# Periods of Drinfeld modules and local shtukas with complex multiplication 

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February 3, 2021


#### Abstract

Colmez [Col93] conjectured a product formula for periods of abelian varieties over number fields with complex multiplication and proved it in some cases. His conjecture is equivalent to a formula for the Faltings height of CM abelian varieties in terms of the logarithmic derivatives at $s=0$ of certain Artin $L$-functions.

In a series of articles we investigate the analog of Colmez's theory in the arithmetic of function fields. There abelian varieties are replaced by Drinfeld modules and their higher dimensional generalizations, so-called $A$-motives. In the present article we prove the product formula for the Carlitz module and we compute the valuations of the periods of a CM $A$-motive at all finite places in terms of Artin $L$-series. The latter is achieved by investigating the local shtukas associated with the $A$-motive. Mathematics Subject Classification (2000): 11G09, (11R42, 11R58, 14L05)


## 1 Introduction

In Col93 P. Colmez considers product formulas for periods of abelian varieties. Let $X$ be an abelian variety defined over a number field $K$ with complex multiplication by the ring of integers in a CM-field $E$ and of CM-type $\Phi$. Let $\mathbb{Q}^{\text {alg }}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, let $H_{E}:=\operatorname{Hom}_{\mathbb{Q}}\left(E, \mathbb{Q}^{\text {alg }}\right)$ be the set of all ring homomorphisms $E \hookrightarrow \mathbb{Q}^{\text {alg }}$ and assume that $K$ contains $\psi(E)$ for every $\psi \in H_{E}$. For a $\psi \in H_{E}$ let $\omega_{\psi} \in \mathrm{H}_{\mathrm{dR}}^{1}(X, K)$ be a non-zero cohomology class such that $a^{*} \omega_{\psi}=\psi(a) \cdot \omega_{\psi}$ for all $a \in E$. For every embedding $\eta: K \hookrightarrow \mathbb{Q}^{\text {alg }}$, let $X^{\eta}$ and $\omega_{\psi}^{\eta}$ be deduced from $X$ and $\omega_{\psi}$ by base extension. Let $\left(u_{\eta}\right)_{\eta} \in \prod_{\eta \in H_{K}} \mathrm{H}_{1}\left(X^{\eta}(\mathbb{C}), \mathbb{Z}\right)$ be a family of cycles compatible with complex conjugation. Let $v$ be a place of $\mathbb{Q}$. If $v=\infty$ the de Rham isomorphism between Betti and de Rham cohomology yields a complex number $\int_{u_{\eta}} \omega_{\psi}^{\eta}$ and its absolute value $\left|\int_{u_{\eta}} \omega_{\psi}^{\eta}\right|_{\infty} \in \mathbb{R}$. If $v$ corresponds to a prime number $p \in \mathbb{Z}$, we fix an inclusion $\mathbb{Q}^{\text {alg }} \hookrightarrow \mathbb{Q}_{p}^{\text {alg }}$. With this data Colmez Col93] associates a period $\int_{u_{\eta}} \omega_{\psi}^{\eta}$ in Fontaine's $p$-adic period field $\mathbf{B}_{\mathrm{dR}}$ and an absolute value $\left|\int_{u_{\eta}} \omega_{\psi}^{\eta}\right|_{v} \in \mathbb{R}$. He considers the product $\prod_{v} \prod_{\eta \in H_{K}}\left|\int_{u_{\eta}} \omega_{\psi}^{\eta}\right|_{v}$ and (after some modifications) conjectures that this product evaluates to 1 ; see Col93, Conjecture 0.1] for the precise formulation. This conjecture is equivalent to a conjectural formula for the Faltings height of a CM abelian variety in terms of the logarithmic derivatives at $s=0$ of certain Artin $L$-functions. Colmez proves the conjectures when $E$ is an abelian extension of $\mathbb{Q}$. On the way, he computes $\prod_{\eta \in H_{K}}\left|\int_{u_{\eta}} \omega_{\psi}^{\eta}\right|_{v}$ at a finite place $v$ in terms of the local factor at $v$ of the Artin $L$-series associated with an Artin character $a_{E, \psi, \Phi}^{0}: \operatorname{Gal}\left(\mathbb{Q}^{\text {alg }} / \mathbb{Q}\right) \rightarrow \mathbb{C}$ that only depends on $E, \psi$ and $\Phi$ but not on $X$ and $v$; see Col93, Théorème I.3.15]. There has been further progress on Colmez's conjecture by Obus Obu13, Yang Yan13, Andreatta, Goren, Howard, Madapusi Pera AGHM15, Yuan, Zhang [YZ15], Barquero-Sanchez, Masri [BSM16] and others.

[^0]Our goal in this article is to develop the analog of Colmez's theory in the "Arithmetic of function fields". Here abelian varieties are replaced by Drinfeld modules Dri76, Gos96] and their higher dimensional generalizations, so-called $A$-motives, which also generalize Anderson's $t$-motives And86]. To define them let $\mathbb{F}_{q}$ be a finite field with $q$ elements, let $C$ be a smooth projective, geometrically irreducible curve over $\mathbb{F}_{q}$, let $\infty \in C$ be a fixed closed point and let $A:=\Gamma\left(C \backslash\{\infty\}, \mathcal{O}_{C}\right)$ be the ring of regular functions on $C$ outside $\infty$. Let $Q$ be the fraction field of $A$ and let $K$ be a finite field extension of $Q$ contained in a fixed algebraic closure $Q^{\text {alg }}$ of $Q$. We write $A_{K}:=A \otimes_{\mathbb{F}_{q}} K$ and consider the endomorphism $\sigma:=\operatorname{id}_{A} \otimes \operatorname{Frob}_{q, K}$ of $A_{K}$, where $\operatorname{Frob}_{q, K}(b)=b^{q}$ for $b \in K$. For an $A_{K}$-module $M$ we set $\sigma^{*} M:=M \otimes_{A_{K}, \sigma} A_{K}$ and for a homomorphism $f: M \rightarrow N$ of $A_{K}$-modules we set $\sigma^{*} f:=f \otimes \operatorname{id}_{A_{K}}: \sigma^{*} M \rightarrow \sigma^{*} N$. Let $\gamma: A \rightarrow K$ be the inclusion $A \subset Q \subset K$, and set $\mathcal{J}:=(a \otimes 1-1 \otimes \gamma(a): a \in A) \subset A_{K}$. Then $\gamma$ can be recovered as the homomorphism $A \rightarrow A_{K} / \mathcal{J}=K$.

Definition 1.1. An $A$-motive of rank $r$ over $K$ is a pair $\underline{M}=\left(M, \tau_{M}\right)$ consisting of a locally free $A_{K^{-}}$ module $M$ of rank $r$ and an isomorphism $\tau_{M}:\left.\left.\sigma^{*} M\right|_{\text {Spec } A_{K} \backslash \mathrm{~V}(\mathcal{J})} \xrightarrow{\sim} M\right|_{\text {Spec } A_{K} \backslash \mathrm{~V}(\mathcal{J})}$ of the associated sheaves outside $\mathrm{V}(\mathcal{J}) \subset \operatorname{Spec} A_{K}$. We write rk $\underline{M}:=r$. A morphism between $A$-motives $f:\left(M, \tau_{M}\right) \rightarrow$ $\left(N, \tau_{N}\right)$ is an $A_{K}$-homomorphism $f: M \rightarrow N$ with $f \circ \tau_{M}=\tau_{N} \circ \sigma^{*} f$.

Let us first give a rough sketch of the function field analog of Colmez's conjecture, before we explain more details and our main results later in this introduction. An $A$-motive has various (co-)homology realizations, for example a de Rham realization $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$, and if it is uniformizable also a Betti realization $\mathrm{H}_{1, \operatorname{Betti}}(\underline{M}, A)$. For every place $v$ of $Q$, that is a closed point $v \in C$ there is a comparison isomorphism between the Betti and de Rham cohomology of $\underline{M}$, which for $\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$ and $u \in$ $\mathrm{H}_{1, \operatorname{Betti}}(\underline{M}, A)$ is given by a pairing $\langle\omega, u\rangle_{v}$ and allows to define the absolute value $\left|\int_{u} \omega\right|_{v}:=\left|\langle\omega, u\rangle_{v}\right|_{v} \in$ $\mathbb{R}$. Now we say that $\underline{M}$ has complex multiplication if $\operatorname{QEnd}_{K}(\underline{M}):=\operatorname{End}_{K}(\underline{M}) \otimes_{A} Q$ contains a commutative, semi-simple $Q$-algebra $E$ of $\operatorname{dimension}^{\operatorname{dim}_{Q} E=r k} \underline{M}$. Here semi-simple means that $E$ is a product of fields and we do not assume that $E$ is itself a field. Let $\underline{M}$ be a uniformizable $A$-motive over a finite Galois extension $K \subset Q^{\text {alg }}$ of $Q$, which has complex multiplication by a separable $Q$-algebra $E$. Let $0 \neq \omega_{\psi} \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$ satisfy $a^{*} \omega_{\psi}=\psi(a) \cdot \omega_{\psi}$ for all $a \in E$, where $\psi: E \rightarrow K$ is a $Q$-homomorphism. Then in Theorem 5.24 and Corollary 5.25 we will for all finite places $v$ compute $\left|\int_{u} \omega_{\psi}\right|_{v}$ and its average over all $Q$-automorphisms of $K$ in terms of the local factor at $v$ of an Artin $L$-series. The question analogous to Col93 is then, whether one can make sense of the product $\prod_{v}\left|\int_{u} \omega_{\psi}\right|_{v}$ over all places $v$ including $\infty$, and whether this product evaluates to 1 .

After this vague sketch let us give more details and precise definitions in order to formulate our main results. We start by introducing the cohomology realizations of an $A$-motive $\underline{M}$ over $K$. First of all, there is the de Rham realization $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K):=\sigma^{*} M / \mathcal{J} \cdot \sigma^{*} M$ and for each maximal ideal $v \subset A$ a $v$-adic étale realization $\mathrm{H}_{v}^{1}\left(\underline{M}, A_{v}\right)$ where $A_{v}$ is the $v$-adic completion of $A$; see Definition 3.3 below. We let $Q_{v}$ be the fraction field of $A_{v}$, and we let $Q_{\infty}$ be the $\infty$-adic completion of $Q$ and $\mathbb{C}_{\infty}$ be the completion of a fixed algebraic closure of $Q_{\infty}$. We fix a $Q$-embedding $Q^{\text {alg }} \hookrightarrow \mathbb{C}_{\infty}$ and consider the base extension of $\underline{M}$ to $\mathbb{C}_{\infty}$. There is a notion of $\underline{M}$ being uniformizable and a uniformizable $\underline{M}$ has a Betti realization $\mathrm{H}_{\text {Betti }}^{1}(\underline{M}, A)$; see [HJ16, §3.5]. These realizations are related by period isomorphisms

$$
\begin{aligned}
h_{\operatorname{Betti}, v}: & \left.\mathrm{H}_{\operatorname{Betti}}^{1} \underline{M}, A\right) \otimes_{A} A_{v} \\
\sim & \mathrm{H}_{v}^{1}\left(\underline{M}, A_{v}\right) \quad \text { and } \\
h_{\operatorname{Betti}, \mathrm{dR}}: & \mathrm{H}_{\operatorname{Betti}}^{1}(\underline{M}, A) \otimes_{A} \mathbb{C}_{\infty} \xrightarrow{\sim} \mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K) \otimes_{K} \mathbb{C}_{\infty}
\end{aligned}
$$

