. an inseparable finite field extension, then $\bar{\mathfrak{C}}_K \otimes_K \tilde{K}$ resp. $\mathfrak{C}_K \otimes_K \tilde{K}$ can be strictly smaller than $\bar{\mathfrak{C}}_{\tilde{K}}$ resp. $\mathfrak{C}_{\tilde{K}}$.

Instead of considering the general Picard functor, we look at the slightly different functor

$$\mathrm{Pic}^0_{X_K/K}\colon \mathfrak{C}_K \to (\mathrm{sets}), \quad (V_K, v_K) \mapsto \mathrm{Pic}^0_{X_K/K}(V_K, v_K)$$

of rigidified line bundles which are trivialized at the given point where

$$\mathrm{Pic}^0_{X_K/K}(V_K, v_K) = \left\{ \mathrm{Isoclass}(\mathscr{L}_K, \lambda)\colon \begin{array}{c} \mathscr{L}_K \text{ line bundle on } X_K \times_K V_K, \\ \lambda\colon \mathscr{O}_{V_K} \xrightarrow{\sim} (x \times \mathrm{id}_{V_K})^* \mathscr{L}_K \text{ isomorphism}, \\ (\mathrm{id}_{X_K} \times v_K)^* \mathscr{L}_K \cong \mathscr{O}_{X_K} \text{ trivial} \end{array} \right\}.$$

These are somehow the deformations of the trivial line bundle or algebraic correspondences in old terminology. Later on it turns out that the representing space of this functor is the 1-component of the usual Picard functor. We can restrict the Picard functor $\mathrm{Pic}^0_{X_K/K}$ associated to (X_K, x_K) to these categories. So we obtain the following functors:

$$\begin{aligned} \mathrm{Pic}^0_{X_K/K} &= \mathrm{Pic}^0_{X_K/K}|\mathfrak{C}_K, \\ \overline{\mathrm{Pic}}^0_{X_K/K} &= \mathrm{Pic}^0_{X_K/K}|\bar{\mathfrak{C}}_K, \\ \widehat{\mathrm{Pic}}^0_{X_K/K} &= \mathrm{Pic}^0_{X_K/K}|\hat{\mathfrak{C}}_K, \end{aligned}$$

and we will study their representability. Moreover we will see that the representing spaces are compatible with finite base field extensions $K \to \tilde{K}$; i.e. solve the similar representation problem also after such a base change.

The procedure is the following: First we will deduce from the fundamental theorem of Artin the representability of $\overline{\mathrm{Pic}}^0_{X_K/K}$ by a space $\bar{P}_K \in \bar{\mathfrak{C}}_K$. We will analyse the structure of $\bar{P}_K$. It is an extension of a formal abelian scheme over $\mathrm{Spf}(R)$ by a formal torus $\bar{T}$. We interpret $\bar{T}$ via the multiplicative line bundles which are induced by $\mathrm{H}^1(X_K, \mathbb{Z})$. This leads to the representability of $\widehat{\mathrm{Pic}}^0_{X_K/K}$ by a space $\hat{P}_K \in \hat{\mathfrak{C}}_K$ which is the pushforward of $\bar{P}_K$ via the open immersion $\bar{T}_K \hookrightarrow T_K$ of the formal torus into the affine torus. Finally $\mathrm{Pic}^0_{X_K/K}$ will be represented by a quotient $\hat{P}_K/M$ of $\hat{P}_K$ by a lattice M of $\hat{P}_K$. As usual the Yoneda lemma gives rise to universal line bundles

$$\begin{array}{lll} \mathscr{P} & \text{on } X_K \times P_K & \text{with rigidificator } \rho, \\ \bar{\mathscr{P}} & \text{on } X_K \times \bar{P}_K & \text{with rigidificator } \bar{\rho}, \\ \hat{\mathscr{P}} & \text{on } X_K \times \hat{P}_K & \text{with rigidificator } \hat{\rho}. \end{array}$$

These line bundles satisfy the universal property and are called *Poincaré bundles*.

3.1. The representability of $\overline{\mathrm{Pic}}^0_{X_K/K}$. Set $R_n := R/(\pi^{n+1})$ and $S_n := \mathrm{Spec}(R_n)$, hence $X_n := X \times_S S_n$ is a proper flat S_n-scheme with geometrically reduced special fiber. The point $x \in X(S)$ induces a point $x_n \in X_n(S_n)$ for all $n \in \mathbb{N}$. So we can introduce the functor $\mathrm{Pic}^0_{X_n/S_n}$ on the category of S_n-schemes locally of finite type. Due to the classical result of M. Artin [Ar 1], Theorem 7.3, this functor is representable by an algebraic space

$$P'_n := \mathrm{Pic}^0_{X_n/S_n}$$

locally of finite type over S_n. This is a group scheme over S_n since an algebraic group space over an Artinian base is a scheme; cf. [Ar 2], Theorem 3.5. It is quasi-compact, since it is a connected group scheme; cf. [SGA 3], I, Exposé $\mathrm{VI_A}$, Proposition 2.4. So it is of finite type. Furthermore we have the Poincaré bundle $\mathscr{P}'_n$ on $X_n \times_{S_n} P'_n$. We set

$$P' = \varinjlim P'_n.$$

The limit is defined via the projections $P'_{n+1} \to P'_n$; notice that $P'_{n+1} \times_{S_{n+1}} S_n = P'_n$ as $X_{n+1} \times_{S_{n+1}} S_n = X_n$. The formal scheme P'/S is of topological finite type over S.

Lemma 3.1. (1) *There is no $\mathbb{G}_{a,\tilde{k}}$ in P'_0 for any finite field extension $\tilde{k}$ of k. In particular the maximal reduced subscheme of P'_0 is smooth.*

(2) *If ℓ is prime to* $\mathrm{char}(k)$, *any ℓ-torsion point of P'_0 lifts to an ℓ-torsion point of P' which is defined over $\tilde{R}$ where $R \to \tilde{R}$ is unramified.*

(3) *The set of torsion points of order prime to* $\mathrm{char}(k)$ *is a family of geometrically reduced points in P'_{rig} which is dense in the formal topology of P'.*

Proof. (1) Any map $\mathbb{G}_{a,\tilde{k}} \to P'_0$ induces a line bundle $\mathscr{L}$ on $X_0 \times_k \mathbb{G}_{a,\tilde{k}}$ via pulling back the Poincaré bundle. This bundle is trivial over $X_0 \times 0$ where 0 is the zero section of $\mathbb{G}_{a,\tilde{k}}$. Due to Proposition 2.2/4 the bundle $\mathscr{L}$ is trivial. So the map $\mathbb{G}_{a,\tilde{k}} \to P'_0$ is constant 1. Then it follows from [FGA], Exposé 236, Proposition 3.1, that the maximal reduced subscheme of P'_0 is smooth.

(2) Every ℓ-torsion point has values in a finite separable field extension $\tilde{k}$ of k. We claim that it lifts to an $\tilde{R}$-valued point of P' where $\tilde{R}$ is the finite unramified extension of R with residue field $\tilde{k}$. Indeed, every such point corresponds to a unique element of $\mathrm{H}^1_{\text{ét}}(X_0 \times_k \tilde{k}, \mu_\ell)$ and therefore to a unique isomorphism class of étale μ_ℓ-Galois coverings $Y_0 \to X_0 \times_k \tilde{k}$ by [FK], Proposition I.2.11, where μ_ℓ denotes the group of ℓ-roots of unity. The covering can compatibly be extended to a μ_ℓ-Galois covering $Y_n \to X_n \times_{R_n} \tilde{R}_n$ by [SGA 1], Exposé I, Corollaire 8.4. This again gives a torsion point of order ℓ in P'_n. In the limit we obtain an $\tilde{R}$-valued point of P'.

(3) P'_0 is of finite type over k. Due to (1) the set of torsion points of order prime to $\mathrm{char}(k)$ is Zariski-dense in the maximal reduced subscheme and thus also in P'_0. Then the assertion follows from (2). □

In general P' does not need to be flat over S. This is related to the fact that there may exist line bundles on X_n which do not lift to line bundles on X_{n+1}. Furthermore we are not interested in nilpotent structures. Thus, it is natural for our purpose to look at the closed subscheme

$$\overline{P'} := P'/(\mathcal{N} : \pi) \hookrightarrow P',$$

defined by dividing out the nilpotent structure and the π-torsion. Thereby we have used the notation

$$\mathcal{N}(U) := \{f \in \mathcal{O}_{P'}(U); f \text{ nilpotent}\},$$
$$(\mathcal{N} : \pi)(U) := \{f \in \mathcal{O}_{P'}(U); \exists n \in \mathbb{N} \text{ such that } \pi^n \cdot f \in \mathcal{N}(U)\}$$

for any open subset U of P'. Then $\overline{P'}$ is reduced and flat over R, since it has no π-torsion.

Since any pointed map from a reduced connected flat formal R-scheme to P' factors through $\overline{P'}$, the group structure of P' induces a group structure on $\overline{P'}$. The generic fiber of the formal scheme $\overline{P'}$ is a rigid analytic group variety $\overline{P'}_{\mathrm{rig}}$. In general a reduced group variety is not geometrically reduced. But if the characteristic of K is zero, this group variety is smooth as the usual argument of Cartier shows. For the general case there is the following example which remains reduced after any finite field extension but not after the extension by the topological algebraic closure.

Example 3.1.1. Let $k := \mathbb{F}_p(t_n, n \in \mathbb{N})$ and $R := k[\![u]\!]$ and $K := k((u))$. This means the variables t_n have absolute value 1 and u is a variable having absolute value less than 1. Set

$$A_K := K\langle S, T\rangle \Big/ \left(S^p - \sum_{n=1}^{\infty} u^n t_n T^{p^n} \right)$$

then $\mathrm{Sp}(A_K)$ is an analytic group subvariety of $\overline{\mathbb{G}}_{a,K}^2$. For the topological algebraic closure $\mathbb{K}$ of K the extension $A_K \hat{\otimes}_K \mathbb{K}$ is not reduced since the p-th root of the sum exists in $A_K \hat{\otimes}_K \mathbb{K}$. On the other hand the p-th root does not exist in $A_K \otimes_K K'$ for any finite field extension K' of K. So $A_K \otimes_K K'$ remains reduced.

In our case however $\overline{P'}_{\mathrm{rig}}$ is geometrically reduced due to the semi-stability of X.

Lemma 3.2. (1) *The generic fiber* $\overline{P'}_{\mathrm{rig}}$ *is smooth.*

(2) *The canonical map* $\overline{P'}_0 \to P'_0$ *is a surjective closed immersion.*

Proof. (1) On the generic fiber the set of torsion points of order prime to $\mathrm{char}(k)$ forms a subset E_K in the 1-component $\bar{P}_K$ of the generic fiber $\overline{P'}_{\mathrm{rig}}$ which is dense in the formal topology due to 3.1. This implies that for any formal open subset $U \subseteqq P'$ the map

$$\mathcal{O}_{P'}(U_{\mathrm{rig}} \cap \bar{P}_K)_{\mathrm{red}} \hookrightarrow \prod_{x \in E_K \cap U_K}' K(x), \quad f \mapsto \big(f(x)\big)_{x \in E_K \cap U_K}$$

is an isometric embedding. Thereby $\prod'$ denotes the restricted product consisting of those elements of the direct product which are uniformly bounded. It is equipped with the norm $\|\big(f(x)\big)\| := \sup\{|f(x)| : x \in E_K \cap U_K\}$. We have to extend this to the topological algebraic closure $\mathbb{K}$ of K. Since K is a complete discretely valued field, it is stable in the sense of [BGR], Definition 3.6.1/1. This implies that the map

$$\mathcal{O}_{P'}(U_{\mathrm{rig}} \cap \bar{P}_K)_{\mathrm{red}} \otimes_K K^{\mathrm{alg}} \hookrightarrow \prod_{x \in E_K \cap U_K}{}' \bigl(K(x) \,\hat{\otimes}_K\, \mathbb{K}\bigr)$$

also is an isometric embedding. Passing to the completion therefore the same is true for the map

$$\mathcal{O}_{P'}(U_{\mathrm{rig}} \cap \bar{P}_K)_{\mathrm{red}} \,\hat{\otimes}_K\, \mathbb{K} \hookrightarrow \prod_{x \in E_K \cap U_K}{}' \bigl(K(x) \,\hat{\otimes}_K\, \mathbb{K}\bigr).$$

Since $K(x)$ is finite separable over K the algebra $K(x) \,\hat{\otimes}_K\, \mathbb{K} = K(x) \otimes_K \mathbb{K}$ is reduced. This shows that $\overline{P'}_{\mathrm{rig}}$ is geometrically reduced. Being a rigid group variety it is smooth; cf. [Ki], Definition 4.4.