see [HJ16, Theorem 3.23]. Also for every place $v$ of $Q$, let $\mathbb{F}_{v}$ be its residue field and set $q_{v}:=\# \mathbb{F}_{v}=$ $q^{\left[\mathbb{F}_{v}: \mathbb{F}_{q}\right]}$. Let $z:=z_{v} \in Q$ be a uniformizing parameter at $v$. Then there is a canonical isomorphism $A_{v}=$ $\mathbb{F}_{v} \llbracket z_{v} \rrbracket$. Let $\zeta:=\zeta_{v}:=\gamma\left(z_{v}\right)$ denote the image of $z_{v}$ in $K$. We simply write $z$, resp. $\zeta$ for the elements $z \otimes 1$, resp. $1 \otimes \zeta$ of $Q{\otimes \mathbb{F}_{q}} K$. Then the power series ring $K \llbracket z-\zeta \rrbracket$ in the "variable" $z-\zeta$ is canonically isomorphic to the completion of the local ring of $C_{K}:=C \times_{\mathbb{F}_{q}} K$ at $\mathrm{V}(\mathcal{J})$; see HJ16, Lemma 1.2 and 1.3], and thus independent of $v$. We always consider the embedding $Q \hookrightarrow K \llbracket z-\zeta \rrbracket$ given by $z \mapsto z=\zeta+(z-\zeta)$. The de Rham realization lifts to $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K \llbracket z-\zeta \rrbracket):=\sigma^{*} M \otimes_{A_{K}} K \llbracket z-\zeta \rrbracket$, which is the analog of the
(conjectural) $q$-de Rham cohomology of Bhatt, Morrow and Scholze BMS15, BMS16, Sch16, and the vector space $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K \llbracket z-\zeta \rrbracket)\left[\frac{1}{z-\zeta}\right]$ over the field $K((z-\zeta)):=K \llbracket z-\zeta \rrbracket\left[\frac{1}{z-\zeta}\right]$ contains the $K \llbracket z-\zeta \rrbracket$ lattice $\mathfrak{q} \underline{\underline{M}}:=\tau_{M}^{-1}\left(M \otimes_{A_{K}} K \llbracket z-\zeta \rrbracket\right)$, which is called the Hodge-Pink lattice of $\underline{M}$ and is the analog of the Hodge-filtration of an abelian variety; see [HK15, Remark 5.13].

If $v \neq \infty$ we also fix a $Q$-embedding of $Q^{\text {alg }}$ into a fixed algebraic closure $Q_{v}^{\text {alg }}$ of $Q_{v}$ and we let $\mathbb{C}_{v}$ be the $v$-adic completion of $Q_{v}^{\text {alg }}$. Again we denote the image of $z_{v}$ in $Q_{v}^{\text {alg }}$ and $\mathbb{C}_{v}$ by $\zeta_{v}$. We let $K_{v} \subset Q_{v}^{\text {alg }}$ be the induced completion of $K$ and we let $R$ be its valuation ring. There is a period isomorphism by HK15, Remark 4.16]

$$
h_{v, \mathrm{dR}}: \mathrm{H}_{v}^{1}\left(\underline{M}, A_{v}\right) \otimes_{A_{v}} \mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right) \xrightarrow{\sim} \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{M}, K \llbracket z_{v}-\zeta_{v} \rrbracket\right) \otimes_{K \llbracket z_{v}-\zeta_{v} \rrbracket} \mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right) .
$$

The field $\mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right)$ is the analog of Fontaine's $p$-adic period field $\mathbf{B}_{\mathrm{dR}}$; see [HK15, Remark 4.17].
Let $\underline{M}$ have complex multiplication by a commutative, semi-simple $Q$-algebra $E$ of dimension $\operatorname{dim}_{Q} E=\operatorname{rk} \underline{M}$. Let $\mathcal{O}_{E}$ be the integral closure of $A$ in $E$. It is a locally free $A$-module of $\mathrm{rk}_{A} \mathcal{O}_{E}=$ $\operatorname{dim}_{Q} E$. We let $H_{E}:=\operatorname{Hom}_{Q}\left(E, Q^{\text {alg }}\right)$ be the set of $Q$-homomorphisms $\psi: E \rightarrow Q^{\text {alg }}$ and we assume that $K$ contains $\psi(E)$ for every $\psi \in H_{E}$. Then by Lemma A.3 in the appendix there is a decomposition $E \otimes_{Q} K \llbracket z-\zeta \rrbracket=\prod_{\psi \in H_{E}} K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket$, where $y_{\psi}$ is a uniformizer at a place of $E$ such that $\psi\left(y_{\psi}\right) \neq 0$. Again by [HJ16, Lemma 1.2 and 1.3] the factors are obtained as the completion of $\mathcal{O}_{E} \otimes_{A} A_{K}=\mathcal{O}_{E} \otimes_{\mathbb{F}_{q}} K$ along the kernels $\left(a \otimes 1-1 \otimes \psi(a): a \in \mathcal{O}_{E}\right)$ of the homomorphisms $\psi \otimes \operatorname{id}_{K}: \mathcal{O}_{E} \otimes_{\mathbb{F}_{q}} K \rightarrow K$ for $\psi \in H_{E}$. Correspondingly $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K \llbracket z-\zeta \rrbracket)$ decomposes into eigenspaces

$$
\mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right):=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K \llbracket z-\zeta \rrbracket) \otimes_{E \otimes_{Q} K \llbracket z-\zeta \rrbracket} K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket
$$

each of which is free of rank 1 over $K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket$. There are integers $d_{\psi}$ such that the Hodge-Pink lattice is $\mathfrak{q} \underline{\underline{M}}=\prod_{\psi}\left(y_{\psi}-\psi\left(y_{\psi}\right)\right)^{-d_{\psi}} \mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$. The tuple $\Phi:=\left(d_{\psi}\right)_{\psi \in H_{E}}$ is the CM-type of $\underline{M}$.

If we fix elements $u \in \mathrm{H}_{1, \operatorname{Betti}}(\underline{M}, Q):=\operatorname{Hom}_{A}\left(\mathrm{H}_{\mathrm{Betti}}^{1}(\underline{M}, A), Q\right)$ and $\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K \llbracket z-\zeta \rrbracket)$ we can define

$$
\begin{align*}
& \langle\omega, u\rangle_{\infty}:=u \otimes \operatorname{id}_{\mathbb{C}_{\infty}}\left(h_{\operatorname{Betti}, \mathrm{dR}}^{-1}(\omega \bmod z-\zeta)\right) \in \mathbb{C}_{\infty} \quad \text { and }  \tag{1.1}\\
& \left|\int_{u} \omega\right|_{\infty}:=\left|\langle\omega, u\rangle_{\infty}\right|_{\infty} \in \mathbb{R}, \tag{1.2}
\end{align*}
$$

where $|\cdot|_{v}$ is the normalized absolute value on $\mathbb{C}_{v}$ with $\left|\zeta_{v}\right|_{v}=\left(\# \mathbb{F}_{v}\right)^{-1}=q_{v}^{-1}$ for every place $v$. We also consider the valuation $v: \mathbb{C}_{v}^{\times} \rightarrow \mathbb{Q}$ on $\mathbb{C}_{v}$ with $v\left(\zeta_{v}\right)=1$. The expressions in (1.1) and (1.2) only depend on the image of $\omega$ in $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$. Also at a finite place $v \neq \infty$ of $Q$ we consider on elements $x \neq 0$ of the discretely valued field $\mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right)$ the valuation $\hat{v}(x):=\operatorname{ord}_{z_{v}-\zeta_{v}}(x)$, and in addition we define

$$
\begin{aligned}
& |x|_{v}:=\left|\left(\left(z_{v}-\zeta_{v}\right)^{-\hat{v}(x)} \cdot x\right) \bmod z_{v}-\zeta_{v}\right|_{v} \quad \text { and } \\
& v(x):=-\log |x|_{v} / \log q_{v} \quad \text { induced from } \\
& \left(\left(z_{v}-\zeta_{v}\right)^{-\hat{v}(x)} \cdot x\right) \bmod z_{v}-\zeta_{v} \in \mathbb{C}_{v} .
\end{aligned}
$$

Note that $|x|_{v}$ and $v(x)$ are not a norm, respectively a valuation, because they do not satisfy the triangle inequality. The value $|x|_{v}$ does not depend on the choice of the uniformizer $z_{v}$ of $A_{v}$, because if $\tilde{z}_{v}=$ $\sum_{n=0}^{\infty} b_{n} z_{v}^{n}=: f\left(z_{v}\right)$ with $b_{n} \in \mathbb{F}_{v}$ is another uniformizer and $\tilde{\zeta}_{v}=f\left(\zeta_{v}\right)$, then $\frac{\tilde{z}_{v}-\tilde{\zeta}_{v}}{z_{v}-\zeta_{v}} \equiv f^{\prime}\left(\zeta_{v}\right) \bmod z_{v}-$ $\zeta_{v}$ in $\mathcal{O}_{\mathbb{C}_{v}} \llbracket z_{v} \rrbracket=\mathbb{F}_{v} \llbracket z_{v} \rrbracket \widehat{\otimes}_{\mathbb{F}_{v}, \gamma} \mathcal{O}_{\mathbb{C}_{v}}$ by Lemma A. 1 in the appendix and $f^{\prime}\left(\zeta_{v}\right) \in \mathbb{F}_{v} \llbracket \zeta_{v} \rrbracket^{\times}$with inverse $\left.\frac{d z_{v}}{d \tilde{z}_{v}}\right|_{\tilde{z}_{v}=\tilde{\xi}_{v}}$.We define

$$
\begin{align*}
& \langle\omega, u\rangle_{v}:=u \otimes_{\mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right)}\left(h_{\operatorname{Betti}, v}^{-1} \circ h_{v, \mathrm{dR}}^{-1}(\omega)\right) \in \mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right) \quad \text { and }  \tag{1.3}\\
& \left|\int_{u} \omega\right|_{v}:=\left|\langle\omega, u\rangle_{v}\right|_{v}:=\left|\left(\left(z_{v}-\zeta_{v}\right)^{-\hat{v}\left(\langle\omega, u\rangle_{v}\right)} \cdot\langle\omega, u\rangle_{v}\right) \bmod z_{v}-\zeta_{v}\right|_{v} \in \mathbb{R} . \tag{1.4}
\end{align*}
$$

We will show in Theorem 5.24 below that if $E$ is separable over $Q$ and if $\omega \in \mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$ has non-zero image in $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$, then the absolute value (1.4) only depends on that image.