(2) Since the torsion points of order prime to $\mathrm{char}(k)$ in P'_0 lift to P' and factor through $\overline{P'}$, the image of $\overline{P'}_0$ is dense in P'_0. Then, as a closed immersion the map must be surjective. □

Proposition 3.3. *The pointed formal R-scheme $(\overline{P'}, 1)$ represents the functor* $\mathrm{Pic}^0_{X/R}$ *of trivialized and rigidified line bundles on the category of pointed formal R-schemes which are admissible, reduced and connected.*

The canonical map $\overline{P'} \to P'$ is finite and induces a homeomorphism on the special fibers. On the generic fibers it gives rise to an isomorphism $\overline{P'}_{\mathrm{rig}} \xrightarrow{\sim} (P'_{\mathrm{rig}})_{\mathrm{red}}$.

The rigid space $\overline{P'}_{\mathrm{rig}}$ is smooth. The 1*-component $\bar{P}_K := (\overline{P'})^0_{\mathrm{rig}}$ has finite index in $\overline{P'}_{\mathrm{rig}}$.*

Proof. Let $\overline{P'}_n \to P'_n$ be the induced infinitesimal neighborhoods. Let further $\overline{\mathcal{P}'}$ be the pullback of the line bundle $\mathcal{P}' := \varprojlim \mathcal{P}'_n$ from $X \times_R P'$ to $X \times_R \overline{P'}$. Now consider a pointed formal R-scheme (V, v) which is admissible, reduced and connected. Let $(\mathcal{L}, \lambda)$ be a rigidified line bundle on $X \times_R V$ which is trivial on $X \times_R v$. This gives rise to a rigidified line bundle $(\mathcal{L}_n, \lambda_n)$ on $X_n \times_{R_n} V_n$. Since $\mathcal{L}_n$ is trivial over v_n and V_n is connected, one obtains a unique morphism $f_n\colon V_n \to P'_n$ and furthermore a unique isomorphism $\varphi_n\colon \mathcal{L} \otimes_R R_n \xrightarrow{\sim} (\mathrm{id}_{X_n} \times f_n)^* \mathcal{P}'_n$. Due to the uniqueness one gets $f_n = f_{n+1} \otimes_{R_{n+1}} R_n$. So they give rise to a morphism $f\colon V \to P'$. Since V is flat over S, reduced and connected, this morphism factors through $\overline{P'}$. Due to the rigidificator there is a unique isomorphism $\varphi\colon \mathcal{L} \xrightarrow{\sim} (\mathrm{id}_X \times f)^* \mathcal{P}'$ where $\mathcal{P}'$ is the Poincaré bundle on $X \times P'$.

The smoothness follows from Lemma 3.2. The index is finite since P' is quasi-compact. □

The candidate for representing $\overline{\mathrm{Pic}}^0_{X_K/K}$ is the 1-component of the rigid group $\overline{P'}_{\mathrm{rig}}$

$$\bar{P}_K := (\overline{P'})^0_{\mathrm{rig}} = (P'_{\mathrm{red}})^0_{\mathrm{rig}}.$$

It remains to show that $\bar{P}_K$ admits a smooth formal model over R.

Due to the theory of formal Néron models, the quasi-compact group $\bar{P}_K$ admits a smooth formal model $\bar{P}$ over R, cf. [L2], Theorem 2.2 or [BS], Theorem 1.2. It can be obtained as the 1-component of the group smoothening of $\overline{P'}$; cf. [BLR], Theorem 7.1/5. It has the properties:

1. (*Group smoothening*) There is a canonical map $\bar{P} \to \overline{P'} \to P'$.

2. (*Néron mapping property*) Any rigid morphism $f_K\colon V_{\mathrm{rig}} \to \bar{P}_K$ of the generic fiber of a smooth formal scheme V factors through a unique morphism $f\colon V \to \bar{P}$.

Therefore every torsion point of P'_0 of order prime to $\mathrm{char}(k)$ lifts to a point of $\bar{P}$. Since these points are dense, the group homomorphism $\bar{P} \to \overline{P'}$ is surjective on special fibers. It is finite, since it is a surjective homomorphism of group schemes of the same dimension. In particular the rigid fiber of $\bar{P}$ is $\bar{P}_{\mathrm{rig}} = \bar{P}_K$. Furthermore, we obtain a universal line bundle $\bar{\mathscr{P}}$ on $X \times_S \bar{P}$ via pull-back. Due to the Néron mapping property, this 1-component represents the functor $\overline{\mathrm{Pic}}^0_{X_K/K}$ on $\bar{\mathfrak{C}}_K$.

Proposition 3.4. *The formal Néron model $\bar{P}$ of $\bar{P}_K = (P'_{\mathrm{red}})^0_{\mathrm{rig}}$ represents the functor $\overline{\mathrm{Pic}}^0_{X_K/K}$ on the category $\bar{\mathfrak{C}}_K$. This Néron model $\bar{P}$ can be obtained as the group smoothening $\bar{P} \to P'_{\mathrm{red}}$. The latter morphism is finite and a* rig-*isomorphism on* 1-*components.*

Proof. Let V_K be a smooth rigid variety which is connected and admits a smooth formal model V over $\mathrm{Spf}(R)$ and let v_K be a rational point of V_K. Then consider a rigidified line bundle $\mathscr{L}_K$ on $X_K \times_K V_K$ which is trivial over $X_K \times_K v_K$. The point v_K extends to an R-point v of V in a unique way. Due to Proposition 2.6 the line bundle $\mathscr{L}_K$ extends to a line bundle $\mathscr{L}$ on $X \times_R V$ since V is smooth over R. By using the sections x and v one can choose a rigidified extension which is also trivial over v. Due to Proposition 3.3 there exists a unique morphism $f\colon V \to \overline{P'}$ with $\mathscr{L} \cong (\mathrm{id}_X \times f)^*\overline{\mathscr{P}'}$. The uniqueness of f follows from the separatedness of $\overline{P'}$; cf. Lemma 3.5. Due to the universal property of $\bar{P}$, the morphism f factors through $\bar{P}$. In particular, we get an isomorphism $\mathscr{L} \cong (\mathrm{id}_X \times f)^*\bar{\mathscr{P}}$ respecting the rigidificators. □

Remark 3.4.1. The proof of 3.4 shows that the pointed formal R-scheme $(\bar{P}, 1)$ represents the functor $\mathrm{Pic}^0_{X/R}$ of rigidified line bundles on the category of pointed formal R-schemes which are smooth and connected.

Lemma 3.5. *A morphism $f\colon V \to P'$ from a smooth connected formal R-scheme V is uniquely determined by its generic fiber.*

In particular, if $\mathscr{L}$ is a formal line bundle on X such that $\mathscr{L}$ belongs to P'; i.e., $\mathscr{L} \otimes_R k$ belongs to $\mathrm{Pic}^0_{X_0/k}$ *and if $\mathscr{L}_{\mathrm{rig}}$ is trivial, then $\mathscr{L}$ is trivial.*

Proof. This follows from the fact that P'_n is separated over R_n for all $n \in \mathbb{N}$ as being a connected group scheme over an Artinian base. □

3.2. The structure of $\bar{P}_K$. Next we want to study the structure of $\bar{P}_K$; cf. Proposition 3.4. Due to the construction $\bar{P}_K$ has a smooth formal model $\bar{P}$ over $\mathrm{Spf}(R)$. We want to show that the special fiber $\bar{P}_0$ is semi-abelian; i.e., it is an extension of an abelian variety over k by a torus. We know that over the algebraic closure $\bar{k}$ of k the base change $\bar{P}_0 \otimes_k \bar{k}$

is an extension of an abelian variety $B_{\bar{k}}$ by a smooth linear group $L_{\bar{k}}$ due to the theorem of Chevalley; cf. [BLR], Theorem 9.2.1:

$$1 \to L_{\bar{k}} \to \bar{P}_0 \otimes_k \bar{k} \to B_{\bar{k}} \to 1.$$

Due to [SGA 3], II, Exposé XVII, Théorème 7.2.1,

$$L_{\bar{k}} = T_{\bar{k}} \times_{\bar{k}} U_{\bar{k}}$$

is the direct product of a group of multiplicative type $T_{\bar{k}}$ and a unipotent group $U_{\bar{k}}$. Since $L_{\bar{k}}$ is smooth over $\bar{k}$, affine and connected, the same holds for $T_{\bar{k}}$ and $U_{\bar{k}}$. Due to [SGA 3], II, Exposé X, Proposition 1.4, $T_{\bar{k}}$ is diagonal; i.e.,

$$T_{\bar{k}} \cong \operatorname{Spec} \bar{k}[M] = \big(\operatorname{Hom}(\mathbb{Z}, M)\big)_{\operatorname{Spec}(\bar{k})}$$

for an abelian group M. Since $T_{\bar{k}}$ is of finite type over $\bar{k}$, the group M is finitely generated; cf. [SGA 3], II, Exposé VIII, Proposition 2.1 and, hence, a direct sum of a free abelian group and a finite group. Due to [SGA 3], II, Exposé VIII, Théorème 3.1, $T_{\bar{k}}$ is the product of a torus and a finite k-group. Since $T_{\bar{k}}$ is connected and smooth, the torsion part of M is trivial; cf. [SGA 3], II, Exposé VIII, Proposition 2.1. Thus $T_{\bar{k}} \cong \mathbb{G}^r_{m,\bar{k}}$ is a torus.

Now we want to show that the unipotent part $U_{\bar{k}}$ is trivial. Due to [SGA 3], II, Exposé XVII, Proposition 4.1.1, this is equivalent to the fact that there is no additive group of type $\mathbb{G}_{a,\bar{k}}$ contained in $\bar{P}_0 \otimes_k \bar{k}$. So assume that there is a closed immersion

$$\mathbb{G}_{a,\bar{k}} \hookrightarrow \bar{P}_0 \to P'_0.$$

By the finiteness of the later map this however is a contradiction to Lemma 3.1. Since the unipotent part of $\bar{P}_0$ is trivial the base change to the algebraic closure $\bar{k}$ was not necessary in the previous discussion. Therefore we have shown the following result.

Proposition 3.6. *The group $\bar{P}_K$ has a smooth model $\bar{P}$ over R with semi-abelian reduction; i.e., the special fiber of $\bar{P}_0$ is an extension*

$$1 \to T_0 \to \bar{P}_0 \to B_k \to 1$$

of an abelian variety B_k by a torus T_0.

The torus T_0 of $\bar{P}_0$ lifts to a torus $T_n \hookrightarrow \bar{P}_n$ in a unique way for each $n \in \mathbb{N}$ and finally to a formal torus $\bar{T}$ of $\bar{P}$; due to [SGA 3], II, Exposé XV, Corollaire 2.3,

$$\bar{T} := \varinjlim T_n \hookrightarrow \bar{P}.$$

The factor group $B_n := \bar{P}_n/T_n$ exists as group scheme of finite type over R_n and the morphism $\bar{P}_n \to B_n$ is faithfully flat. In particular, one has

$$B_{n-1} \cong B_n \times_{R_n} R_{n-1}.$$

So there exists the formal R-group scheme

$$B := \varinjlim B_n$$

which is smooth and proper over $\mathrm{Spf}(R)$; it is a formal abelian scheme. For further details see [BL], Section 3 and [Se], Theorem VII.3.6.

Proposition 3.7. *The group $\bar{P}_K$ has a smooth model $\bar{P}$ over R. The formal group scheme $\bar{P}$ is an extension*

$$1 \to \bar{T} \to \bar{P} \to B \to 1$$

of a formal abelian scheme B by a formal torus $\bar{T}$.

$\bar{P}$ is a $\bar{T}$-torsor over B. The extension is equivalent to a group homomorphism $\chi(\bar{T}) \to B^\vee$ of the character group of $\bar{T}$ to the dual formal abelian scheme of B.

The next point is to understand the torus part in $\bar{P}$. It can be described by the topology of the formal scheme X; i.e. by the combinatorial configuration of the smooth irreducible components of X_0.