With these definitions we can now consider the product $\prod_{v}\left|\int_{u} \omega\right|_{v}$, or equivalently its logarithm $\log \prod_{v}\left|\int_{u} \omega\right|_{v}=-\sum_{v} v\left(\int_{u} \omega\right) \log q_{v}$. Like in Colmez's theory, these products or sums do not converge and one has to give a convergent interpretation to their finite parts $\prod_{v \neq \infty}\left|\int_{u} \omega\right|_{v}$, respectively $-\sum_{v \neq \infty} v\left(\int_{u} \omega\right) \log q_{v}$; see Convention 1.4 below. To formulate the convention we make the following

Definition 1.2. For $F=Q$ or $F=Q_{v}$ let $F^{\text {sep }}$ be the separable closure of $F$ in $F^{\text {alg }}$ and let $\mathscr{G}_{F}:=\operatorname{Gal}\left(F^{\mathrm{sep}} / F\right)$. For a finite field extension $F^{\prime}$ of $F$ let $H_{F^{\prime}}:=\operatorname{Hom}_{F}\left(F^{\prime}, F^{\text {alg }}\right)$ be the set of $F$-homomorphisms $\psi: F^{\prime} \rightarrow F^{\text {alg }}$. Let $\mathcal{C}\left(\mathscr{G}_{F}, \mathbb{Q}\right)$ be the $\mathbb{Q}$-vector space of locally constant functions $a: \mathscr{G}_{F} \rightarrow \mathbb{Q}$ and let $\mathcal{C}^{0}\left(\mathscr{G}_{F}, \mathbb{Q}\right)$ be the subspace of those functions which are constant on conjugacy classes, that is, which satisfy $a\left(h^{-1} g h\right)=a(g)$ for all $g, h \in \mathscr{G}_{F}$. Then the $\mathbb{C}$-vector space $\mathcal{C}^{0}\left(\mathscr{G}_{F}, \mathbb{Q}\right) \otimes_{\mathbb{Q}} \mathbb{C}$ is spanned by the traces of representations $\rho: \mathscr{G}_{F} \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ with open kernel for varying $n$ by Ser77, $\S 2.5$, Theorem 6]. Via the fixed embedding $Q^{\text {sep }} \hookrightarrow Q_{v}^{\text {sep }}$ we consider the induced inclusion $\mathscr{G}_{Q_{v}} \subset \mathscr{G}_{Q}$ and morphism $\mathcal{C}\left(\mathscr{G}_{Q}, \mathbb{Q}\right) \rightarrow \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$. If $\chi$ is the trace of a representation $\rho: \mathscr{G}_{Q} \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ with open kernel we let $L(\chi, s):=\prod_{\text {all } v} L_{v}(\chi, s)$, respectively $L^{\infty}(\chi, s):=\prod_{v \neq \infty} L_{v}(\chi, s)$ be the Artin $L$-function of $\rho$, respectively without the factor at $\infty$. It only depends on $\chi$ and converges for all $s \in \mathbb{C}$ with $\mathcal{R} e(s)>1$; see [Ros02, pp. 126ff]. We also set

$$
\begin{align*}
Z(\chi, s) & :=\frac{\frac{d}{d s} L(\chi, s)}{L(\chi, s)}=-\sum_{\text {all } v} Z_{v}(\chi, s) \log q_{v} \quad \text { and }  \tag{1.5}\\
Z^{\infty}(\chi, s) & :=\frac{\frac{d}{d s} L^{\infty}(\chi, s)}{L^{\infty}(\chi, s)}=-\sum_{v \neq \infty} Z_{v}(\chi, s) \log q_{v} \quad \text { with }  \tag{1.6}\\
Z_{v}(\chi, s) & :=\frac{\frac{d}{d s} L_{v}(\chi, s)}{-L_{v}(\chi, s) \cdot \log q_{v}}=\frac{\frac{d}{d q_{v}^{-s}} L_{v}(\chi, s)}{q_{v}^{s} \cdot L_{v}(\chi, s)} . \tag{1.7}
\end{align*}
$$

Moreover, we let $\mathfrak{f}_{\chi}$ be the Artin conductor of $\chi$. It is an effective divisor $\mathfrak{f}_{\chi}=\sum_{v} \mu_{\text {Art, }}(\chi) \cdot(v)$ on $C$; see [Ser79, Chapter VI, $\S \S 2,3$ ], where $\mu_{\text {Art }, v}(\chi)$ is denoted $f(\chi, v)$. We set

$$
\begin{align*}
& \mu_{\mathrm{Art}}(\chi):=\operatorname{deg}\left(\mathfrak{f}_{\chi}\right) \log q:=\sum_{\text {all } v} \mu_{\mathrm{Art}, v}(\chi)\left[\mathbb{F}_{v}: \mathbb{F}_{q}\right] \log q=\sum_{\text {all } v} \mu_{\mathrm{Art}, v}(\chi) \log q_{v} \text { and }  \tag{1.8}\\
& \mu_{\mathrm{Art}}^{\infty}(\chi):=\sum_{v \neq \infty} \mu_{\mathrm{Art}, v}(\chi) \log q_{v} . \tag{1.9}
\end{align*}
$$

In particular, only finitely many values $\mu_{\text {Art, } v}(\chi)$ are non-zero. By linearity we extend $Z^{\infty}(., s)$ and $\mu_{\mathrm{Art}}^{\infty}$ to all $a \in \mathcal{C}^{0}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ and $Z_{v}(., s)$ and $\mu_{\mathrm{Art}, v}$ to all $a \in \mathcal{C}^{0}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$. The map $Z_{v}(., s)$ takes values in $\mathbb{Q}\left(q_{v}^{-s}\right)$.

In terms of this definition we prove in this article a formula for $\left|\int_{u} \omega\right|_{v}$ with $v \neq \infty$ for a uniformizable $A$-motive $\underline{M}$ over $K$ with complex multiplication by a semi-simple separable CM-algebra $E$ of CM-type $\Phi=\left(d_{\varphi}\right)_{\varphi \in H_{E}}$ as follows. Let us assume that $\mathcal{O}_{E} \subset \operatorname{End}_{K}(\underline{M})$, that $K$ is a finite Galois extension of $Q$ which contains $\psi(E)$ for all $\psi \in H_{E}$, and that $\underline{M}$ has good reduction at all primes of $K$. (By unpublished results of Schindler Sch09 this is no restriction of generality, because for every $A$-motive $\underline{M}^{\prime}$ with complex multiplication by a semi-simple separable CM-algebra $E$ there is an $A$-motive $\underline{M}$ isogenous to $\underline{M}^{\prime}$ such that the integral closure $\mathcal{O}_{E}$ of $A$ in $E$ is contained in $\operatorname{End}_{K}(\underline{M})$ and $\underline{M}^{\prime}$ and $\underline{M}$ have good reduction everywhere after replacing $K$ by a finite separable extension. Moreover, every $A$-motive over a field extension of $Q$ with $\mathcal{O}_{E} \subset \operatorname{End}_{K}(\underline{M})$ is already defined over a finite separable
extension $K$ of $Q$ ). For $\psi \in H_{E}$ we define the functions

$$
\begin{array}{ll}
a_{E, \psi, \Phi}: \mathscr{G}_{Q} \rightarrow \mathbb{Z}, & g \mapsto d_{g \psi} \quad \text { and } \\
a_{E, \psi, \Phi}^{0}: \mathscr{G}_{Q} \rightarrow \mathbb{Q}, & g \mapsto \frac{1}{\# H_{K}} \sum_{\eta \in H_{K}} d_{\eta^{-1} g \eta \psi} \tag{1.11}
\end{array}
$$

which factor through $\operatorname{Gal}(K / Q)$. In particular, $a_{E, \psi, \Phi} \in \mathcal{C}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ and $a_{E, \psi, \Phi}^{0} \in \mathcal{C}^{0}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ is independent of $K$.
 $\mathrm{H}^{\psi}(\underline{M}, K):=\mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right) /\left(y_{\psi}-\psi\left(y_{\psi}\right)\right) \mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$ in $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$ of the character $\psi: E \rightarrow K$ is a $K$-vector space of dimension 1 by Proposition 4.9 below. For an $E$-generator $\left.u \in \mathrm{H}_{1, \operatorname{Betti}(\underline{M}}, Q\right)$ and a generator $\omega_{\psi} \in \mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$ as $K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket$-module we next define integers $v\left(\omega_{\psi}\right)$ and $v_{\psi}(u)$ for all $v \neq \infty$ which are all zero except for finitely many. Let $\mathcal{O}_{E_{v}}:=\mathcal{O}_{E} \otimes_{A} A_{v}$ and let $c \in E_{v}:=E \otimes_{Q} Q_{v}$ be such that $c^{-1} u$ is an $\mathcal{O}_{E_{v}}$-generator of $\mathrm{H}_{1, \operatorname{Betti}}(\underline{M}, A) \otimes_{A} A_{v}$, which exists because $\mathcal{O}_{E_{v}}$ is a product of discrete valuation rings. Then $c$ is unique up to multiplication by an element of $\mathcal{O}_{E_{v}}^{\times}$and we set

$$
\begin{equation*}
v_{\psi}(u):=v(\psi(c)) \in \mathbb{Z} \tag{1.12}
\end{equation*}
$$

## (NOTE THAT $v_{\psi}(u) \in \mathbb{Q}$ IN GENERAL; SEE ERRATUMB.1)

where we extend $\psi \in H_{E}$ by continuity to $\psi: E_{v} \rightarrow Q_{v}^{\text {alg }}$. Also let $\underline{\mathcal{M}}=\left(\mathcal{M}, \tau_{\mathcal{M}}\right)$ be an $A$-motive over $R:=\mathcal{O}_{K_{v}}$ with good reduction and $\underline{\mathcal{M}} \otimes_{\mathcal{O}_{K v}} K_{v} \cong \underline{M} \otimes_{K} K_{v}$; see Example 3.2. Then there is an element $x \in K_{v}^{\times}$, unique up to multiplication by $R^{\times}$, such that $x^{-1} \omega_{\psi} \bmod y_{\psi}-\psi\left(y_{\psi}\right)$ is an $R$-generator of the free $R$-module of rank one

$$
\mathrm{H}^{\psi}(\underline{\mathcal{M}}, R):=\left\{\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{M}}, R):=\sigma^{*} \mathcal{M} \otimes_{A_{R}, \gamma \otimes \operatorname{id}_{R}} R:[b]^{*} \omega=\psi(b) \cdot \omega \forall b \in \mathcal{O}_{E}\right\},
$$

and we set

$$
\begin{equation*}
v\left(\omega_{\psi}\right):=v(x) \in \mathbb{Z} \tag{1.13}
\end{equation*}
$$

(NOTE THAT THIS DEFINITION IS WRONG; SEE ERRATUM B.2)
This value only depends on the image of $\omega_{\psi}$ in $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, K)$. It also does not depend on the choice of the model $\underline{\mathcal{M}}$ with good reduction, because all such models are isomorphic over $R$ by Gar03, Proposition 2.13(ii)]. In this situation our first main result is the following

Theorem 1.3. Let $\omega_{\psi}$ be a generator of the $K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket$-module $\mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$. For every $\eta \in H_{K}$ let $\underline{M}^{\eta}$ and $\omega_{\psi}^{\eta} \in \mathrm{H}^{\eta \psi}\left(\underline{M}^{\eta}, K \llbracket y_{\eta \psi}-\eta \psi\left(y_{\eta \psi}\right) \rrbracket\right)$ be obtained by extension of scalars via $\eta$, and choose an E-generator $u_{\eta} \in \mathrm{H}_{1, \operatorname{Betti}}\left(\underline{M}^{\eta}, Q\right)$. Then for every place $v \neq \infty$ of $C$ we have

$$
\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}} v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E, \psi, \Phi}^{0}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E, \psi, \Phi}^{0}\right)-\frac{v\left(\mathfrak{d}_{\psi(E) / Q}\right)}{[\psi(E): Q]}+\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}}\left(v\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right)\right),
$$

where $\mathfrak{d}_{\psi(E) / Q}$ is the discriminant of the field extension $\psi(E) / Q$.
We will prove this theorem at the end of Section 5 by using the local shtuka at $v$ attached to $\underline{M}$. The latter is an analog of the Dieudonné-module of the $p$-divisible group attached to an abelian variety; see Har09, §3.2]. The theorem allows us to make the following convention which is the analog of Col93, Convention 0].

Convention 1.4. Let $\left(x_{v}\right)_{v \neq \infty}$ be a tuple of complex numbers indexed by the finite places $v$ of $Q$. We will give a sense to the (divergent) series $\Sigma \stackrel{?}{=} \sum_{v \neq \infty} x_{v}$ in the following situation. We suppose that there exists an element $a \in \mathcal{C}^{0}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ such that $x_{v}=-Z_{v}(a, 1) \log q_{v}$ for all $v$ except for finitely many. Then we let $a^{*} \in \mathcal{C}^{0}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ be defined by $a^{*}(g):=a\left(g^{-1}\right)$. We further assume that $Z^{\infty}\left(a^{*}, s\right)$ does not have a pole at $s=0$, and we define the limit of the series $\sum_{v \neq \infty} x_{v}$ as

$$
\begin{equation*}
\Sigma:=-Z^{\infty}\left(a^{*}, 0\right)-\mu_{\mathrm{Art}}^{\infty}(a)+\sum_{v \neq \infty}\left(x_{v}+Z_{v}(a, 1) \log q_{v}\right) \tag{1.14}
\end{equation*}
$$

inspired by Weil's Wei48, p. 82] functional equation

$$
Z(\chi, 1-s)=-Z\left(\chi^{*}, s\right)-(2 \cdot \operatorname{genus}(C)-2) \chi(1) \log q-\mu_{\operatorname{Art}}(\chi)
$$

deprived of the summands at $\infty$, where the genus term is considered as belonging to $\infty$.
The Convention 1.4 and the Theorem 1.3 allow us to give a convergent interpretation to the sum $-\sum_{v} \sum_{\eta \in H_{K}} v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right) \log q_{v}$ and the product $\prod_{v} \prod_{\eta \in H_{K}}\left|\int_{u_{\eta}} \omega_{\psi}^{\eta}\right|_{v}$, and we can ask whether this product is 1 . In Section 2 we prove our second main result, namely that the answer to the question is "yes" in the easiest case of the Carlitz motive which is related to the zeta function of $\mathbb{F}_{q}[t]$ and is the analog of the multiplicative group $\mathbb{G}_{m, \mathbb{Q}}$ considered by Colmez. For general CM $A$-motives we plan to address the question in a sequel to this article and also discuss its consequences for the Faltings height of CM $A$-motives similar to [Col93, Théorème 0.3 and Conjecture 0.4] and conditions under which $Z^{\infty}\left(a^{*}, s\right)$ does not have a pole at $s=0$.

Let us describe the structure of this article. In Section 3 we recall from HK15 the definition of local shtukas, how to attach a local shtuka at $v \subset A$ to an $A$-motive $\underline{M}$ over $L$ with good reduction, and we discuss its cohomology realizations. In Section 4 we define the notions of complex multiplication and CM-type of a local shtuka, and in Section 5 we compute the periods and their valuations of a local shtuka with complex multiplication, and we prove Theorem 1.3. Finally in Appendix A we prove the facts used above.

## 2 The Carlitz-Motive

Let $A=\mathbb{F}_{q}[t]$ and $C=\mathbb{P}_{\mathbb{F}_{q}}^{1}$. Let $K=\mathbb{F}_{q}(\vartheta)$ be the rational function field in the variable $\vartheta$ and let $\gamma: A \rightarrow K$ be given by $\gamma(t)=\vartheta$. We also set $z:=z_{\infty}:=\frac{1}{t}$ and $\zeta:=\zeta_{\infty}:=\frac{1}{\vartheta}$. It satisfies $|\zeta|_{\infty}=q^{-1}<1$. The Carlitz motive over $K$ is the $A$-motive $\underline{\mathcal{C}}=\left(K[t], \tau_{\mathcal{C}}=t-\vartheta\right)$ which is associated with the Carlitz module; see Car35] or Gos96, Chapter 3]. It has rank 1 and dimension 1, and complex multiplication by the ring of integers $A$ in $E:=Q$ with CM-type $\Phi=\left(d_{\mathrm{id}}\right)$, where $H_{E}=\{\mathrm{id}\}$ and $d_{\mathrm{id}}=1$. As is well known, its cohomology satisfies $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{C}}, K \llbracket z-\zeta \rrbracket)=K \llbracket z-\zeta \rrbracket$ and $\mathrm{H}_{\text {Betti }}^{1}(\mathcal{C}, A)=A \cdot \beta \ell_{\bar{\zeta}}$, where $\beta \in \mathbb{C}_{\infty}$ satisfies $\beta^{q-1}=-\zeta$ and $\ell_{\zeta}^{-}:=\prod_{i=0}^{\infty}\left(1-\zeta^{q^{i}} t\right)$; see for example HJ16, Example 3.34]. We denote the generator 1 of $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{C}}, K \llbracket z-\zeta \rrbracket)$ by $\omega$ and we take $u \in \mathrm{H}_{1, \operatorname{Betti}}\left(\underline{\mathcal{C}}, \mathbb{F}_{q}[t]\right)$ as the generator which is dual to $\beta \ell_{\zeta}^{-} \in \mathrm{H}_{\mathrm{Betti}}^{1}\left(\underline{\mathcal{C}}, \mathbb{F}_{q}[t]\right)$. The de Rham isomorphism $h_{\mathrm{Betti}, \mathrm{dR}}$ sends $\beta \ell_{\zeta}^{-}$to

$$
\sigma^{*}\left(\beta \ell_{\zeta}^{-}\right) \cdot \omega=\beta^{q} \sigma^{*}\left(\ell_{\zeta}^{-}\right) \cdot \omega \in \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{M}, \mathbb{C}_{\infty} \llbracket z-\zeta \rrbracket\right)=\mathbb{C}_{\infty} \llbracket z-\zeta \rrbracket \cdot \omega,
$$

respectively to $\left.\beta^{q} \sigma^{*}\left(\ell_{\zeta}^{-}\right)\right|_{t=\vartheta} \cdot \omega=\beta^{q} \prod_{i=1}^{\infty}\left(1-\zeta^{q^{i}-1}\right) \cdot \omega \in \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{M}, \mathbb{C}_{\infty}\right)=\mathbb{C}_{\infty} \cdot \omega$. Here the coefficient $\beta^{q} \prod_{i=1}^{\infty}\left(1-\zeta^{q^{i}-1}\right)$ is the function field analog of the complex number $(2 i \pi)^{-1}$, the inverse of the period of the multiplicative group $\mathbb{G}_{m, \mathbb{Q}}$. We obtain

$$
\left|\int_{u} \omega\right|_{\infty}=\left|\left(\beta^{q} \prod_{i=1}^{\infty}\left(1-\zeta^{q^{i}-1}\right)\right)^{-1}\right|_{\infty}=|\beta|_{\infty}^{-q}=q^{q /(q-1)} .
$$

At a finite place $v \subset \mathbb{F}_{q}[t]$ let $v=\left(z_{v}\right)$ and $\zeta_{v}=\gamma\left(z_{v}\right)$. Then $\mathrm{H}_{v}^{1}\left(\underline{M}, A_{v}\right)=A_{v} \cdot\left(\ell_{\zeta_{v}}^{+}\right)^{-1}$, where $\ell_{\zeta_{v}}^{+}:=\sum_{n=0}^{\infty} \ell_{n} z_{v}^{n} \in \mathbb{C}_{v} \llbracket z_{v} \rrbracket$ with $\ell_{0}^{q_{v}-1}=-\zeta_{v}$ and $\ell_{n}^{q_{v}}+\zeta_{v} \ell_{n}=\ell_{n-1}$; see HK15, Example 4.19]. This implies $\left|\ell_{n}\right|=\left|\zeta_{v}\right|^{q_{v}^{-n} /\left(q_{v}-1\right)}<1$. The $v$-adic comparison isomorphism $h_{v, \mathrm{dR}}$ sends the generator $\ell_{\zeta_{v}}^{+}$of $\mathrm{H}_{v}^{1}\left(\underline{M}, A_{v}\right)$ to

$$
\left(z_{v}-\zeta_{v}\right)^{-1}\left(\ell_{\zeta_{v}}^{+}\right)^{-1} \cdot \omega \in \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{M}, \mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right)\right)=\mathbb{C}_{v}\left(\left(z_{v}-\zeta_{v}\right)\right) \cdot \omega,
$$

where the coefficient of $\omega$ is the $v$-adic Carlitz period which has a pole of order one at $z_{v}=\zeta_{v}$. So $\langle\omega, u\rangle_{v}=\left(z_{v}-\zeta_{v}\right) \ell_{\zeta_{v}}^{+}$has $\hat{v}\left(\langle\omega, u\rangle_{v}\right)=1$ and

$$
\left|\int_{u} \omega\right|_{v}=\left|\ell_{\zeta_{v}}^{+} \bmod z_{v}-\zeta_{v}\right|_{v}=\left|\sum_{n=0}^{\infty} \ell_{n} \zeta_{v}^{n}\right|_{v}=\left|\ell_{0}\right|_{v}=q_{v}^{-1 /\left(q_{v}-1\right)}
$$

So the product $\prod_{v}\left|\int_{u} \omega\right|_{v}$ of the norms at all places has logarithm

$$
\begin{equation*}
\log \prod_{\text {all } v}\left|\int_{u} \omega\right|_{v}=\log \left|\int_{u} \omega\right|_{\infty}+\log \prod_{v \neq \infty}\left|\int_{u} \omega\right|_{v}=\frac{q}{q-1} \log q+\sum_{v \neq \infty} \frac{-1}{q_{v}-1} \log q_{v} \tag{2.1}
\end{equation*}
$$

Note that this series is not convergent, but that the sum over $v \neq \infty$ is equal to $\frac{\zeta_{A}^{\prime}(1)}{\zeta_{A}(1)}$ and the summand at $\infty$ is equal to $\frac{\zeta_{A}^{\prime}(0)}{\zeta_{A}(0)}$, where $\zeta_{A}$ is the zeta function associated with $A$, which does not converge at $s=1$. Namely, the zeta functions are defined as the following products which converge for $s \in \mathbb{C}$ with $\mathcal{R} e(s)>1$

$$
\begin{aligned}
\zeta_{C}(s) & :=\prod_{\text {all } v}\left(1-\left(\# \mathbb{F}_{v}\right)^{-s}\right)^{-1}=\prod_{\text {all } v}\left(1-q_{v}^{-s}\right)^{-1}=\frac{1}{\left(1-q^{-s}\right)\left(1-q^{1-s}\right)} \quad \text { and } \\
\zeta_{A}(s) & :=\prod_{v \neq \infty}\left(1-\left(\# \mathbb{F}_{v}\right)^{-s}\right)^{-1}=\prod_{v \neq \infty}\left(1-q_{v}^{-s}\right)^{-1}=\frac{1}{1-q^{1-s}}
\end{aligned}
$$

see for example Sil86, Chapter V, Example 2.1]. In particular $\left(1-q_{v}^{-s}\right)^{-1}=L_{v}(\mathbb{1}, 1)$ and $\zeta_{A}(s)=$ $L^{\infty}(\mathbb{1}, s)$ where $\mathbb{1}: g \mapsto 1$ is the trivial character in $\mathcal{C}^{0}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$, which equals $a_{Q, \mathrm{id}, \Phi}=a_{Q, \mathrm{id}, \Phi}^{0}$. Then $\frac{1}{q_{v}^{s}-1}=Z_{v}(\mathbb{1}, s)$ and $\frac{\zeta_{A}^{\prime}(s)}{\zeta_{A}(s)}=Z^{\infty}(\mathbb{1}, s)$ in the notation of Definition 1.2, Applying Convention 1.4 with $a=\mathbb{1}$ and $\mu_{\text {Art }}(\mathbb{1})=0$ and $\operatorname{genus}\left(\mathbb{P}_{\mathbb{F}_{q}}^{1}\right)=0$ we define the limit of the series as

$$
\begin{aligned}
\frac{q}{q-1} \log q+\sum_{v \neq \infty} \frac{-1}{q_{v}-1} \log q_{v} & :=\frac{q}{q-1} \log q-\sum_{v \neq \infty} Z_{v}(\mathbb{1}, 1) \log q_{v} \\
& =\frac{q}{q-1} \log q-Z^{\infty}(\mathbb{1}, 0) \\
& =\frac{q}{q-1} \log q-\frac{\zeta_{A}^{\prime}(0)}{\zeta_{A}(0)} \\
& =0 .
\end{aligned}
$$

So the value of the product $\prod_{v}\left|\int_{u} \omega\right|_{v}$ is 1 for the Carlitz motive.