Proposition 3.8. *Assume that the residue field k of R is separably closed. There exists a commutative diagram of canonical group homomorphisms*

$$\begin{array}{ccc} \mathrm{H}^1(X_0, \mathbb{Z}) & \longrightarrow & \mathrm{H}^1(X_K, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathbb{G}_{m,k}, P_0') & \longleftarrow & \mathrm{Hom}(\overline{\mathbb{G}}_{m,R}, \bar{P}). \end{array}$$

The vertical maps are given by sending a cocycle $n := (n_{ij})$ to the group homomorphism which sends a point t of $\mathbb{G}_m$ to the line bundle given by the cocycle $(t^{n_{ij}})$. The upper horizontal map is induced by the continuous reduction map $X_K \to X_0$. The lower horizontal one is given by reducing morphisms.

All these homomorphisms are bijective. In particular

$$\mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^1(X_K, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\overline{\mathbb{G}}_{m,R}, \bar{T}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_{m,k}, \bar{T}_0)$$

are free of finite rank r equal to the rank of the torus parts.

Proof. This follows from the following facts where ζ denotes a coordinate on $\mathbb{G}_m$:

1.1. Any rigid line bundle on $X_K \times \mathbb{G}_{m,K}$ is locally trivial over X_K. Namely, due to Proposition 2.6 the line bundle extends to a formal line bundle on $X \times \overline{\mathbb{G}}_{m,R}$. Then its reduction is locally trivial over X_0 due to Proposition 2.2. Then a Nakayama argument yields the assertion.

1.2. Any invertible rigid function ε on $U_K \times \overline{\mathbb{G}}_{m,K}$ for an open affinoid subvariety U_K of X_K decomposes uniquely into $\varepsilon = u \cdot \zeta^n \cdot (1 + h)$ with function u on U_K and a function of absolute value $|h| < 1$. The integer $n \in \mathbb{Z}$ is uniquely determined by ε; cf. [BGR], Lemma 9.7.1/1.

2.1. Any algebraic line bundle on $X_0 \times \mathbb{G}_{m,k}$ is locally trivial over X_0; cf. Proposition 2.2(2).

2.2. Any invertible function ε on $U_0 \times \mathbb{G}_{m,k}$ for an open affine subvariety of X_0 decomposes uniquely into $\varepsilon = u \cdot \zeta^n$ with a function u on U_0. The integer $n \in \mathbb{N}$ is uniquely determined by ε.

Using 1.1 and 1.2 one shows easily the injectivity of the vertical map on the right. From 2.1 and 2.2 follows that the map on the left is bijective. Moreover the horizontal map on the bottom is easily seen to be injective by looking at points of finite order. $\square$

3.3. The universal covering $\hat{P}_K$. The formal torus $\bar{T}_{\mathrm{rig}}$ is embedded into the affine torus T_K; namely by the map associated to the inclusion $K[\chi(\bar{T})] \hookrightarrow K\langle\chi(T_K)\rangle$ of their coordinate rings after identifying the character groups

$$\chi(\bar{T}) = \mathrm{Hom}(\bar{T}, \bar{\mathbb{G}}_{m,R}) = \mathrm{Hom}(T_K, \mathbb{G}_{m,K}) = \chi(T_K).$$

Thus the morphism $\varphi \colon \chi(\bar{T}) \to B^\vee$ of Proposition 3.7 induces a morphism $\chi(T_K) \to B^\vee_{\mathrm{rig}}$ and, hence, an extension of B_{rig} by the affine torus T_K

$$1 \to T_K \to \hat{P}_K \to B_{\mathrm{rig}} \to 1.$$

This is the push-forward of $\bar{P}_{\mathrm{rig}}$ associated to the open immersion $\bar{T}_{\mathrm{rig}} \hookrightarrow T_K$. So we obtain the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bar{T}_{\mathrm{rig}} & \longrightarrow & \bar{P}_{\mathrm{rig}} & \xrightarrow{\bar{q}} & B_{\mathrm{rig}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T_K & \longrightarrow & \hat{P}_K & \xrightarrow{q} & B_{\mathrm{rig}} & \longrightarrow & 1. \end{array}$$

Both vertical arrows on the left are open immersions. If the torus is split, e.g. if k is separably closed, the extension can be described in an easy way. Namely, locally over B, such an extension splits; i.e., there exists an open covering $\{U^1, \dots, U^n\}$ of B such that

$$\bar{P} \times_B U^\nu \cong \bar{T} \times_R U^\nu \cong \bar{\mathbb{G}}^r_{m,R} \times_R U^\nu,$$

$$\hat{P}_K \times_{B_{\mathrm{rig}}} U^\nu_{\mathrm{rig}} \cong T_K \times_K U^\nu_{\mathrm{rig}} \cong \mathbb{G}^r_{m,K} \times_K U^\nu_{\mathrm{rig}}.$$

The pasting is given in both cases by $\varphi \colon \chi(\bar{T}) \to B^\vee$. Recall that there is a rigidified Poincaré bundle on $(\bar{\mathscr{P}}, \bar{\rho})$ on $X \times_R \bar{P}$.

Lemma 3.9. *The rigidified line bundle $(\bar{\mathscr{P}}, \bar{\rho})$ on $X \times_R \bar{P}$ extends to a rigidified line bundle $(\hat{\mathscr{P}}_K, \hat{\rho})$ on $X_K \times_K \hat{P}_K$ in a unique way as cubical line bundle.*

Proof. From the formal R-group extension

$$1 \to \bar{T} \to \bar{P} \to B \to 1$$

we obtain by the base change $X \to \mathrm{Spf}(R)$ a formal X-group extension

$$1 \to \bar{T}_X \to \bar{P}_X \xrightarrow{\bar{q}} B_X \to 1.$$

The Poincaré bundle $\overline{\mathscr{P}}$ on $\bar{P}_X$ has a canonical cubical structure; cf. [Mo]. Namely, the cubical structure is induced by its universal property. To explain this, consider the line bundle

$$\mathscr{D}_3\overline{\mathscr{P}} := \mu_{123}^*\overline{\mathscr{P}} \otimes \mu_{12}^*\overline{\mathscr{P}}^\vee \otimes \mu_{13}^*\overline{\mathscr{P}}^\vee \otimes \mu_{23}^*\overline{\mathscr{P}}^\vee \otimes \mu_1^*\overline{\mathscr{P}} \otimes \mu_2^*\overline{\mathscr{P}} \otimes \mu_3^*\overline{\mathscr{P}}$$

on the threefold product $\bar{P}_X^3$ where $\mu_I\colon \bar{P}_X^3 \to \bar{P}_X$ is the morphism $(p_1, p_2, p_3) \mapsto \sum_{i \in I} p_i$ induced by adding points. A cubical structure on $\overline{\mathscr{P}}$ is a trivialization τ of $\mathscr{D}_3\overline{\mathscr{P}}$ which satisfies certain symmetry and cocycle conditions; cf. [Mo], Définition I.2.4.5. Due to the universal property of $\overline{\mathscr{P}}$ we know

$$\mu_I^*\overline{\mathscr{P}} \cong \bigotimes_{i \in I} \mu_i^*\overline{\mathscr{P}};$$

cf. Remark 3.4.1. So there exists a trivialization τ of $\mathscr{D}_3\overline{\mathscr{P}}$. Since $\mathscr{D}_3\overline{\mathscr{P}}$ is rigidified, τ satisfies the symmetry and cocycle condition; cf. [Mo], Définition I.2.4.5. Thus there is a cubical structure on $\overline{\mathscr{P}}$. If $\overline{\mathscr{P}}|\bar{T}_X$ is trivial, the cubical line bundle $\overline{\mathscr{P}}$ descends to a line bundle on B_X. This of course applies to any open subvariety U of X. Due to Proposition 2.6, there exists an open formal covering $\{X^1, \dots, X^N\}$ of X and trivializations

$$\delta^\nu\colon \overline{\mathscr{P}}|_{X^\nu \times_R \bar{T}} \xrightarrow{\sim} \mathscr{O}_{X^\nu \times_R \bar{T}}$$

for $\nu = 1, \dots, N$. Now consider the line bundle $\overline{\mathscr{P}}_n$ with the cubical structure τ_n. Restricted to $X_n^\nu \times_{R_n} \bar{P}_n$ there is the trivialization δ_n^ν. Due to [Mo], Proposition I.7.2.2, there exists a uniquely determined line bundle $\mathscr{B}_n^\nu$ on $X_n^\nu \times_{R_n} B_n$ such that $\overline{\mathscr{P}}_n \cong \bar{q}_n^*\mathscr{B}_n^\nu$ where this isomorphism is compatible with the chosen trivialization δ_n^ν. It follows from the uniqueness that there is a canonical isomorphism

$$\mathscr{B}_{n+1}^\nu \otimes_{R_{n+1}} R_n \xrightarrow{\sim} \mathscr{B}_n^\nu.$$

So the inverse limit

$$\mathscr{B}^\nu := \varprojlim \mathscr{B}_n^\nu$$

is a line bundle on $X^\nu \times_R B$ satisfying

$$\overline{\mathscr{P}}|_{X^\nu \times_R \bar{P}} \xrightarrow{\sim} \bar{q}^*\mathscr{B}^\nu.$$

Denote by $q\colon X_K \times_K \hat{P}_K \to X_K \times_K B_{\mathrm{rig}}$ the map which is induced by the projection $\hat{P}_K \to B_{\mathrm{rig}}$. Then we obtain the pull-back $q^*\mathscr{B}_{\mathrm{rig}}^\nu$ on $X_K^\nu \times_K \hat{P}_K$ for $\nu = 1, \dots, N$. The restriction of such a line bundle to $X_K^\nu \times_K \bar{P}_{\mathrm{rig}}$ is $\bar{q}^*\mathscr{B}_{\mathrm{rig}}^\nu$. Due to [BL], Propositions 4.1 and 4.2, the gluing of the line bundles $\bar{q}^*\mathscr{B}^\mu$ and $\bar{q}^*\mathscr{B}^\nu$ over $(X^\mu \cap X^\nu) \times_R B$ is given by a character

$$\chi_{\mu\nu} \in \mathrm{Hom}(\bar{T}, \overline{\mathbb{G}}_{\mu,P}) = \mathrm{Hom}(T_K, \mathbb{G}_{\mu,K}).$$

So it can be regarded as a gluing of $q^*\mathscr{B}^{\mu}_{\mathrm{rig}}$ and $q^*\mathscr{B}^{\nu}_{\mathrm{rig}}$ over $(X^{\mu} \cap X^{\nu})_{\mathrm{rig}} \times_K \hat{P}_K$. Thus we obtain a line bundle $\hat{\mathscr{P}}_K$ over $X_K \times \hat{P}_K$. This line bundle is trivial over $X_K \times \{1\}$. The rigidificator $\bar{\rho}: \mathscr{O}_{\bar{P}} \xrightarrow{\sim} (x \times \mathrm{id}_{\bar{P}})^*\overline{\mathscr{P}}$ is given by a character due to [BL], Propositions 4.1 and 4.2. So it extends to a rigidificator $\hat{\rho}: \mathscr{O}_{\hat{P}_K} \xrightarrow{\sim} (x_K \times \mathrm{id}_{\hat{P}_K})^*\hat{\mathscr{P}}_K$. Thus we obtain a rigidified line bundle $(\hat{P}_K, \hat{\rho})$ on $X_K \times_K \hat{P}_K$ whose restriction to $X_K \times_K \bar{P}_K$ is isomorphic to $(\bar{P}_{\mathrm{rig}}, \bar{\rho}_{\mathrm{rig}})$ and which is trivial over the unit element. □

Remark 3.9.1. Due to Proposition 3.4 there is a canonical isomorphism of the groups

$$\overline{\mathrm{Pic}}^0_{X_K/K}(\bar{P}_K^2) \xrightarrow{\sim} \mathrm{Hom}_{\bar{\mathfrak{C}}_K}(\bar{P}_K^2, \bar{P}_K).$$

The tensor product of line bundles on the left corresponds to the group law on $\bar{P}_K$ on the right. Applying this correspondence to the map $\mu_I: \bar{P}_K^2 \to \bar{P}_K$ which sends a point $(p_1, p_2) \mapsto \sum_{i \in I} p_i$, one gets an isomorphism of the line bundles

$$\mu_1^*\overline{\mathscr{P}}_K \otimes \mu_2^*\overline{\mathscr{P}}_K \cong \mu_{12}^*\overline{\mathscr{P}}_K$$

on $X_K \times_K \bar{P}_K \times_K \bar{P}_K$. Since the maps μ_I^* are compatible with the cubical structure, this isomorphism extends to an isomorphism of the line bundles

$$\mu_1^*\hat{\mathscr{P}}_K \otimes \mu_2^*\hat{\mathscr{P}}_K \cong \mu_{12}^*\hat{\mathscr{P}}_K$$

on $X_K \times_K \hat{P}_K \times_K \hat{P}_K$.