## 3 Local Shtukas

In the rest of the article we fix a place $v \neq \infty$ of $Q$. We keep the notation from the introduction, except that we write $L=\kappa\left(\left(\pi_{L}\right)\right)$ for the field $K_{v}$ and let $R=\kappa \llbracket \pi_{L} \rrbracket$ be its valuation ring. We write $z:=z_{v}$. Then $A_{v}=\mathbb{F}_{v} \llbracket z \rrbracket$ and $Q_{v}=\mathbb{F}_{v}((z))$. The homomorphism $\gamma: A \rightarrow K$ extends by continuity
to $\gamma: A_{v} \rightarrow L$ and factors through $\gamma: A_{v} \rightarrow R$ with $\zeta:=\zeta_{v}=\gamma(z) \in \pi_{L} R \backslash\{0\}$. Let $R \llbracket z \rrbracket$ be the power series ring in the variable $z$ over $R$ and $\hat{\sigma}$ the endomorphism of $R \llbracket z \rrbracket$ with $\hat{\sigma}(z)=z$ and $\hat{\sigma}(b)=b^{q_{v}}$ for $b \in R$, where $q_{v}=\# \mathbb{F}_{v}$. For an $R \llbracket z \rrbracket$-module $\hat{M}$ we let $\hat{\sigma}^{*} \hat{M}:=\hat{M} \otimes_{R \llbracket z \rrbracket, \hat{\sigma}^{*}} R \llbracket z \rrbracket$ as well as $\hat{M}\left[\frac{1}{z-\zeta}\right]:=\hat{M} \otimes_{R \llbracket z \rrbracket} R \llbracket z \rrbracket\left[\frac{1}{z-\zeta}\right]$ and $\hat{M}\left[\frac{1}{z}\right]:=\hat{M} \otimes_{R \llbracket z \rrbracket} R \llbracket z \rrbracket\left[\frac{1}{z}\right]$.
Definition 3.1. A local $\hat{\sigma}$-shtuka of rank $r$ over $R$ is a pair $\underline{\hat{M}}=\left(\hat{M}, \tau_{\hat{M}}\right)$ consisting of a free $R \llbracket z \rrbracket$ module $\hat{M}$ of rank $r$, and an isomorphism $\tau_{\hat{M}}: \hat{\sigma}^{*} \hat{M}\left[\frac{1}{z-\zeta}\right] \xrightarrow{\sim} \hat{M}\left[\frac{1}{z-\zeta}\right]$. It is effective if $\tau_{\hat{M}}\left(\hat{\sigma}^{*} \hat{M}\right) \subset \hat{M}$ and étale if $\tau_{\hat{M}}\left(\hat{\sigma}^{*} \hat{M}\right)=\hat{M}$. We write $\operatorname{rk} \underline{\hat{M}}$ for the rank of $\underline{\hat{M}}$.

A morphism of local shtukas $f: \underline{\hat{M}}=\left(\hat{M}, \tau_{\hat{M}}\right) \rightarrow \underline{\hat{N}}=\left(\hat{N}, \tau_{\hat{N}}\right)$ over $R$ is a morphism of the underlying modules $f: \hat{M} \rightarrow \hat{N}$ which satisfies $\tau_{\hat{N}} \circ \hat{\sigma}^{*} f=f \circ \tau_{\hat{M}}$. We denote the $A_{v}$-module of homomorphisms $f: \underline{\hat{M}} \rightarrow \underline{\hat{N}}$ by $\operatorname{Hom}_{R}(\underline{\hat{M}}, \underline{\hat{N}})$ and $\operatorname{write}_{\operatorname{End}}^{R}(\underline{\hat{M}})=\operatorname{Hom}_{R}(\underline{\hat{M}}, \underline{\hat{M}})$.

A quasi-morphism between local shtukas $f:\left(\hat{M}, \tau_{\hat{M}}\right) \rightarrow\left(\hat{N}, \tau_{\hat{N}}\right)$ over $R$ is a morphism of $R \llbracket z \rrbracket\left[\frac{1}{z}\right]-$ modules $f: M\left[\frac{1}{z}\right] \xrightarrow{\sim} N\left[\frac{1}{z}\right]$ with $\tau_{\hat{N}} \circ \hat{\sigma}^{*} f=f \circ \tau_{\hat{M}}$. It is called a quasi-isogeny if it is an isomorphism of $R \llbracket z \rrbracket\left[\frac{1}{z}\right]$-modules. We denote the $Q_{v}$-vector space of quasi-morphisms from $\underline{\hat{M}}$ to $\underline{\hat{N}}$ by $\mathrm{QHom}_{R}(\underline{\hat{M}}, \underline{\hat{N}})$ and write $\operatorname{QEnd}_{R}(\underline{\hat{M}})=\operatorname{QHom}_{R}(\underline{\hat{M}}, \underline{\hat{M}})$.

Note that $\operatorname{Hom}_{R}(\underline{\hat{M}}, \underline{\hat{N}})$ is a finite free $A_{v}$-module of rank at most $\mathrm{rk} \underline{\hat{M}} \cdot \mathrm{rk} \hat{\underline{N}}$ by HK15, Corollary 4.5] and $\operatorname{QHom}_{R}(\underline{\hat{M}}, \underline{\hat{N}})=\operatorname{Hom}_{R}(\underline{\hat{M}}, \underline{\hat{N}}) \otimes_{A_{v}} Q_{v}$. Also every quasi-isogeny $f: \underline{\hat{M}} \rightarrow \underline{\hat{N}}$ induces an isomorphism of $Q_{v}$-algebras $\operatorname{QEnd}_{R}(\underline{\hat{M}}) \xrightarrow{\sim} \operatorname{QEnd}_{R}(\underline{\hat{N}}), g \mapsto f g f^{-1}$.

Example 3.2. We assume that the $A$-motive $\underline{M}=\left(M, \tau_{M}\right)$ has good reduction, that is, there exist a pair $\underline{\mathcal{M}}=\left(\mathcal{M}, \tau_{\mathcal{M}}\right)$ consisting of a locally free module $\mathcal{M}$ over $A_{R}:=A \otimes_{\mathbb{F}_{q}} R$ of finite rank and an isomorphism $\tau_{\mathcal{M}}:\left.\left.\sigma^{*} \mathcal{M}\right|_{\text {Spec } A_{R} \backslash V(\mathcal{J})} \xrightarrow{\longrightarrow} \mathcal{M}\right|_{\operatorname{Spec} A_{R} \backslash V(\mathcal{J})}$ of the associated sheaves outside the vanishing locus $\mathrm{V}(\mathcal{J}) \subset \operatorname{Spec} A_{R}$ of the ideal $\mathcal{J}:=(a \otimes 1-1 \otimes \gamma(a): a \in A) \subset A_{R}$, such that $\underline{\mathcal{M}} \otimes_{R} L \cong \underline{M}$. The reduction $\underline{\mathcal{M}} \otimes_{R} \kappa$ is an $A$-motive over $\kappa$ of $A$-characteristic $v=\operatorname{ker}(\gamma: A \rightarrow \kappa)$. The pair $\underline{\mathcal{M}}$ is called an $A$-motive over $R$ and a good model of $\underline{M}$.

We consider the $v$-adic completions $A_{v, R}$ of $A_{R}$ and $\underline{\mathcal{M}} \otimes_{A_{R}} A_{v, R}:=\left(\mathcal{M} \otimes_{A_{R}} A_{v, R}, \tau_{\mathcal{M}} \otimes \mathrm{id}\right)$ of $\underline{\mathcal{M}}$. We let $d_{v}:=\left[\mathbb{F}_{v}: \mathbb{F}_{q}\right]$ and discuss the two cases $d_{v}=1$ and $d_{v}>1$ separately. If $d_{v}=1$, and hence $q_{v}=q$ and $\hat{\sigma}=\sigma$, we have $A_{v, R}=R \llbracket z \rrbracket$, and $\underline{\mathcal{M}} \otimes_{A_{R}} A_{v, R}$ is a local $\hat{\sigma}$-shtuka over Spec $R$ which we denote by $\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ and call the local shtuka at $v$ associated with $\underline{\mathcal{M}}$.

If $d_{v}>1$, the situation is more complicated, because $\mathbb{F}_{v} \otimes_{\mathbb{F}_{q}} R$ and $A_{v, R}$ fail to be integral domains. Namely,

$$
\mathbb{F}_{v} \otimes_{\mathbb{F}_{q}} R=\prod_{\operatorname{Gal}\left(\mathbb{F}_{v} / \mathbb{F}_{q}\right)} \mathbb{F}_{v} \otimes_{\mathbb{F}_{v}} R=\prod_{i \in \mathbb{Z} / d_{v} \mathbb{Z}} \mathbb{F}_{v} \otimes_{\mathbb{F}_{q}} R /\left(a \otimes 1-1 \otimes \gamma(a)^{q^{i}}: a \in \mathbb{F}_{v}\right)
$$

and $\sigma$ transports the $i$-th factor to the $(i+1)$-th factor. In particular $\hat{\sigma}$ stabilizes each factor. Denote by $\mathfrak{a}_{i}$ the ideal of $A_{v, R}$ generated by $\left\{a \otimes 1-1 \otimes \gamma(a)^{q^{i}}: a \in \mathbb{F}_{v}\right\}$. Then

$$
A_{v, R}=\prod_{\operatorname{Gal}\left(\mathbb{F}_{v} / \mathbb{F}_{q}\right)} A_{v} \widehat{\otimes}_{\mathbb{F}_{v}} R=\prod_{i \in \mathbb{Z} / d_{v} \mathbb{Z}} A_{v, R} / \mathfrak{a}_{i} .
$$

Note that each factor is isomorphic to $R \llbracket z \rrbracket$ and the ideals $\mathfrak{a}_{i}$ correspond precisely to the places $v_{i}$ of $C_{\mathbb{F}_{v}}$ lying above $v$. The ideal $\mathcal{J}$ decomposes as follows $\mathcal{J} \cdot A_{v, R} / \mathfrak{a}_{0}=(z-\zeta)$ and $\mathcal{J} \cdot A_{v, R} / \mathfrak{a}_{i}=(1)$ for $i \neq 0$. We define the local shtuka at $v$ associated with $\underline{\mathcal{M}}$ as $\underline{\hat{M}}_{v}(\underline{\mathcal{M}}):=\left(\hat{M}, \tau_{\hat{M}}\right):=\left(\mathcal{M} \otimes_{A_{R}} A_{v, R} / \mathfrak{a}_{0},\left(\tau_{\mathcal{M}} \otimes 1\right)^{d_{v}}\right)$, where $\tau_{\mathcal{M}}^{d_{v}}:=\tau_{\mathcal{M}} \circ \sigma^{*} \tau_{\mathcal{M}} \circ \ldots \circ \sigma^{\left(d_{v}-1\right) *} \tau_{\mathcal{M}}$. Of course if $d_{v}=1$ we get back the definition of $\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ given above. Also note if $\underline{\mathcal{M}}$ is effective, then $\mathcal{M} / \tau_{\mathcal{M}}\left(\sigma^{*} \mathcal{M}\right)=\hat{M} / \tau_{\hat{M}}\left(\hat{\sigma}^{*} \hat{M}\right)$.

The local shtuka $\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ allows to recover $\underline{\mathcal{M}} \otimes_{A_{R}} A_{v, R}$ via the isomorphism

$$
\bigoplus_{i=0}^{d_{v}-1}\left(\tau_{\mathcal{M}} \otimes 1\right)^{i} \bmod \mathfrak{a}_{i}:\left(\bigoplus_{i=0}^{d_{v}-1} \sigma^{i *}\left(\mathcal{M} \otimes_{A_{R}} A_{v, R} / \mathfrak{a}_{0}\right),\left(\tau_{\mathcal{M}} \otimes 1\right)^{d_{v}} \oplus \bigoplus_{i \neq 0} \mathrm{id}\right) \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_{R}} A_{v, R}
$$

because for $i \neq 0$ the equality $\mathcal{J} \cdot A_{v, R} / \mathfrak{a}_{i}=(1)$ implies that $\tau_{\mathcal{M}} \otimes 1$ is an isomorphism modulo $\mathfrak{a}_{i}$; see BH11, Propositions 8.8 and 8.5] for more details.

Next we define the $v$-adic realization and the de Rham realization of a local shtuka $\underline{\hat{M}}=\left(\hat{M}, \tau_{\hat{M}}\right)$ over $R$. Since $\tau_{\hat{M}}$ induces an isomorphism $\tau_{\hat{M}}: \hat{\sigma}^{*} \hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z \rrbracket \xrightarrow{\sim} \hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z \rrbracket$, because $z-\zeta \in L \llbracket z \rrbracket^{\times}$, we can think of $\underline{\hat{M}} \otimes_{R \llbracket z \rrbracket} L \llbracket z \rrbracket$ as an étale local shtuka over $L$.
Definition 3.3. The $v$-adic realization $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right)$ of a local $\hat{\sigma}$-shtuka $\underline{\hat{M}}=\left(\hat{M}, \tau_{\hat{M}}\right)$ is the $\mathscr{G}_{L}:=$ $\operatorname{Gal}\left(L^{\text {sep }} / L\right)$-module of $\tau$ - invariants

$$
\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right):=\left(\hat{M} \otimes_{R \llbracket z \rrbracket} L^{\mathrm{sep}} \llbracket z \rrbracket\right)^{\tau}:=\left\{m \in \hat{M} \otimes_{R \llbracket z \rrbracket} L^{\mathrm{sep}} \llbracket z \rrbracket: \tau_{\hat{M}}\left(\hat{\sigma}_{\hat{M}}^{*} m\right)=m\right\},
$$

where we set $\hat{\sigma}_{\hat{M}}^{*} m:=m \otimes 1 \in \hat{M} \otimes_{R \llbracket z \rrbracket, \hat{\sigma}} R \llbracket z \rrbracket=: \sigma^{*} M$ for $m \in M$. One also writes sometimes $\check{T}_{v} \underline{\hat{M}}=\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right)$ and calls this the dual Tate module of $\underline{\hat{M}}$. By HK15, Proposition 4.2] it is a free $A_{v}$-module of the same rank as $\hat{M}$. We also write $\mathrm{H}_{v}^{1}(\underline{\hat{M}}, B):=\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \otimes_{A_{v}} B$ for an $A_{v}$-algebra $B$.

If $\underline{M}=\left(M, \tau_{M}\right)$ is an $A$-motive over $L$ with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}}=\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ is the local shtuka at $v$ associated with $\underline{\mathcal{M}}$, then $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right)$ is by [HK15, Proposition 4.6] canonically isomorphic as a representation of $\mathscr{G}_{L}$ to the $v$-adic realization of $\underline{M}$, which is defined as

$$
\mathrm{H}_{v}^{1}\left(\underline{M}, A_{v}\right):=\left\{m \in M \otimes_{A_{L}} A_{v, L^{\text {sep }}}: \tau_{M}\left(\sigma_{M}^{*} m\right)=m\right\},
$$

where we set $\sigma_{M}^{*} m:=m \otimes 1 \in M \otimes_{A_{R}, \sigma} A_{R}=: \sigma^{*} M$ for $m \in M$ and where $A_{v, L^{\text {sep }}}$ is the $v$-adic completion of $A_{L^{\text {sep }}}$.

Definition 3.4. Let $\underline{\hat{M}}=\left(\hat{M}, \tau_{\hat{M}}\right)$ be a local $\hat{\sigma}$-shtuka over $R$. We define the de Rham realizations of $\underline{\hat{M}}$ as

$$
\begin{aligned}
\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R) & :=\hat{\sigma}^{*} \hat{M} /(z-\zeta) \hat{\sigma}^{*} \hat{M}=\hat{\sigma}^{*} \hat{M} \otimes_{R \llbracket z \rrbracket, z \mapsto \zeta} R, \quad \text { as well as } \\
\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket) & :=\hat{\sigma}^{*} \hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z-\zeta \rrbracket \text { and } \\
\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L) & :=\hat{\sigma}^{*} \hat{M} \otimes_{R \llbracket z \rrbracket, z \mapsto \zeta} L=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket) \otimes_{L \llbracket z-\zeta \rrbracket} L \llbracket z-\zeta \rrbracket /(z-\zeta) \\
& =\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R) \otimes_{R} L .
\end{aligned}
$$

It carries the Hodge-Pink lattice $\mathfrak{q}^{\hat{\hat{M}}}:=\tau_{\hat{M}}^{-1}\left(\hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z-\zeta \rrbracket\right) \subset H_{d R}^{1}(\underline{\underline{M}}, L \llbracket z-\zeta \rrbracket)\left[\frac{1}{z-\zeta}\right]$. We also write $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, B):=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket) \otimes_{L \llbracket z-\zeta \rrbracket} B$ for an $L \llbracket z-\zeta \rrbracket$-algebra $B$.

If $\underline{M}=\left(M, \tau_{M}\right)$ is an $A$-motive over $L$ with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}}=\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ is the local shtuka at $v$ associated with $\underline{\mathcal{M}}$ and $d_{v}=\left[\mathbb{F}_{v}: \mathbb{F}_{q}\right]$ is as in Example 3.2, the map

$$
\sigma^{*} \tau_{M}^{d_{v}-1}=\sigma^{*} \tau_{M} \circ \sigma^{2 *} \tau_{M} \circ \cdots \circ \sigma^{\left(d_{v}-1\right) *} \tau_{M}: \sigma^{d_{v} *} M \otimes_{A_{R}} A_{v, R} / \mathfrak{a}_{0} \xrightarrow{\sim} \sigma^{*} M \otimes_{A_{R}} A_{v, R} / \mathfrak{a}_{0}
$$

is an isomorphism, because $\tau_{M}$ is an isomorphism over $A_{v, R} / \mathfrak{a}_{i}$ for all $i \neq 0$. Therefore it defines canonical isomorphisms of the de Rham realizations

$$
\begin{aligned}
& \sigma^{*} \tau_{M}^{d_{v}-1}: \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket) \xrightarrow{\sim} \mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, L \llbracket z-\zeta \rrbracket) \quad \text { and } \\
& \sigma^{*} \tau_{M}^{d_{v}-1}: \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L) \xrightarrow{\longrightarrow} \mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, L),
\end{aligned}
$$

which are compatible with the Hodge-Pink lattices.
Remark 3.5. By HK15, Theorem 4.15] there is a canonical comparison isomorphism

$$
\begin{equation*}
h_{v, \mathrm{dR}}: \mathrm{H}_{v}^{1}\left(\underline{\underline{M}}, \mathbb{C}_{v}((z-\zeta))\right) \xrightarrow{\sim} \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\underline{M}}, \mathbb{C}_{v}((z-\zeta))\right) \tag{3.1}
\end{equation*}
$$

which is equivariant for the action of $\mathscr{G}_{L}$. For our computations below we need an explicit description of $h_{v, \mathrm{dR}}$. It is constructed as follows. The natural inclusion $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \subset \hat{M} \otimes_{R \llbracket z \rrbracket} L^{\text {sep }} \llbracket z \rrbracket$ defines a canonical isomorphism of $L^{\text {sep }} \llbracket z \rrbracket$-modules

$$
\begin{equation*}
\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \otimes_{A_{v}} L^{\mathrm{sep}} \llbracket z \rrbracket \xrightarrow{\sim} \hat{M} \otimes_{R \llbracket z \rrbracket} L^{\mathrm{sep}} \llbracket z \rrbracket, \tag{3.2}
\end{equation*}
$$

which is $\mathscr{G}_{L}$ and $\tau$-equivariant, where on the left module $\mathscr{G}_{L}$ acts on both factors and $\tau$ is id $\otimes \hat{\sigma}$ and on the right module $\mathscr{G}_{L}$ acts only on $L^{\text {sep }} \llbracket z \rrbracket$ and $\tau$ is $\left(\tau_{\hat{M}} \circ \hat{\sigma}_{\hat{M}}^{*}\right) \otimes \hat{\sigma}$. Since $\left(L^{\text {sep }}\right)^{\mathscr{G}_{L}}=L$ we obtain

$$
\hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z \rrbracket=\left(\mathrm{H}_{v}^{1}\left(\underline{\underline{M}}, A_{v}\right) \otimes_{A_{v}} L^{\mathrm{sep}} \llbracket z \rrbracket\right)^{\mathscr{G}_{L}} .
$$

It turns out, see HK15, Remark 4.3], that the isomorphism (3.2) extends to an equivariant isomorphism

$$
\begin{equation*}
h: \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \otimes_{A_{v}} L^{\operatorname{sep}}\left\langle\frac{z}{\zeta}\right\rangle \xrightarrow{\sim} \hat{M} \otimes_{R \llbracket z \rrbracket} L^{\operatorname{sep}}\left\langle\frac{z}{\zeta}\right\rangle, \tag{3.3}
\end{equation*}
$$

where for an $r \in \mathbb{R}_{>0}$ we use the notation

$$
L^{\text {sep }}\left\langle\frac{z}{\zeta^{r}}\right\rangle:=\left\{\sum_{i=0}^{\infty} b_{i} z^{i}: b_{i} \in L^{\text {sep }},\left|b_{i}\right||\zeta|^{\mid i} \rightarrow 0(i \rightarrow+\infty)\right\} .
$$

These are subrings of $L^{\text {sep }} \llbracket z \rrbracket$ and the endomorphism $\hat{\sigma}: \sum_{i} b_{i} z^{i} \mapsto \sum_{i} b_{i}^{q_{v}} z^{i}$ of $L^{\text {sep }} \llbracket z \rrbracket$ restricts to a homomorphism $\hat{\sigma}: L^{\operatorname{sep}}\left\langle\frac{z}{\zeta^{r}}\right\rangle \rightarrow L^{\operatorname{sep}}\left\langle\frac{z}{\zeta^{r q v}}\right\rangle$. Note that the $\tau$-equivariance of $h$ means $h \otimes \operatorname{id}_{L^{\operatorname{sep} p}}\left\langle\frac{z}{\zeta^{q v}}\right\rangle=$ $\tau_{\hat{M}} \circ \hat{\sigma}^{*} h$. Now the period isomorphism is defined as

$$
\begin{align*}
h_{v, \mathrm{dR}}:=\left(\tau_{\hat{M}}^{-1} \circ h\right) \otimes \operatorname{id}_{\mathbb{C}_{v}((z-\zeta))}: \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, \mathbb{C}_{v}((z-\zeta))\right) & \xrightarrow{\longrightarrow} \hat{\sigma}^{*} \hat{M} \otimes_{R \llbracket z \rrbracket} \mathbb{C}_{v}((z-\zeta))  \tag{3.4}\\
& =\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}, \mathbb{C}_{v}((z-\zeta))\right) .
\end{align*}
$$

## 4 Local Shtukas with Complex Multiplication

Definition 4.1. Let $\underline{\hat{M}}$ be a local $\hat{\sigma}$-shtuka over $R$ and assume that there is a commutative, semi-simple $Q_{v}$-algebra $E_{v} \subset \operatorname{QEnd}_{R}(\underline{\hat{M}}):=\operatorname{End}_{R}(\underline{\hat{M}}) \otimes_{A_{v}} Q_{v}$ with $\operatorname{dim}_{Q_{v}} E_{v}=\operatorname{rk} \underline{\hat{M}}$. Then we say that $\underline{\hat{M}}$ has complex multiplication (by $E_{v}$ ).