We now want to exhibit a suitable lattice $M \subseteq \hat{P}_K$ such that the quotient $\hat{P}_K/M$ represents the functor $\mathrm{Pic}^0_{X_K/K}$. As usual denote by $\mathbb{K}$ a topological algebraic closure of K. Then we can consider

$$M := \{p \in \hat{P}_K(\mathbb{K}) : (\mathrm{id}_{X_K} \times p)^*\hat{\mathscr{P}} \text{ is trivial}\}.$$

Then M is a subgroup of $\hat{P}_K(\mathbb{K})$ consisting of K-rational points. Namely, if $p_1, p_2 \in M$ it follows from Remark 3.9.1, that

$$(\mathrm{id}_{X_K} \times p_1)^*\hat{\mathscr{P}} \cong \big(\mathrm{id}_{X_K} \times (p_1 - p_2)\big)^*\hat{\mathscr{P}} \otimes (\mathrm{id}_{X_K} \times p_2)^*\hat{\mathscr{P}}$$

and, hence that $\big(\mathrm{id}_{X_K} \times (p_1 - p_2)\big)^*\hat{\mathscr{P}}$ is trivial. Thus we see that $(p_1 - p_2) \in M$ and, hence, that M is a subgroup of $\hat{P}_K(\mathbb{K})$. The intersection $M \cap \bar{P}_K(\mathbb{K})$ consists only of the unit element. Namely, any point p_K with field of definition K' of the intersection extends to an R'-valued point of $\bar{P}$ where R' is the ring of integers of K'. So the line bundle associated to p_K extends to a formal line bundle on $X \times_R R'$. Due to Lemma 3.5 this line bundle is trivial and, hence, p_K is the unit element.

In the following we will show that the points of M are K-rational and form a discrete subgroup if the torus T is split. Therefore we have to introduce the following notions.

If $\mathscr{E}$ is a formal line bundle on the smooth formal variety B over R, there exists a formal trivialization of $\mathscr{E}$ on an open covering of B such that the transition functions associated to $\mathscr{E}$ are formal units; i.e., these functions take absolute value 1 at each point of B. Any character χ of the torus T_K of $\hat{P}_K$ defines a morphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_K & \longrightarrow & \hat{P}_K & \xrightarrow{q} & B_{\mathrm{rig}} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi_P} & & \| & & \\ 1 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & \hat{E}_K & \xrightarrow{q} & B_{\mathrm{rig}} & \longrightarrow & 1 \end{array}$$

where $\hat{E}_K$ just as $\hat{P}_K$ is the extension of B_{rig} associated to the image of χ under the homomorphism $\chi(T_K) \to B^\vee$ from Proposition 3.7. Using the absolute value on $\hat{E}_K$ we obtain a well defined pairing between the group of characters $\chi(T_K)$ of the torus and the points $\hat{P}_K(\mathbb{K})$ of $\hat{P}_K$ in a topological algebraic closure $\mathbb{K}$ of K

$$\chi(T_K) \times \hat{P}_K(\mathbb{K}) \to \mathbb{R}; \quad (\chi, p) \mapsto -\log_{|\pi|}|\chi_P(p)|;$$

cf. [BL], Section 3. Thereby we have chosen the logarithm to the base $|\pi|$ in order to send the rational points of $\hat{P}_K$ to $\mathbb{Z}$. If we choose a basis $\zeta_1, \ldots, \zeta_r$ of the character group $\chi(T_K)$, we obtain a group homomorphism

$$\mathrm{val}\colon \hat{P}_K(\mathbb{K}) \to \mathbb{R}^r; \quad p \mapsto \bigl(-\log_{|\pi|}|\zeta_1(p)|, \ldots, -\log_{|\pi|}|\zeta_r(p)|\bigr).$$

The kernel of val is $\bar{P}_K(\mathbb{K})$. Moreover we obtain a non-degenerate pairing

$$\chi(T_K) \times \hat{P}_K(\mathbb{K})/\bar{P}_K(\mathbb{K}) \to \mathbb{R}.$$

The pairing takes integer values on the K-rational points of $\hat{P}_K$. A subgroup M of $\hat{P}_K(\mathbb{K})$ is called a lattice if under the group homomorphism val it is mapped bijectively onto a lattice in $\mathbb{R}^r$.

Lemma 3.10. *If the torus T is split, the subgroup*

$$M := \{p \in \hat{P}_K(\mathbb{K}) : (\mathrm{id}_{X_K} \times p)^* \hat{\mathscr{P}} \text{ is trivial}\}$$

is a K-rational lattice in $\hat{P}_K(\mathbb{K})$.

Proof. Since $M \cap \bar{P}_K = \{1\}$ the group M is étale over K. We consider the action of the Galois group $G := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\hat{P}_K$. Since the torus is split, the Galois group stabilizes the factors of a splitting. Since K is henselian, the Galois group G respects absolute values. Thus we see that G stabilizes the subset $\bar{P}_K$ and any translate of it. In particular G stabilizes every point of M and so M consists only of K-rational points. Under the map val the set of K-rational points is mapped to the lattice $\mathbb{Z}^r$, so M is mapped to a sub-lattice. □

Remark 3.10.1. The analog of the Hopf surface in the p-adic case is an example where the rank of M is strictly smaller than the rank of the torus; cf. Remark 0.1.3(4).

The next topic is the universal property of $\hat{P}_K$. For its proof we need the following descent lemma.

Lemma 3.11. *Let q_K: $V'_K \to V_K$ be a faithfully flat quasi-compact morphism of rigid analytic varieties over K and let $\mathscr{L}_K$ be a rigidified line bundle on $X_K \times_K V_K$. Consider a morphism f'_K: $V'_K \to \hat{P}_K$ such that $(\mathrm{id}_{X_K} \times f'_K)^* \hat{\mathscr{P}}_K \cong (\mathrm{id}_{X_K} \times q_K)^* \mathscr{L}_K$. Then, locally on V_K with respect to the Grothendieck topology, there exists a factorization f_K: $V_K \to \hat{P}_K$ such that $f'_K = f_K \circ q_K$ and $(\mathrm{id}_{X_K} \times f_K)^* \hat{\mathscr{P}}_K \cong \mathscr{L}_K$. The local maps f_K are uniquely determined up to a translation by a lattice point; i.e., a point of M.*

Proof. The problem is local on $\hat{P}_K$ and V_K. So let $0 < \varepsilon < 1/2$ and

$$\hat{P}_K(\varepsilon) := \{p \in \hat{P}_K; \mathrm{val}(p) \in [-\varepsilon, \varepsilon]^r\}$$

the relative polyannulus in $\hat{P}_K$ over B_K with radius $\pi^{2\varepsilon}$; recall the explanation before Lemma 3.10. Let U_K be a nonempty affine formal open subvariety of B_K on which $\hat{P}_K$ is trivial. So one can cover $\hat{P}_K$ by translates of $\hat{P}_K(\varepsilon) \times_{B_K} U_K$. Let H_K be one of these translates. This is an affinoid subvariety of $\hat{P}_K$. It remains to find a morphism f_K making the following diagram commutative:

$$\begin{array}{ccc}
W'_K := (f'_K)^{-1}(H_K) & \xrightarrow{\;f'_K\;} & H_K \subseteq \mathbb{D}^n_K \\
\Big\downarrow q_K & \nearrow_{f_K} & \\
W_K := q_K\big((f'_K)^{-1}(H_K)\big). & &
\end{array}$$

Since H_K is affinoid, we can regard it as a closed subscheme of some n-dimensional polydisc $\mathbb{D}^n_K$. Thus we are reduced to prove a descent theorem for functions.

Since q_K is flat, W_K is open in V_K due to [FRG], II, Corollary 5.11. It follows from [FRG], I, Theorem 4.1 and [FRG], II, Theorem 5.2 that there exist formal models W and W' of W_K and W'_K such that q_K extends to a flat and quasi-compact morphism $q: W' \to W$ and f'_K to a formal map f': $W' \to \mathbb{D}^n_R$. Since q_K is surjective, q is surjective and, hence, faithfully flat. Now we can replace W by an open affine subvariety $\mathrm{Spf}(A)$. Since q is quasi-compact, $q^{-1}(W)$ is a finite union of affine open subvarieties. Replacing W' by the disjoint union of these subvarieties, we may assume that $W' = \mathrm{Spf}(A')$ is affine and faithfully flat over W. Thus we are in the formal descent situation:

$$A \to A' \rightrightarrows A' \hat{\otimes}_A A' =: A''.$$

This sequence is exact. Namely, after tensoring with R_n over R, the sequence is exact for all $n \in \mathbb{N}$, since q_n is faithfully flat; cf. [BLR], Lemma 6.1.2. Tensoring with K we obtain the exact sequence

$$A_K \to A'_K \rightrightarrows A'_K \hat{\otimes}_{A_K} A'_K =: A''_K.$$

Via the projections

$$q^1_K; q^2_K : \mathrm{Sp}(A''_K) := \mathrm{Sp}(A'_K \hat{\otimes}_{A_K} A'_K) \rightrightarrows \mathrm{Sp}(A'_K)$$

the morphism f'_K induces the morphisms $f'_K \circ q^1_K$ and $f'_K \circ q^2_K$ from $\mathrm{Sp}\, A''_K$ to H_K. They satisfy the following condition:

$$\big(\mathrm{id}_{X_K} \times (f'_K \circ q^1_K)\big)^* \hat{\mathscr{P}}_K \cong \big(\mathrm{id}_{X_K} \times (f'_K \circ q^2_K)\big)^* \hat{\mathscr{P}}_K.$$

Thus the difference

$$f'_K \circ q^1_K - f'_K \circ q^2_K : \mathrm{Sp}(A''_K) \to \hat{P}_K(2\varepsilon)$$

factors through M due to our assumption. Since $M \cap \hat{P}_K(2\varepsilon) = \{1\}$ due to Lemma 3.10, the maps $f'_K \circ q^1_K = f'_K \circ q^2_K$ coincide. As a map from W'_K to $\mathbb{D}^n_K$ the morphism f'_K is given by an n-tuple of functions $(a'_1, \dots, a'_n) \in (A'_K)^n$. The coincidence of $f'_K \circ q^1_K = f'_K \circ q^2_K$ implies the cocycle condition for the descent of the functions $(a'_1, \dots, a'_n)$ and, hence, they are induced from functions $(a_1, \dots, a_n) \in A^n_K$ giving rise to a map $f_K \colon W_K \to \mathbb{D}^n_K$. Now, as H_K is a closed subvariety of $\mathbb{D}^n_K$, the functions $(a'_1, \dots, a'_n)$ have to satisfy certain equations which we will not specify. Then $(a_1, \dots, a_n)$ will satisfy these equations also and, hence, yield a factorization of f_K through H_K. So we obtain the morphism $f_K \colon W_K \to \hat{P}_K$ such that $f'_K | W_K = f_K \circ q_K | W'_K$.

It remains to show that there is an isomorphism $\mathscr{L}_K \xrightarrow{\sim} (\mathrm{id}_{X_K} \times f_K)^* \hat{\mathscr{P}}_K$. On $X_K \times_K V'_K$ there is an isomorphism

$$\varphi' \colon (\mathrm{id}_{X_K} \times q_K)^* \mathscr{L}_K \xrightarrow{\sim} (\mathrm{id}_{X_K} \times q_K)^* (\mathrm{id}_{X_K} \times f_K)^* \hat{\mathscr{P}}_K$$

respecting the rigidificators. So this isomorphism is uniquely determined. Thus it satisfies the cocycle condition for descent. It remains to prove the descent for sections in a line bundle. Denote by $\mathscr{H}_K$ the line bundle of homomorphisms from $\mathscr{L}_K$ to $(\mathrm{id}_{X_K} \times f_K)^* \hat{\mathscr{P}}_K$. In the affine situation we can replace the line bundles by their modules of global sections which are invertible modules. So it suffices to show the descent for sections in $H_K := \Gamma(Y_{\mathrm{rig}} \times_K W_K, \mathscr{H}_K)$ where $Y = \mathrm{Spf}(C)$ is an affine open formal part of X. Setting $B := C \hat{\otimes}_R A$, we obtain a descent situation as above where A is replaced by B. Then we obtain the exact sequence

$$H_K \to H_K \otimes_{B_K} B'_K \rightrightarrows H_K \otimes_{B_K} B''_K.$$

The map φ' belongs to the kernel of the double arrows on the right. So it has a unique preimage from the left which is an isomorphism over $Y_{\mathrm{rig}} \times_K W_K$. Due to the uniqueness of descent they fit together to build an isomorphism over $X_{\mathrm{rig}} \times_K W_K$. □

We will use this lemma to show the universal mapping property of $\hat{P}_K$.