Here again semi-simple means that $E_{v}$ is a direct product $E_{v}=E_{v, 1} \times \cdots \times E_{v, s}$ of finite field extensions of $Q_{v}$. We do not assume that $E_{v}$ is itself a field and in Section 4 we do not assume that the $E_{v, i}$ are separable over $Q_{v}$. We let $\mathcal{O}_{E_{v}}$ be the integral closure of $A_{v}$ in $E_{v}$. It is a product $\mathcal{O}_{E_{v}}=\mathcal{O}_{E_{v, 1}} \times \cdots \times \mathcal{O}_{E_{v, s}}$ of complete discrete valuation rings where $\mathcal{O}_{E_{v, i}}$ is the integral closure of $A_{v}$ in the field $E_{v, i}$. For every $i$ we write $\left.\mathcal{O}_{E_{v, i}}=\mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i}\right]$ and set $f_{i}:=\left[\mathbb{F}_{\tilde{v}_{i}}: \mathbb{F}_{v}\right]$ and $e_{i}:=\operatorname{ord}_{y_{i}}(z)$. Then $f_{i} e_{i}=\left[E_{v, i}: Q_{v}\right]$ and $e_{i}$ is divisible by the inseparability degree of $E_{v, i}$ over $Q_{v}$. Also we write $\tilde{q}_{i}:=\# \mathbb{F}_{\tilde{v}_{i}}=q_{v}^{f_{i}}$.
Proposition 4.2. If $\underline{\hat{M}}$ has complex multiplication by $E_{v}$, then there is a local shtuka $\underline{\hat{M}}^{\prime}$ over $R$ quasi-isogenous to $\underline{\hat{M}}$ with $\mathcal{O}_{E_{v}} \subset \operatorname{End}_{R}\left(\underline{\hat{M}}^{\prime}\right)$.

Proof. The $A_{v}$-submodule $T^{\prime}:=\mathcal{O}_{E_{v}} \cdot \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \subset \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, Q_{v}\right)$ is $\mathscr{G}_{L}$-invariant and contains $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right)$. Since $\mathcal{O}_{E_{v}} \subset \operatorname{QEnd}_{R}(\underline{\hat{M}})=\operatorname{End}_{R}(\underline{\hat{M}}) \otimes_{A_{v}} Q_{v}$ there is an element $a \in A_{v}$ with $a \cdot \mathcal{O}_{E_{v}} \subset \operatorname{End}_{R}(\underline{\hat{M}})$, and therefore $a \cdot T^{\prime} \subset \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right)$ is a finitely generated $A_{v}$-module, that is an $A_{v}$-lattice. By HK15, Proposition 4.22] there is a local shtuka $\underline{\hat{M}}^{\prime}$ and a quasi-isogeny $f: \underline{\hat{M}} \rightarrow \underline{\hat{M}}^{\prime}$ which maps $T^{\prime}$ isomorphically onto $\mathrm{H}_{v}^{1}\left(\hat{\underline{M}}^{\prime}, A_{v}\right)$. In particular $\mathcal{O}_{E_{v}}$ acts as $\mathscr{G}_{L}$-equivariant endomorphisms of $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}^{\prime}, A_{v}\right)$. Since the functor $\underline{\hat{M}}^{\prime} \mapsto \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}^{\prime}, A_{v}\right)$ from local shtukas to $A_{v}\left[\mathscr{G}_{L}\right]$-modules is fully faithful by HK15, Theorem 4.20], we see that $\mathcal{O}_{E_{v}} \subset \operatorname{End}_{R}\left(\underline{\hat{M}}^{\prime}\right)$.

Definition 4.3. If $\mathcal{O}_{E_{v}} \subset \operatorname{End}_{R}(\underline{\hat{M}})$ we say that $\underline{\hat{M}}$ has complex multiplication by $\mathcal{O}_{E_{v}}$. This makes the underlying module $\hat{M}$ into a module over the ring $\mathcal{O}_{E_{v}, R}:=\mathcal{O}_{E_{v}} \otimes_{A_{v}} R \llbracket z \rrbracket=\mathcal{O}_{E_{v}} \widehat{\otimes}_{\mathbb{F}_{v}} R$. For $a \in \mathcal{O}_{E_{v}}$ note that $a \otimes 1 \in \mathcal{O}_{E_{v}, R}$ acts on $\hat{M}$ as the endomorphism $a$ and on $\hat{\sigma}^{*} \hat{M}$ as the endomorphism $\hat{\sigma}^{*} a$ and $\tau_{\hat{M}}$ is $\mathcal{O}_{E_{v}}$-linear because $a \circ \tau_{\hat{M}}=\tau_{\hat{M}} \circ \hat{\sigma}^{*} a$.
4.4. Let us assume that $L$ contains $\psi\left(E_{v}\right)$ for every $\psi \in H_{E_{v}}$. This implies $\psi\left(\mathcal{O}_{E_{v}}\right) \subset R$ for every $\psi \in H_{E_{v}}$. Under this assumption let us describe the ring $\mathcal{O}_{E_{v}, R}=\prod_{i=1}^{s} \mathcal{O}_{E_{v, i}, R}$. Fix an $i$ and choose and fix an $\mathbb{F}_{v}$-homomorphism $\mathbb{F}_{\tilde{v}_{i}} \hookrightarrow \kappa$. Then $H_{\tilde{v}_{i}}:=\operatorname{Hom}_{\mathbb{F}_{v}}\left(\mathbb{F}_{\tilde{v}_{i}}, \kappa\right) \cong \mathbb{Z} / f_{i} \mathbb{Z}$ under the map that sends $j \in \mathbb{Z} / f_{i} \mathbb{Z}$ to the homomorphism $\left(\lambda \mapsto \lambda^{q_{v}^{j}}\right) \in H_{\tilde{v}_{i}}$. Also

$$
\mathbb{F}_{\tilde{v}_{i}} \otimes_{\mathbb{F}_{v}} R=\prod_{H_{\tilde{v}_{i}}} R=\prod_{j \in \mathbb{Z} / f_{i} \mathbb{Z}} \mathbb{F}_{\tilde{v}_{i}} \otimes_{\mathbb{F}_{v}} R /\left(\lambda \otimes 1-1 \otimes \lambda^{q_{v}^{j}}: \lambda \in \mathbb{F}_{\tilde{v}_{i}}\right)
$$

and $\hat{\sigma}^{*}$ transports the $j$-th factor to the $(j+1)$-th factor. Denote by $\mathfrak{b}_{i, j} \subset \mathcal{O}_{E_{v, i}}$ the ideal generated by $\left(\lambda \otimes 1-1 \otimes \lambda^{q_{v}^{j}}: \lambda \in \mathbb{F}_{\tilde{v}_{i}}\right)$. Then

$$
\begin{equation*}
\mathcal{O}_{E_{v, i}, R}:=\mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket \otimes_{A_{v}} R \llbracket z \rrbracket=\prod_{j \in \mathbb{Z} / f_{i} \mathbb{Z}} \mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket \otimes_{\mathbb{F}_{v} \llbracket z \rrbracket} R \llbracket z \rrbracket / \mathfrak{b}_{i, j}=\prod_{H_{\tilde{v}_{i}}} R \llbracket y_{i} \rrbracket . \tag{4.1}
\end{equation*}
$$

Definition 4.5. For every $\psi \in H_{E_{v}}$ we let $i(\psi)$ be such that $\psi$ factors through the quotient $E_{v} \rightarrow E_{v, i(\psi)}$ and we let $j(\psi) \in \mathbb{Z} / f_{i(\psi)} \mathbb{Z}$ be the element such that $\psi(\lambda)=\lambda^{q_{v}^{j(\psi)}}$ for all $\lambda \in \mathbb{F}_{\tilde{v}_{i(\psi)}}$. Then the morphism $\psi: \mathcal{O}_{E_{v}} \rightarrow R$ equals the composition $\mathcal{O}_{E_{v}} \hookrightarrow \mathcal{O}_{E_{v}, R} \rightarrow \mathcal{O}_{E_{v, i}(\psi), R} /\left(\mathfrak{b}_{i(\psi), j(\psi)}, y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)$ and $H_{E_{v, i}}=\left\{\psi \in H_{E_{v}}: i(\psi)=i\right\}$.
Lemma 4.6. Let $p^{m}$ be the inseparability degree of $E_{v, i}$ over $Q_{v}$. Then in the $j$-th component $R \llbracket y_{i} \rrbracket$ of (4.1) we have

$$
\begin{equation*}
z-\zeta=\varepsilon \cdot \prod_{\psi \in H_{E_{v}}:(i, j)(\psi)=(i, j)}\left(y_{i}-\psi\left(y_{i}\right)\right)^{p^{m}} \tag{4.2}
\end{equation*}
$$