Lemma 3.12. *Assume that the residue field k of R is separably closed. Let $(V_K, v_K) \in \mathfrak{C}_K$ be a rigid analytic variety and let $\mathscr{L}_K$ be a rigidified line bundle on $X_K \times_K V_K$ which is trivial on $X_K \times_K v_K$. Then there exists an admissible covering V^i_K of V_K and morphisms $f^i \colon V^i_K \to \hat{P}_K$ such that $(\mathrm{id}_{X_K} \times f^i)^* \hat{\mathscr{P}}_K \cong \mathscr{L}|_{X_K \times_K V^i_K}$. The obstruction for gluing the morphisms f^i to build a morphism $f \colon V_K \to \hat{P}_K$ satisfying $(\mathrm{id}_{X_K} \times f)^* \hat{\mathscr{P}}_K \cong \mathscr{L}_K$ and $f(v_K) = 1$ is given by a cohomology class*

$$(f^i - f^j) \in \mathrm{H}^1(V_K, M) \cong \mathrm{H}^1(V_K, \mathbb{Z})^d.$$

Proof. Let us first consider the special case where V_K admits a formal model V which is formally smooth over $R\langle\zeta_1, \dots, \zeta_{t+1}\rangle/(\zeta_1 \cdot \ldots \cdot \zeta_{t+1} - \pi)$ and is small in the sense of 1.4. Moreover assume that the point $v_K \in V_K(K)$ lies above the open part $\{\zeta_1 = \cdots = \zeta_t = 1\}$. In this case the line bundle $\mathscr{L}_K$ decomposes into a tensor product

$$\mathscr{L}_K \cong \mathscr{M}_K \otimes \mathscr{N}_K$$

where $\mathscr{N}$ is a formal line bundle on $X \times V$ and where $M_K \cong (\zeta_1^{n_1} \otimes \cdots \otimes \zeta_t^{n_t})$ is a multiplicative line bundle; cf. Proposition 2.6. The cohomology classes $n_1, \dots, n_t \in \mathrm{H}^1(X_K, \mathbb{Z})$ give rise to group homomorphisms $\mu_1, \dots, \mu_t \in \mathrm{Hom}(\mathbb{G}_{m,K}, \hat{P}_K)$ due to Proposition 3.8. First they are defined as map $\overline{\mathbb{G}}_{m,K} \to \bar{P}_K$ and, hence, due to the construction of $\hat{P}_K$ this map extends to a map $\mathbb{G}_{m,K} \to \hat{P}_K$. Thus we obtain a morphism

$$\mu := \mu_1 \circ \zeta_1 \cdot \ldots \cdot \mu_t \circ \zeta_t : V_K \to \hat{P}_K$$

where " $\cdot$ " denotes the group law on $\hat{P}_K$. The pull-back of the Poincaré bundle satisfies

$$\mathscr{M}_K \cong (\mathrm{id}_{X_K} \times \mu)^* \hat{\mathscr{P}}$$

and $\mu(v_K) = 1$. Due to Proposition 3.3 the line bundle $\mathscr{N}$ on $X \times_R V$ gives rise to a morphism $V \to P'$. This morphism factors through $\overline{P'}$, since V is flat over R and reduced. On the generic fiber it induces a morphism

$$v\colon V_K \to \bar{P}_K \to \hat{P}_K$$

such that

$$\mathscr{N}_K \cong (\mathrm{id}_{X_K} \times v)^* \hat{\mathscr{P}}_K$$

satisfying $v(v_K) = 1$. Altogether we obtain a morphism

$$\lambda := \mu \cdot v : V_K \to \hat{\mathscr{P}}_K$$

satisfying $\lambda(v_K) = 1$ such that

$$(\mathrm{id}_{X_K} \times \lambda)^* \hat{\mathscr{P}}_K \cong (\mathrm{id}_{X_K} \times \mu)^* \hat{\mathscr{P}}_K \otimes (\mathrm{id}_{X_K} \times v)^* \hat{\mathscr{P}}_K \cong \mathscr{M}_K \otimes \mathscr{N}_K \cong \mathscr{L}_K.$$

For the general case of a smooth rigid variety V_K we need a desingularization argument to reduce this case to the special case. We use a formal geometry analog of de Jong's alterations result [dJ]. Namely, after a suitable finite separable field extension $\tilde{K}$ of K, there exists an étale surjective morphism $\tilde{V}_K \to V_K$ such that $\tilde{V}_K$ admits a strict semi-stable model $\tilde{V}$ over $\mathrm{Spf}(R)$. Indeed one can write V_K locally as a smooth curve fibration. In [L2], Theorems 5.2 and 5.3 it was shown that étale locally on V_K this curve fibration can be embedded into a smooth projective curve fibration which has a semi-stable formal model. Then we apply the methods of de Jong to obtain the desired strict semi-stable model of V_K.

Furthermore we may assume that each connected component $\tilde{V}^i$ of $\tilde{V}$ is small as in the special case, cf. Remark 1.4.1; and that it is punctured by a K-rational point $\tilde{v}^i$ lying in a position as required in the special case. Then for the line bundle $\tilde{\mathscr{L}}_K := \mathscr{L}_K \otimes_{\mathcal{O}_{V_K}} \mathcal{O}_{\tilde{V}_K^i}$ we obtain by the case discussed above a morphism

$$\lambda^i \colon \tilde{V}_K^i \to \hat{P}_K$$

satisfying

$$(\mathrm{id}_{X_K} \times \lambda^i)^* \hat{\mathscr{P}}_K \cong \tilde{\mathscr{L}}_K \otimes p^* (\mathrm{id}_{X_K} \times \tilde{v}^i)^* \tilde{\mathscr{L}}_K^\vee$$

where $p \colon X_K \times \tilde{V}_K^i \to X_K$ is the projection. Due to Lemma 3.11, there exists an open covering V_K^i of V_K and maps

$$f^i \colon V_K^i \to \hat{P}_K$$

satisfying $f^i(v_K^i) = [(\mathrm{id}_{X_K} \times v^i)^* \mathscr{L}_K]$ such that there exists a morphism of rigidified line bundles

$$\varphi^i \colon (\mathrm{id}_{X_K} \times f^i)^* \hat{\mathscr{P}}_K \xrightarrow{\sim} \mathscr{L}_K | X_K \times_K V_K^i.$$

Thereby v^i is the image of the point $\tilde{v}^i$ and the brackets indicate the isomorphism class of the line bundle. By a connectedness argument one shows that these isomorphism classes belong to $\hat{P}_K$. On the overlaps V_K^{ij} associated to the covering $\{V^i; i \in I\}$ the morphisms $(f^i - f^j) \colon V_K^{ij} \to \hat{P}_K$ factor through the lattice M. Therefore they define a cohomology class in

$$(f^i - f^j) \in \mathrm{H}^1(V_K, M) \cong \mathrm{H}^1(V_K, \mathbb{Z})^d. \qquad \square$$

If the cohomology $\mathrm{H}^1(V_K, \mathbb{Z}) = 0$ vanishes, we have seen that there exist coboundaries $m^i \in C^1(\{V_K^i\}, M)$ such that

$$(f^i - m^i) \colon V_K^i \to \hat{P}_K$$

fit together to build a morphism $f \colon V_K \to \hat{P}_K$ satisfying $f(v_K) = 1$. Since $\mathscr{L}_K$ is rigidified, the local isomorphisms φ^i are uniquely determined and, hence, they can be glued together. Therefore our last statement implies the representability of the functor $\widehat{\mathrm{Pic}}^0_{X_K/K}$ on the category $\hat{\mathfrak{C}}_K$.

Proposition 3.13. *$(\hat{P}_K, 1)$ represents the functor $\widehat{\mathrm{Pic}}^0_{X_K/K}$ on the category $\hat{\mathfrak{C}}_K$.*

In the proof we have used the fact that the residue field is separably closed. So we have established the statement after base change to the maximal unramified extension $K \to K^{\mathrm{un}}$. The Galois group $G(K^{\mathrm{un}}/K)$ acts on all objects in a compatible way. Since Galois descent is effective in our situation, the whole construction descends to the given field.

3.4. The construction of P_K. We divide out the lattice M in $\hat{P}_K$ and obtain a rigid group variety

$$p \colon \hat{P}_K \to P_K = \hat{P}_K / M.$$

The residue map $\hat{P}_K \to P_K$ is a local isomorphism and, hence, P_K is smooth. Next we want to construct a Poincaré bundle on $X_K \times P_K$ by dividing out the canonical M-action on $\hat{\mathscr{P}}_K$. Due to Remark 3.9.1 there is an isomorphism

$$\lambda\colon \mu_{12}^*\hat{\mathscr{P}}_K \xrightarrow{\sim} \mu_1^*\hat{\mathscr{P}}_K \otimes \mu_2^*\hat{\mathscr{P}}_K$$

of line bundles on $X_K \times_K \hat{P}_K^2$. This isomorphism is compatible with the rigidificators of the Poincaré bundles $\hat{\rho}_K\colon \mathscr{O}_{\hat{P}_K} \xrightarrow{\sim} (x_K \times \mathrm{id}_{\hat{P}_K})^*\hat{\mathscr{P}}_K$; i.e.,

$$(x_K \times \mathrm{id}_{\hat{P}_K^2})^*\lambda \circ \mu_{12}^*\hat{\rho}_K = \mu_1^*\hat{\rho}_K \otimes \mu_2^*\hat{\rho}_K.$$

Via pull-back by the section $(\mathrm{id}_{\hat{P}_K} \times m)\colon \hat{P}_K \to \hat{P}_K^2$ for $m \in M$ we obtain an isomorphism

$$\varphi_m := (\mathrm{id}_{X_K} \times \mathrm{id}_{\hat{P}_K} \times m)^*\lambda\colon \tau_m^*\hat{\mathscr{P}}_K \xrightarrow{\sim} \hat{\mathscr{P}}_K;$$

namely $(\mathrm{id}_{X_K} \times \mathrm{id}_{\hat{P}_K} \times m)^*\mu_2^*\hat{\mathscr{P}}_K$ is trivial due to the definition of M. Thereby τ_m denotes the translation by the point $m \in \hat{P}_K$. For the rigidificator we get

$$(x_K \times \mathrm{id}_{\hat{P}_K})^*\varphi_m(\tau_m^*\hat{\rho}_K) = \hat{\rho}_K$$

where $\tau_m^*\hat{\rho}_K$ is the canonical rigidificator of the line bundle $\tau_m^*\hat{\mathscr{P}}_K$. Since the following isomorphisms of line bundles

$$\tau_{m+m'}^*\hat{\mathscr{P}}_K \xrightarrow{\sim} \hat{\mathscr{P}}_K$$

are compatible with the rigidificators, they coincide

$$\varphi_{m'} \circ \tau_{m'}^*\varphi_m = \varphi_{m+m'}.$$

This is the cocycle condition for the isomorphisms φ_m. Thus they define an M-linearization on $\hat{\mathscr{P}}_K$. Now we can divide out the M-linearization on $\hat{\mathscr{P}}_K$ and, hence, we obtain a line bundle

$$\mathscr{P}_K = \hat{\mathscr{P}}_K/M \quad \text{on } (X_K \times \hat{P}_K)/M = X_K \times P_K$$

which is equipped with a canonical rigidificator induced from $\hat{\rho}_K$. Moreover $\mathscr{P}_K$ is trivial over $X_K \times \{1\}$.

To give a local description, each point of P_K has an open neighborhood W_K which is isomorphic to a translate U_K of $\hat{P}_K(1/2)$ under the quotient map; cf. the proof of Proposition 3.11. For the restriction of the Poincaré bundles one has a canonical isomorphism

$$\hat{\mathscr{P}}_K|X_K \times U_K \xrightarrow{\sim} \mathscr{P}_K|X_K \times W_K.$$

Thus we can prove the first statement of the main theorem.

Theorem 3.14. *The smooth connected rigid group variety* $(P_K, 1)$ *represents the Picard functor* $\mathrm{Pic}^0_{X_K/K}$ *on the category* $\mathfrak{C}_K$. *The universal line bundle on* $X_K \times P_K$ *is the Poincaré bundle constructed above.*

Proof. Let us first assume that the residue field k is separably closed. Consider a couple $(V_K, v_K) \in \mathfrak{C}_K$ and a rigidified line bundle $\mathscr{L}_K$ on $X_K \times V_K$ which is trivial on $X_K \times v_K$. Due to Lemma 3.12 there exists an admissible covering V_K^i of V_K and morphisms

$$f^i \colon V_K^i \to \hat{P}_K \to P_K$$

associated to $\mathscr{L}_K$. The maps f^i and f^j differ on $V_K^i \cap V_K^j$ by an element of M. So they fit together to a morphism

$$f \colon V_K \to P_K$$

satisfying $f(v_K) = 1$ such that there is an isomorphism

$$\mathscr{L}_K \xrightarrow{\sim} (\mathrm{id}_{X_K} \times f)^* \mathscr{P}_K$$

of rigidified line bundles. This morphism is unique due to the definition of the lattice. By the usual Galois descent argument as mentioned at the end of Section 3.4 one descends the result to the given base field. $\square$

Remark 3.14.1. The construction commutes with base change by a finite extension of discrete valuation rings $R \to \tilde{R}$. In particular, the following holds:

$P_K \otimes_K \tilde{K}$ represents the Picard functor $\mathrm{Pic}^0_{X_{\tilde{K}}/\tilde{K}}$ on the category $\mathfrak{C}_{\tilde{K}}$.

$\bar{P} \otimes_R \tilde{R}$ represents the functor $\overline{\mathrm{Pic}}^0_{\tilde{X}_{\tilde{K}}/\tilde{K}}$ on the category $\bar{\mathfrak{C}}_{\tilde{K}}$.

$\hat{P}_K \otimes_K \tilde{K}$ represents the Picard functor $\widehat{\mathrm{Pic}}^0_{\tilde{X}_{\tilde{K}}/\tilde{K}}$ on the category $\hat{\mathfrak{C}}_{\tilde{K}}$ as well.

Namely, we can do the same construction starting with $\tilde{X} := X \otimes_R \tilde{R}$. Since the initial object $\mathrm{Pic}^0_{\tilde{X}_0/\tilde{k}} = \mathrm{Pic}^0_{X_0/k} \otimes \tilde{k}$ commutes with base change, the statement follows from the construction. We did not make use of the fact that the initial X was regular as we said just in the beginning.

The structure of P_K is described in the following theorem.

Theorem 3.15. *Assume that the residue field k of R is separably closed. Then P_K is an extension*

$$1 \to T'_K \to P_K \to Q_K \to 1$$

of an abeloid rigid analytic group Q_K by an affine torus T'_K; i.e., Q_K is smooth connected and proper.

Proof. Due to Lemma 3.10 the lattice M can be regarded as a subgroup of $\hat{P}_K(K)/\bar{P}_K(K)$. Then look at the non-degenerate pairing

$$\chi(T_K) \times \hat{P}_K(\mathbb{K})/\bar{P}_K(\mathbb{K}) \to \mathbb{R};$$

cf. [BL], Section 3. The orthogonal complement of M with respect to this pairing is a subgroup $\chi(T'_K)$ of $\chi(T_K)$. This subgroup has a direct complement $\chi(T''_K)$ in the group of characters, since the Galois action is trivial as k is assumed to be separably closed,

$$\chi(T_K) = \chi(T'_K) \oplus \chi(T''_K).$$

According to this splitting, the torus T_K of $\hat{P}_K$ splits into a product of subtori

$$T_K = T'_K \times T''_K.$$

Due to the structure of $\hat{P}_K$ we obtain an extension via push-out with respect to the projection $T_K \to T''_K$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_K & \longrightarrow & \hat{P}_K & \longrightarrow & B_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T''_K & \longrightarrow & \hat{Q}_K & \longrightarrow & B_K & \longrightarrow & 1. \end{array}$$

The canonically induced map $\hat{P}_K \to \hat{Q}_K$ gives rise to an isomorphism of lattices

$$\begin{array}{ccc} \hat{P}_K & \longrightarrow & \hat{Q}_K \\ \cup & & \cup \\ M & \xrightarrow{\sim} & N \end{array}$$

where N is now a lattice of $\hat{Q}_K$ of full rank r''. In particular, the quotient

$$Q_K = \hat{Q}_K / N$$

is an abeloid variety. Thus we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & M & = & N & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T'_K & \longrightarrow & \hat{P}_K & \longrightarrow & \hat{Q}_K & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T'_K & \longrightarrow & P_K & \longrightarrow & Q_K & \longrightarrow & 1 \end{array}$$

making P_K an extension of an abeloid variety Q_K by an affine torus T'_K. □

This settles the demonstration of the theorem as far as the identity component is concerned.

3.5. The Néron-Severi group. Keep the notations of above. So let $X \to \mathrm{Spf}(R)$ be the strict semi-stable formal model of X_K we started with at the beginning. Let X_0^σ for $\sigma = 1, \dots, s$ be the irreducible components of X_0. We always assume that the residue field k of R is separably closed. Thus $X_0 \otimes_k \tilde{k} \to X_0$ is a homeomorphism for any finite field extension $k \to \tilde{k}$.

Let $\mathbb{K}$ be the topological algebraic closure of K. We want to study the Néron-Severi group of X_K which is by definition the quotient

$$\mathrm{NS}_{X_K/K}(\mathbb{K}) = \mathrm{Pic}_{X_K/K}(\mathbb{K})/\mathrm{Pic}^0_{X_K/K}(\mathbb{K})$$

where $\mathrm{Pic}_{X_K/K}(K')$ denotes the group of isomorphism classes of line bundles on $X_K \hat{\otimes}_K K'$ and where $\mathrm{Pic}^0_{X_K/K}(K')$ is the set of K'-rational points of P_K for any field extension $K \to K'$. We want to relate this group to the Néron-Severi group of the special fiber X_0; i.e., to the group

$$\mathrm{NS}_{X_0/k} = \mathrm{Pic}_{X_0/k}(\bar{k})/\mathrm{Pic}^0_{X_0/k}(\bar{k})$$

where $\bar{k}$ is the algebraic closure of the residue field k of R. Let us first study what happens after a finite field extension. So, for a finite field extension $K \to \tilde{K}$, set

$$\mathrm{NS}_{X_K/K}(\tilde{K}) = \mathrm{Pic}_{X_K/K}(\tilde{K})/\mathrm{Pic}^0_{X_K/K}(\tilde{K}).$$

Furthermore let $R \to \tilde{R}$ be the associated extension of discrete valuation rings and set $\tilde{X} = X \times_R \tilde{R}$. If $\tilde{\pi}$ is the uniformizer of $\tilde{R}$ and $e \in \mathbb{N}$ the ramification index then

$$\pi = \tilde{u} \cdot \tilde{\pi}^e$$

for a unit $\tilde{u}$ of $\tilde{R}$. For the following, fix a line bundle $\tilde{\mathscr{L}}_K$ on $X_K \times_K \tilde{K}$. We want to discuss the obstructions for extending $\tilde{\mathscr{L}}_K$ to a line bundle $\tilde{\mathscr{L}}$ on $\tilde{X}$.

It was shown in Proposition 2.5 that there exists an open covering

$$\mathfrak{U} := \{U^1, \ldots, U^n\}$$

of X and line bundles $\tilde{\mathscr{L}}^i$ on $\tilde{U}^i := U^i \times_R \tilde{R}$ for $i = 1, \ldots, n$ extending $\tilde{\mathscr{L}}_K$. After a refinement of $\mathfrak{U}$ we may assume that the line bundles $\tilde{\mathscr{L}}^i$ are isomorphic to $\mathscr{O}_{\tilde{X}}|\tilde{U}^i$ for $i = 1, \ldots, n$. Therefore $\tilde{\mathscr{L}}_K$ is given by a cocycle

$$\tilde{\lambda}^{ij}_K \in Z^1(\tilde{\mathfrak{U}}, \mathscr{O}^\times_{\tilde{X}_{\mathrm{rig}}}).$$

If W is an open affine part of X such that the irreducible components are principal Cartier divisors on W, there exist functions ξ_σ on W satisfying

$$|\xi_\sigma|_{X_0^\sigma} = |\pi|,$$
$$|\xi_\tau|_{X_0^\sigma} = 1 \quad \text{for } \sigma \neq \tau$$

where $|\xi_\tau|_{X_0^\sigma}$ denotes the sup-norm of the function ξ_τ on a formal open part of X_{rig} which specializes to a dense open part of X_0^σ. Later on we will use this fact to glue local models $\mathscr{L}^i$. We assume that the open affine parts U^i are chosen in such a way that the irreducible components X_0^σ are principal Cartier divisors on any U^i.

Lemma 3.16. *Let W be an affine open subvariety of X such that the irreducible components X_0^σ induce principal Cartier divisors on W. Then the ring of invertible functions on*

$\tilde{W}_{\mathrm{rig}}$ has a canonical filtration

$$\mathcal{O}^{\times}_{\tilde{X}_K}(\tilde{W}_{\mathrm{rig}}) =: G_2(W) \supset G_1(W) \supset G_0(W) := \mathcal{O}^{\times}_{\tilde{X}}(\tilde{W})$$

where

$$G_1(W) := \mathcal{O}^{\times}_{\tilde{X}}(\tilde{W}) \oplus \xi_{i_1}^{\Gamma(W_0,\mathbb{Z})} \oplus \cdots \oplus \xi_{i_r}^{\Gamma(W_0,\mathbb{Z})}$$

and

$$\Gamma(W_0,\mathbb{Z})/\Gamma(W_0,\mathbb{Z}) \cdot e \xrightarrow{\sim} G_2(W)/G_1(W); \quad 1 \mapsto \tilde{\pi}$$

if W_0 meets r irreducible components $X_0^{i_\rho}$ of X_0. For the functions ξ_{i_ρ} one can choose any generator of the ideals defined by the irreducible component $X_0^{i_\rho}$ of X_0 meeting W_0. The group G_1 is independent of the choice of these generators.

Proof. If f is an invertible function on W_{rig}, for any connected component of W, we can find an integer n such that $f \cdot \tilde{\pi}^{-n}$ has sup-norm 1 on each connected component of W_{rig}. The irreducible components of W_{rig} correspond one-to-one to the connected components of W_0. So we can view the collection of these integers as an element of $\Gamma(W_0,\mathbb{Z})$. Any irreducible component of X_0 can meet only a single connected component of W_0. Thus we may assume that W_{rig} is connected. The product of the functions satisfies

$$\xi_{i_1} \cdot \ldots \cdot \xi_{i_r} = u \cdot \pi = u \cdot \tilde{u} \cdot \tilde{\pi}^e$$

where $u \in \mathcal{O}^{\times}_{X}(W)$ is a unit. The generic point of any irreducible component of the singular locus of W_0 lies on precisely two irreducible components of X_0. Then it follows from Lemma 1.5 that the difference of the absolute values on the two components containing that point is given by a power of $|\pi|$. Thus we see that there exist integers $n_\rho \in \mathbb{Z}$ such that

$$f = \bar{f} \cdot \tilde{\pi}^n \cdot \xi_{i_1}^{n_1} \cdot \ldots \cdot \xi_{i_r}^{n_r}$$

where $\bar{f}$ is an invertible function on $\tilde{W}_{\mathrm{rig}}$ taking absolute value 1 on each irreducible component. Therefore $\bar{f}$ is an invertible formal function and belongs to $\mathcal{O}^{\times}_{\tilde{X}}(\tilde{W})$. The remaining assertions are clear. □

Remark 3.16.1. Keep the situation of the last Lemma 3.16. If $\tilde{\mathscr{L}}$ is a line bundle on $\tilde{W}$, the set of generators of $\mathscr{L}_{\mathrm{rig}}$ decomposes in a similar way.

Now go back to the line bundles $\tilde{\mathscr{L}}^i \cong \mathcal{O}_{\tilde{X}}\lfloor\tilde{U}^i$ for $i = 1,\ldots,n$ assumed to be trivial. So the line bundle is given by a cocycle $\tilde{\lambda}_K^{ij} \in Z^1(\mathfrak{U}, \mathcal{O}^{\times}_{\tilde{X}_{\mathrm{rig}}})$. Over the overlaps $U^{ij} := U^i \cap U^j$, this cocycle induces elements

$$\mathrm{cyc}(\tilde{\mathscr{L}}_K)^{ij} := [\tilde{\lambda}_K^{ij}] \in G_2(U^{ij})/G_1(U^{ij}) \cong (\mathbb{Z}/\mathbb{Z}e)_{\tilde{X}}(U^{ij})$$

giving rise to a cocycle

$$\mathrm{cyc}(\tilde{\mathscr{L}}_K) := \left(\mathrm{cyc}(\tilde{\mathscr{L}}_K)^{ij}\right) \in \mathrm{Z}^1(\mathfrak{U}, \mathbb{Z}/\mathbb{Z}e),$$

and hence to a cohomology class

$$\overline{\mathrm{cyc}}(\tilde{\mathscr{L}}_K) \in \mathrm{H}^1(X_0, \mathbb{Z}/\mathbb{Z}e).$$

Now consider the exact sequence of constant sheaves on X_0 given by the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot e} \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}e \to 0.$$

It induces the long exact cohomology sequence

$$0 \to \mathrm{H}^1(X_0, \mathbb{Z}) \xrightarrow{\cdot e} \mathrm{H}^1(X_0, \mathbb{Z}) \to \mathrm{H}^1\big(X_0, (\mathbb{Z}/\mathbb{Z}e)\big)$$
$$\xrightarrow{\delta} \mathrm{H}^2(X_0, \mathbb{Z}) \xrightarrow{\cdot e} \mathrm{H}^2(X_0, \mathbb{Z})$$

where δ: $\mathrm{H}^1\big(X_0, (\mathbb{Z}/\mathbb{Z}e)\big) \to \mathrm{H}^2(X_0, \mathbb{Z})$ is the connecting homomorphism.

We want to investigate the vanishing of the class $\overline{\mathrm{cyc}}(\tilde{\mathscr{L}}_K)$. Therefore we need the following definition.

Definition 3.17. A line bundle $\mathscr{M}_K$ which is associated to a cocycle $c^n \in \mathrm{H}^1(X_K, \mathcal{O}_{X_K}^{\times})$ for some $c \in \mathbb{K}^{\times}$ and $n \in \mathrm{H}^1(X_{\mathrm{rig}}, \mathbb{Z})$ is called a *multiplicative line bundle*.

Such line bundles correspond to points of P_K which are induced form the $\mathbb{K}$-valued points of the torus T_K under the canonical morphism $T_K \to P_K$.

Lemma 3.18. *Keep the situation of above.*

(1) *$\tilde{\mathscr{L}}_K$ extends to a line bundle $\tilde{\mathscr{L}}$ on $\tilde{X}$ if and only if $\overline{\mathrm{cyc}}(\tilde{\mathscr{L}}_K) = 0$ in $\mathrm{H}^1\big(X_0, (\mathbb{Z}/\mathbb{Z}e)\big)$.*

(2) $\delta\big(\overline{\mathrm{cyc}}(\tilde{\mathscr{L}}_K)\big) = 0$ *vanishes if and only if there exists a multiplicative line bundle $\tilde{\mathscr{M}}_K$ such that $\tilde{\mathscr{L}}_K \otimes \tilde{\mathscr{M}}_K$ extends to a line bundle on $\tilde{X}$.*

Proof. (1) If $\tilde{\mathscr{L}}_K$ extends to a line bundle $\tilde{\mathscr{L}}$ on $\tilde{X}$, then $\tilde{\mathscr{L}}_K$ can be represented by a cocycle $(\tilde{\lambda}_K^{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_{\tilde{X}}^{\times})$. So the functions $\tilde{\lambda}_K^{ij}$ belong to $\mathcal{O}_{\tilde{X}}^{\times}(\tilde{U}^{ij}) \subset G_1(U^{ij})$ and, hence, $\mathrm{cyc}(\tilde{\mathscr{L}}_K) = [(\tilde{\lambda}_K^{ij})] = 0$.

If $\overline{\mathrm{cyc}}(\tilde{\mathscr{L}}_K) = 0$, the line bundle $\tilde{\mathscr{L}}_K$ can be represented by a cocycle

$$(\tilde{\lambda}_K^{ij}) \in Z^1(\mathfrak{U}, \mathcal{O}_{\tilde{X}_{\mathrm{rig}}}^{\times})$$

for a suitable open covering $\mathfrak{U}$ of X_0 which takes values

$$|\tilde{\lambda}_K^{ij}|_{X_0^{\sigma}} \in |\pi|^{\mathbb{Z}} \quad \text{for } \sigma = 1, \ldots, s.$$

So these values can also be taken by powers of the generator ξ_{σ} of the ideal associated to X_0^{σ}. Therefore we can stepwise adjust the cocycle. Namely let $\tilde{\mathscr{L}}(m-1)$ be an extension of $\tilde{\mathscr{L}}_K$ to $U(m-1) := U^1 \cup \cdots \cup U^{m-1}$. Then the pasting of $\tilde{\mathscr{L}}(m-1)_{\mathrm{rig}}$ with $\tilde{\mathscr{L}}_{\mathrm{rig}}^m$ over

$\tilde{U}(m-1)_{\text{rig}} \cap \tilde{U}^m_{\text{rig}}$ can be adjusted to have value 1 on each irreducible component by replacing the model $\tilde{\mathscr{L}}^m$ by the model $(\xi_{i_1}^{n_1} \cdot \ldots \cdot \xi_{i_r}^{n_r}) \cdot \tilde{\mathscr{L}}^m$ where the exponents $n_1, \ldots, n_r$ have to be chosen in such a way that the value of the pasting over X_σ is $|\pi|^{n_\sigma}$.

(2) If $\tilde{\mathscr{M}}_K$ is a multiplicative line bundle on $\tilde{X}$, there exists an element $c \in \tilde{K}$ and a cocycle $n := (n^{ij}) \in Z^1(\mathfrak{U}, \mathbb{Z})$ such that $\tilde{\mathscr{M}}_K$ is associated to the cocycle $c^n \in \mathrm{H}^1(\mathfrak{U}, \mathscr{O}^\times_{\tilde{X}_{\text{rig}}})$. We can write $c = \tilde{\pi}^s \cdot \tilde{\varepsilon}$ for some $\tilde{\varepsilon} \in \tilde{R}^\times$. Then the cocycle of $\tilde{\mathscr{M}}_K$ is given by the residue class

$$\overline{\text{cyc}}(\tilde{\mathscr{M}}_K) = \bar{s} \cdot \bar{n} \in \mathrm{H}^1(X, \mathbb{Z}/\mathbb{Z} \cdot e)$$

which is the image of $s \cdot n \in \mathrm{H}^1(X_0, \mathbb{Z})$ and, hence, $\delta\big(\overline{\text{cyc}}(\tilde{\mathscr{M}}_K)\big) = 0$ vanishes in $\mathrm{H}^2(X_0, \mathbb{Z})$. Then it follows from (1) that $\delta\big(\overline{\text{cyc}}(\tilde{\mathscr{L}}_K)\big) = 0$ vanishes if there exists a multiplicative line bundle $\tilde{\mathscr{M}}_K$ such that $\tilde{\mathscr{L}}_K \otimes \tilde{\mathscr{M}}_K$ extends to a line bundle on $\tilde{X}$.

Conversely, if $\delta\big(\overline{\text{cyc}}(\tilde{\mathscr{L}}_K)\big) = 0$ vanishes, there exists a cocycle $n := (n^{ij}) \in \mathrm{H}^1(X_0, \mathbb{Z})$ such that its residue class satisfies $\bar{n} = \overline{\text{cyc}}(\tilde{\mathscr{L}}_K)$ in $\mathrm{H}^1(X_0, \mathbb{Z}/\mathbb{Z} \cdot e)$. Let $\tilde{\mathscr{M}}_K$ be the multiplicative line bundle associated to the cocycle $\tilde{\pi}^{-n} \in \mathrm{H}^1(X, \mathscr{O}^\times_{\tilde{X}_{\text{rig}}})$. Then the cycle

$$\overline{\text{cyc}}(\tilde{\mathscr{L}}_K \otimes \tilde{\mathscr{M}}_K) = 0$$

vanishes and, hence the claim follows from the first statement. □

Remark 3.18.1. It is an open question whether $\delta\big(\overline{\text{cyc}}(\tilde{\mathscr{L}}_K)\big) = 0$ vanishes for all line bundles $\tilde{\mathscr{L}}_K$ on $X \times_R \tilde{R}$ and for all finite extensions $R \to \tilde{R}$. It seems to us that line bundles on $X \times_R \tilde{R}$ with $\delta\big(\text{cyc}(\tilde{\mathscr{L}}_K)\big) \neq 0$ are exotic.

Proposition 3.19. *There exists a finite extension $R \to \tilde{R}$ and finitely many exotic line bundles $\tilde{\mathscr{E}}^1_K, \ldots, \tilde{\mathscr{E}}^m_K$ on $\tilde{X}$ such that $\delta\big(\overline{\text{cyc}}(\tilde{\mathscr{E}}^1_K)\big), \ldots, \delta\big(\overline{\text{cyc}}(\tilde{\mathscr{E}}^m_K)\big)$ give all elements in $\mathrm{H}^2(X_0, \mathbb{Z})$ which are induced by line bundles on any extension $\bar{X} = X \times_R \bar{R}$ for any finite extension $\tilde{R} \to \bar{R}$.*

Proof. For any topological space T of finite combinatorial dimension such as X_0, the groups $\mathrm{H}^n(T, \mathbb{Z})$ are finitely generated for all $n \in \mathbb{N}$. A topological space is said to have combinatorial dimension less or equal to N if any strictly increasing sequence of irreducible closed subsets of Z consists of at most $N+1$ elements. Therefore the group $\mathrm{H}^2(X_0, \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module and, hence there are only finitely many torsion elements in $\mathrm{H}^2(X_0, \mathbb{Z})$. □

Theorem 3.20. *The Néron-Severi group of X_{rig} is finitely generated.*

Proof. For any finite separable field extension $K \to \tilde{K}$ with residue field $\tilde{k}$, we have an exact sequence

$$1 \longrightarrow \mathrm{NS}^1_{X_K/K}(\tilde{K}) \longrightarrow \mathrm{NS}_{X_K/K}(\tilde{K}) \xrightarrow{\delta \circ \overline{\text{cyc}}} \mathrm{H}^2(X_0, \mathbb{Z})$$

where $\mathrm{NS}^1_{X_K/K}(\tilde{K})$ is defined as the kernel of the cycle map. Due to Lemma 3.18 any representative of an element in $\mathrm{NS}^1_{X_K/K}(\tilde{K})$ extends to a line bundle on $\tilde{X}$ after twisting by a multiplicative line bundle. Since the multiplicative line bundles belong to $\mathrm{Pic}^0_{X_K}(\tilde{K})$, we get a group homomorphism by taking the reduction of the formal extension

$$\mathrm{NS}^1_{X_K/K}(\tilde{K}) \to \mathrm{NS}_{X_0/k}(\tilde{k})/M(\tilde{R})$$

where $M(\tilde{R})$ is the image of the set of formal line bundles on $X \times_R \tilde{R}$ which induce multiplicative line bundles on the generic fiber. Elements in the kernel of this map can be represented by formal line bundles $\tilde{\mathscr{L}}$ on $\tilde{X}$ such that the reduction $\tilde{\mathscr{L}}_0 := \tilde{\mathscr{L}} \otimes_{\tilde{R}} \tilde{k}$ belongs to $\mathrm{Pic}^0_{X_0/k}(\tilde{k})$. Due to our construction, the class of $\tilde{\mathscr{L}}$ is therefore an $\tilde{R}$-rational point of $\overline{P'}$; cf. Proposition 3.4. Thus we obtain an exact sequence

$$\overline{P'}(\tilde{R})/\bar{P}(\tilde{R}) \to \mathrm{NS}^1_{X_K/K}(\tilde{K}) \to \mathrm{NS}_{X_0/k}(\tilde{k})/M(\tilde{R}).$$

So $\mathrm{NS}^1_{X_K/K}(\tilde{K})$ is an extension of a subquotient of the Néron-Severi group of the special fiber by a finite group; cf. Proposition 3.3. This description is compatible with field extensions, since the map $\delta \circ \mathrm{cyc}$ is compatible with field extensions. Since the Néron-Severi group $\mathrm{NS}_{X_0/k}$ is finitely generated; cf. [SGA 6], Exp. XIII, Thm. 5.1, the group $\mathrm{NS}^1_{X_K/K}(\overline{K})$ is finitely generated where $\overline{K}$ is the algebraic closure of K. Due to Proposition 3.19 the Néron-Severi group $\mathrm{NS}_{X_K/K}(\overline{K})$ is finitely generated.

Finally we have to show that any line bundle over the completion of the algebraic closure $\mathbb{K}$ of K is represented by a point on a translate of P_K by a point which is defined over a finite extension of K. Consider a line bundle $\mathscr{L}_{\mathbb{K}}$ on $X \hat{\otimes}_K \mathbb{K}$. Then there exists an admissible blowing up

$$\mathfrak{Y} \to X \hat{\otimes}_R R_{\mathbb{K}}$$

of a coherent open ideal $\mathscr{J}$ on $X \hat{\otimes}_R R_{\mathbb{K}}$ such that $\mathscr{L}_{\mathbb{K}}$ extends to a line bundle $\mathscr{L}$ on $\mathfrak{Y}$ where $R_{\mathbb{K}}$ is the ring of integers of $\mathbb{K}$; cf. [L1], Lemma 2.9. Since $\mathscr{J}$ is open and X is quasi-compact, there exists a finite extension $\tilde{R}$ of R such that $\mathscr{J}$ is induced by an open ideal $\mathscr{I}$ on $X \times_R \tilde{R}$. Thus $\mathfrak{Y}$ is obtained from a model Y of $\tilde{X}_K := X_K \otimes_K \tilde{K}$ and, hence, we may assume that Y is a blowing-up of $X \otimes_R \tilde{R}$.

Thus we see that $\mathscr{L}$ is an $R_{\mathbb{K}}$-valued point of

$$Q = \big((\varinjlim Q_n)/(\pi\text{-torsion})\big)_{\mathrm{red}}$$

where Q_n is the Picard scheme $\mathrm{Pic}_{Y_n/\tilde{S}_n}$ of the model Y of $\tilde{X}_{\mathrm{rig}}$ over $\tilde{S} = \mathrm{Spf}(\tilde{R})$. We may assume that there is a $\tilde{K}$-valued point on the connected component of Q_{rig} which contains the class of $\mathscr{L}_{\mathrm{rig}}$; this point can be represented by the class of a line bundle $\tilde{\mathscr{F}}$ on Y. Then for our purpose, we can assume that $\mathscr{F}$ is trivial so that we have to consider the 1-component Q^0_{rig} of Q_{rig}. Thus we are concerned with a rigid-analytic group object over $\tilde{K}$. The group $Q_K := Q^0_{\mathrm{rig}}$ is geometrically reduced and hence smooth over $\tilde{K}$. Indeed, the rig-isomorphism $Y \to \tilde{X}$ induces a map $\bar{P}_K \to Q_K$. This map does not factor through any closed subgroup of lower dimension and thus we see that the points which are separable over $\tilde{K}$ are dense in Q_K. Thus we see that $\mathscr{L}_{\mathbb{K}}$ is a fiber of the Poincaré bundle on $Y_{\mathrm{rig}} \times Q_K$ over the smooth base Q_K. Since $Y_{\mathrm{rig}} = X_{\mathrm{rig}}$, the point associated to $\mathscr{L}_{\mathbb{K}}$ belongs to $P_K(\mathbb{K})$, due to Theorem 3.14 resp. Remark 3.14.1. □

3.6. The end of the proof. After these preparations it is easy to finish the proof of the main theorem announced in the introduction. So start with the strict semi-stable formal R-scheme X which is assumed to be proper and connected.

Furthermore let $x \in X(R)$ be a rational point. Then let $R \to \tilde{R}$ be a finite extension of discrete valuation rings and let $\tilde{R}$ be chosen so large that a generating system of the Néron-Severi group is already given by the classes of line bundles on $X_{\tilde{K}} := X_K \times_K \tilde{K}$. This is possible since the Néron-Severi group is finitely generated due to Theorem 3.20. Moreover we assume that the residue field k of R is separably closed to avoid decompositions of irreducible components of X_0 under base change. Then all classes in the Néron-Severi group can be represented by line bundles on $X_{\tilde{K}}$. Let

$$\mathrm{NS}_{X_K/K}(\mathbb{K}) = \{\mathscr{L}^i; i \in I\}$$

be a set of representatives of all classes in the Néron-Severi group $\mathrm{NS}_{X_K/K}(\mathbb{K})$. Then define the rigid analytic variety

$$\mathrm{Pic}_{X_{\tilde{K}}/\tilde{K}} := \coprod_{i \in I} \mathscr{L}^i \otimes P_K$$

equipped with the line bundle $p_1^* \mathscr{L}^i \otimes \mathscr{P}_K$ over $\mathscr{L}^i \otimes P_K$ where $p_1 \colon X_K \times P_K \to X_K$ is the first projection.

Consider a smooth rigid-analytic variety V_K over K and a rigidified line bundle $(\mathscr{L}, \lambda)$ on $X_K \times_K V_K$. We may assume that V_K is connected. Let $K \to \tilde{K} \to K'$ be a finite field extension such that V_K admits a K'-rational point v. Again we may assume that $V_K \times_K K'$ is connected. Then we twist the line bundle $\mathscr{L}$ by the inverse of the line bundle

$$\mathscr{L}_v := (\mathrm{id}_{X_K} \times v)^* \mathscr{L}$$

on the fiber so that we may assume that $\mathscr{L}$ is trivial above v. It follows from Theorem 3.14 that there exists a unique morphism $\varphi \colon V_K \otimes K' \to P_K$ and a unique isomorphism $(p_1^* \mathscr{L}_v^{-1} \otimes \mathscr{L}, \lambda) \xrightarrow{\sim} (\mathrm{id} \times \varphi)^* (\mathscr{P}, \lambda_{\mathscr{P}})$. Moreover there exists a unique $i \in I$ such that $\mathscr{L}_v$ lies in the class of $\mathscr{L}^i$ mod $P_K(K')$. Now one can compose φ with the translation map given by the element $\mathscr{L}_v \otimes (\mathscr{L}^i)^{-1} \in P_K(K')$. Thus we obtain the desired morphism

$$\Phi \colon V_K \to \mathscr{L}^i \otimes P_K \hookrightarrow \mathrm{Pic}_{X_{\tilde{K}}/\tilde{K}}$$

satisfying the universal property.

Of course we may assume that K'/K and also $\tilde{K}/K$ are Galois, since any smooth connected variety admits a closed point which is separable over K. The Galois group $G(\tilde{K}/K)$ acts on $\mathrm{Pic}_{X_{\tilde{K}}/\tilde{K}}$. Each orbit is finite since the group is finite. On P_K the Galois group acts in a formal way; i.e., it stabilizes the subgroup $\bar{P}$ and the formal torus part $\bar{T}$. Furthermore, since the reduction of $\bar{P}$ is quasi-projective, the Galois descent is effective. Thus we see that $\hat{P}_K$ is defined over K. Then it is clear that P_K is defined over K. From this it follows that the group variety $\mathrm{Pic}_{X_{\tilde{K}}/\tilde{K}}$ descends to a rigid analytic K-group variety $\mathrm{Pic}_{X_K/K}$. This group variety represents the Picard functor over K since the mapping property is compatible with Galois descent under the Galois group $G(K'/K)$ of the extension K'/K. $\square$

References

[Ar 1] *Artin, M.*, Algebraization of formal moduli I, Global Analysis, Papers in Honor of K. Kodaira, Princeton University Press (1969), 21–71.

[Ar 2] *Artin, M.*, The Implicit Function Theorem in Algebraic Geometry, Arithmetical Algebraic Geometry, Proceedings Paper of the Conference on Arithmetic Algebraic Geometry at Purdue University in 1963, Harper and Row, New York (1965), 13–34.

[BM] *Bass, H., Murthy, M. P.*, Grothendieck Groups and Picard Groups of Abelian Group Rings, Ann. Math. **86** (1967), 16–73.

[B] *Blanchard, A.*, Les variétés analytiques complexes, E.N.S. **73** (1956), 157–202.

[BGR] *Bosch, S., Güntzer, U., Remmert, R.*, Nonarchimedian Analysis, Springer Grundl. **261**, Berlin-Heidelberg 1984.

[BL] *Bosch, S., Lütkebohmert, W.*, Degenerating Abelian Varieties, Topology **30**, No. 4 (1991), 653–698.

[BLR] *Bosch, S., Lütkebohmert, W., Raynaud, M.*, Néron Models, Springer Ergeb. 3. Folge **21**, Berlin-Heidelberg 1990.

[FRG] *Bosch, S., Lütkebohmert, W., Raynaud, M.*, Formal and Rigid Geometry; I. Rigid Spaces, Math. Ann. **295** (1993), 291–317; II. Flattening Techniques, Math. Ann. **296** (1993), 403–429; III. The Relative Maximum Principle, Math. Ann. **302** (1995), 1–29; IV. The Reduced Fiber Theorem, Invent. Math. **119** (1995), 361–398.

[BS] *Bosch, S., Schlöter, K.*, Néron models in the setting of formal and rigid geometry, Math. Ann. **301** (1995), 339–362.

[Bo] *Bourbaki, N.*, Commutative Algebra, Elements of Mathematics, Hermann, Paris 1972.

[dJ] *de Jong, A. J.*, Smoothness, Semi-stability and Alterations, Publ. Math. IHES **83** (1996), 51–93.

[SGA 3] *Demazure, M., Grothendieck, A.*, et al., Séminaire de Géométrie Algébrique 3: Schémas en Groupes I, II, III, Springer Lect. Notes Math. **151**, **152**, **153**, Berlin-Heidelberg 1970.

[FK] *Freitag, E., Kiehl, R.*, Etale Cohomology and the Weil Conjecture, Springer Ergeb. 3. Folge **13**, Berlin-Heidelberg 1988.

[Ge] *Gerritzen, L.*, Zerlegung der Picard-Gruppe nichtarchimedischer holomorpher Räume, Compos. Math. **25** (1977), 23–38.

[EGA] *Grothendieck, A., Dieudonné, J.*, et al., Eléments de Géométrie Algébrique, Publ. Math. IHES **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1960–67) and Springer Grundl. **166**, Berlin-Heidelberg 1971.

[FGA] *Grothendieck, A.*, Fondements de la Géométrie Algébrique, Séminaire Bourbaki 1957–62, Secrétariat Mathématiques, Paris 1962.

[TCGA] *Grothendieck, A.*, Techniques de construction en géométrie analytique. I–X, Sém. Cartan **13** (1960/61), n^0 7–16.

[SGA 1] *Grothendieck, A.*, et al., Séminaire de Géométrie Algébrique 1: Revêtements Etales et Groupe Fondamental, Springer Lect. Notes Math. **224**, Berlin-Heidelberg 1971.

[SGA 6] *Grothendieck, A.*, et al., Séminaire de Géométrie Algébrique 6: Théorie des Intersections et Théorème de Riemann-Roch, Springer Lect. Notes Math. **225**, Berlin-Heidelberg 1971.

[Ha] *Hartl, U.*, Semi-stability and base change, Arch. Math., to appear.

[TE] *Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.*, Toroidal embeddings I, Springer Lect. Notes Math. **339**, Berlin-Heidelberg-New York 1973.

[Ki] *Kiehl, R.*, Analytische Familien affinoider Algebren, Heidelberger Sitzungsber., Springer-Verlag (1968), 25–49.

[L1] *Lütkebohmert, W.*, Formal-algebraic and rigid-analytic geometry, Math. Ann. **286** (1990), 341–371.

[L2] *Lütkebohmert, W.*, The structure of proper rigid groups, J. reine angew. Math. **468** (1995), 167–219.

[Mo] *Moret-Bailly, L.*, Pinceaux de Variétés Abéliennes, Astérisque **129** (1985).

[Ms] *Mustafin, G. A.*, *p*-adic Hopf varieties, Funct. Anal. Appl. **11** (1977), 234–235.

[Ra 1] *Raynaud, M.*, Variétés abéliennes et géométrie rigide, Actes du congrès international de Nice **1** (1970), 473–477.

[Ra 2] *Raynaud, M.*, Géométrie analytique rigide d'après Tate, Kiehl, ... Table ronde d'analyse non archimedienne, Mém. Soc. Math. France **39/40** (1974), 319–327.

[Se] *Serre, J.-P.*, Groupes algébriques et corps de classes, Hermann, Paris 1959.

[Ta] *Tate, J.*, Rigid analytic spaces, Invent. Math. **12** (1971), 257–289.

Abteilung Reine Mathematik, Universität Ulm, D-89069 Ulm, Germany

Eingegangen 7. September 1999, in revidierter Fassung 4. Februar 2000