for a unit $\varepsilon \in R \llbracket y_{i} \rrbracket$.
Proof. Set $y_{i}^{\prime}:=y_{i}^{p^{m}}$ and let $P=P(z, Y)=\sum_{\mu, \nu} b_{\mu \nu} z^{\mu} Y^{\nu} \in \mathbb{F}_{\tilde{v}_{i}} \llbracket z \rrbracket[Y]$ with $b_{\mu \nu} \in \mathbb{F}_{\tilde{v}_{i}}$ be the minimal polynomial of $y_{i}^{\prime}$ over $\mathbb{F}_{\tilde{v}_{i}}((z))$. It is an Eisenstein polynomial of degree $e_{i} / p^{m}$, because $\mathbb{F}_{\tilde{v}_{i}}\left(\left(y_{i}^{\prime}\right)\right)$ is purely ramified and separable over $\mathbb{F}_{\tilde{v}_{i}}((z))$ by Lemma A.2 in the appendix. In particular $b_{0, \nu}=0$ for $0 \leq$ $\nu<e_{i} / p^{m}$, and $b_{1,0} \neq 0$. Consider the polynomials $P^{(j)}(z, Y):=\sum_{\mu, \nu} b_{\mu \nu}^{q_{j}^{3}} z^{\mu} Y^{\nu} \in \mathbb{F}_{\tilde{v}_{i}} \llbracket z \rrbracket[Y] \subset R \llbracket z \rrbracket[Y]$ and $P^{(j)}(\zeta, Y) \in R[Y]$. If $\psi \in H_{E_{v}}$ satisfies $(i, j)(\psi)=(i, j)$ then $P^{(j)}\left(\zeta, \psi\left(y_{i}^{\prime}\right)\right)=\psi\left(P\left(z, y_{i}^{\prime}\right)\right)=$ $\psi(0)=0$. These zeroes $\psi\left(y_{i}^{\prime}\right)$ of $P^{(j)}(\underset{\sim}{\zeta}, Y)$ in $L$ are pairwise different, because if $\psi\left(y_{i}^{\prime}\right)=\widetilde{\psi}\left(y_{i}^{\prime}\right)$ then $(i, j)(\psi)=(i, j)(\widetilde{\psi})$ implies that $\psi$ and $\widetilde{\psi}$ coincide on $E_{v, i}$ and hence on $E_{v}$. It follows that $P^{(j)}(\zeta, Y)=$ $\prod_{\psi:(i, j)(\psi)=(i, j)}\left(Y-\psi\left(y_{i}^{\prime}\right)\right)$ in $L[Y]$, whence already in $R[Y]$. In the $j$-th component $R \llbracket y_{i} \rrbracket$ of (4.1) we have $0=\sum_{\mu, \nu} b_{\mu \nu} z^{\mu}\left(y_{i}^{\prime}\right)^{\nu} \otimes 1=\sum_{\mu, \nu}\left(y_{i}^{\prime}\right)^{\nu} \otimes b_{\mu \nu}^{q_{\nu}^{j}} z^{\mu}=P^{(j)}\left(z, y_{i}^{\prime}\right)$, and we compute

$$
\begin{aligned}
\prod_{\psi:(i, j)(\psi)=(i, j)}\left(y_{i}-\psi\left(y_{i}\right)\right)^{p^{m}} & =P^{(j)}\left(\zeta, y_{i}^{\prime}\right) \\
& =P^{(j)}\left(\zeta, y_{i}^{\prime}\right)-P^{(j)}\left(z, y_{i}^{\prime}\right) \\
& =\sum_{\mu, \nu}^{b_{\mu \nu}^{j}}\left(\zeta^{\mu}-z^{\mu}\right)\left(y_{i}^{\prime}\right)^{\nu} \\
& =(\zeta-z) \cdot \sum_{\mu, \nu} b_{\mu \nu}^{q_{v}^{j}}\left(\zeta^{\mu-1}+\zeta^{\mu-2} z+\ldots+z^{\mu-1}\right)\left(y_{i}^{\prime}\right)^{\nu}
\end{aligned}
$$

The factor $\sum_{\mu, \nu} b_{\mu \nu}^{q_{\nu}^{j}}\left(\zeta^{\mu-1}+\zeta^{\mu-2} z+\ldots+z^{\mu-1}\right)\left(y_{i}^{\prime}\right)^{\nu}$ is congruent to $b_{1,0}^{q_{v}^{j}} \neq 0$ modulo the maximal ideal $\left(\pi_{L}, y_{i}\right) \subset R \llbracket y_{i} \rrbracket$ and therefore a unit in $R \llbracket y_{i} \rrbracket$. This finishes the proof. Note that since

$$
P^{(j)}\left(\zeta, y_{i}^{\prime}\right)-P^{(j)}\left(z, y_{i}^{\prime}\right)=\sum_{n \geq 1} \frac{1}{n!} \frac{\partial^{n} P^{(j)}}{\partial z^{n}}\left(z, y_{i}^{\prime}\right) \cdot(\zeta-z)^{n}
$$

the proof could also be phrased by saying that $\frac{\partial P^{(j)}}{\partial z}\left(z, y_{i}^{\prime}\right)=\sum_{\mu, \nu} \mu b_{\mu \nu}^{q_{\nu}^{j}} z^{\mu-1}\left(y_{i}^{\prime}\right)^{\nu}$ lies in $\mathcal{O}_{E_{v, i}^{\prime}}^{\times}$. In fact this partial derivative is congruent to $b_{1,0}^{q_{v}^{j}} \neq 0$ modulo $y_{i}^{\prime} \cdot \mathcal{O}_{E_{v, i}^{\prime}}$.

Let us draw a direct corollary from the proof of this lemma. To formulate it, recall that if $E_{v, i}$ is separable over $Q_{v}$, the different $\mathfrak{D}_{E_{v, i} / Q_{v}}$ of $E_{v, i}$ over $Q_{v}$ is defined as the ideal in $\mathcal{O}_{E_{v, i}}$ which annihilates the module $\Omega_{\mathcal{O}_{E_{v, i}} / A_{v}}^{1}$ of relative differentials.
Corollary 4.7. If $E_{v, i}$ is separable over $Q_{v}$ then $\mathfrak{D}_{\varphi\left(E_{v, i}\right) / Q_{v}}=\left(\left.\frac{z-\zeta}{y_{i}-\varphi\left(y_{i}\right)}\right|_{y_{i}=\varphi\left(y_{i}\right)}\right)$ in $\mathcal{O}_{\varphi\left(E_{v, i}\right)}$ for every $\varphi \in H_{E_{v, i}}$.
Proof. By [Ser79, Chapter III, §4, Proposition 8] the different is multiplicative, that is $\mathfrak{D}_{E_{v, i} / Q_{v}}=$ $\mathfrak{D}_{E_{v, i} / \mathbb{F}_{\tilde{v}_{i}}((z))} \cdot \mathfrak{D}_{\mathbb{F}_{\tilde{v}_{i}}((z)) / Q_{v}}$. Moreover, $\mathfrak{D}_{\mathbb{F}_{\tilde{v}_{i}}((z)) / Q_{v}}=1$ because $\mathbb{F}_{\tilde{v}_{i}} \llbracket z \rrbracket$ is unramified over $A_{v}$. As in the proof of the preceding lemma let $P(z, Y)$ be the minimal polynomial of $y_{i}^{\prime}$ over $\mathbb{F}_{\tilde{v}_{i}}((z))$ and note that $y_{i}^{\prime}=y_{i}$ under our separability assumption. Then $\frac{\partial P}{\partial z}\left(z, y_{i}\right) \in \mathcal{O}_{E_{v, i}}^{\times}$and

$$
\Omega_{\mathcal{O}_{E_{v, i}} / / \mathbb{F}_{\tilde{v}_{i}} \llbracket z \rrbracket}=\mathcal{O}_{E_{v, i}}\left\langle d z, d y_{i}\right\rangle /\left(d z, \frac{\partial P}{\partial z}\left(z, y_{i}\right) d z+\frac{\partial P}{\partial Y}\left(z, y_{i}\right) d y_{i}\right)=\mathcal{O}_{E_{v, i}} \cdot d y_{i} /\left(\frac{\partial P}{\partial Y}\left(z, y_{i}\right) d y_{i}\right)
$$

We write $z=f\left(y_{i}\right) \in \mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket$. Then $0=\frac{d}{d y_{i}} P\left(f\left(y_{i}\right), y_{i}\right)=\frac{\partial P}{\partial z}\left(f\left(y_{i}\right), y_{i}\right) \frac{d f\left(y_{i}\right)}{d y_{i}}+\frac{\partial P}{\partial Y}\left(f\left(y_{i}\right), y_{i}\right)$ and $\mathfrak{D}_{E_{v, i} / Q_{v}}=\left(\frac{\partial P}{\partial Y}\left(z, y_{i}\right)\right)=\left(\frac{d f\left(y_{i}\right)}{d y_{i}}\right)$. Now Lemma A.1 in the appendix implies that $\mathfrak{D}_{\varphi\left(E_{v, i}\right) / Q_{v}}=$ $\varphi\left(\frac{d f\left(y_{i}\right)}{d y_{i}}\right)=\left.\left(\frac{z-\zeta}{y_{i}-\varphi\left(y_{i}\right)}\right)\right|_{y_{i}=\varphi\left(y_{i}\right)}$.

Now we explore the consequences of these decompositions for local shtukas with complex multiplication.

Proposition 4.8. Let $\underline{\hat{M}}=\left(\hat{M}, \tau_{\hat{M}}\right)$ have complex multiplication by $\mathcal{O}_{E_{v}}$. Then the $\mathcal{O}_{E_{v}, R}$-module $\hat{M}$ is free of rank 1. In particular, $\underline{\hat{M}}_{i}:=\underline{\hat{M}} \otimes_{\mathcal{O}_{E_{v}}} \mathcal{O}_{E_{v, i}}$ is a local $\hat{\sigma}$-shtuka over $R$ with $\mathrm{rk} \underline{\hat{M}}_{i}=\left[E_{v, i}: Q_{v}\right]$ and $\underline{\hat{M}}=\bigoplus_{i=1}^{s} \underline{\underline{M}}_{i}$.
Proof. By faithfully flat descent EGA, $\mathrm{IV}_{2}$, Proposition 2.5.2], we may replace $R$ by a finite extension of discrete valuation rings. Therefore it suffices to prove the proposition in the case where $R$ contains $\psi\left(\mathcal{O}_{E_{v}}\right)$ for all $\psi \in H_{E_{v}}$. In this case $\mathcal{O}_{E_{v}, R}$ is a product of two dimensional regular local rings $R \llbracket y_{i} \rrbracket$ by (4.1). By Ser58, §6, Lemme 6] a finitely generated module $M$ over such a ring is free if and only if it is reflexive, that is $M$ is isomorphic to its bidual $M^{\vee \vee}$, where $M^{\vee}=\operatorname{Hom}_{R \llbracket y_{i} \rrbracket}\left(M, R \llbracket y_{i} \rrbracket\right)$. In particular $M^{\vee \vee}$, which is isomorphic to $\left(M^{\vee \vee}\right)^{\vee \vee}$ is free. We apply this to $M:=\hat{M} \otimes_{\mathcal{O}_{E_{v}, R}} R \llbracket y_{i} \rrbracket$ and consider the base changes $M \otimes_{R \llbracket y_{i} \rrbracket} L \llbracket y_{i} \rrbracket=M \otimes_{R \llbracket z \rrbracket} L \llbracket z \rrbracket$ and $M \otimes_{R \llbracket y_{i} \rrbracket} R \llbracket y_{i} \rrbracket\left[\frac{1}{y_{i}}\right]=M \otimes_{R \llbracket z \rrbracket} R \llbracket z \rrbracket\left[\frac{1}{z}\right]$. Like $L \llbracket y_{i} \rrbracket$ also $R \llbracket y_{i} \rrbracket\left[\frac{1}{y_{i}}\right]$ is a principal ideal domain, because it is a factorial ring of dimension 1. Using Eis95, Proposition 2.10] and that both base changes of $M$ are torsion free, whence free, we see that the canonical morphism $M \rightarrow M^{\vee \vee}$ is an isomorphism after both base changes. Since $R \llbracket z \rrbracket=L \llbracket z \rrbracket \cap R \llbracket z \rrbracket\left[\frac{1}{z}\right] \subset L((z))$ and $M$ and $M^{\vee \vee}$ are free $R \llbracket z \rrbracket$-modules, $M$ equals the intersection $\left(M \otimes_{R \llbracket z \rrbracket} L \llbracket z \rrbracket\right) \cap\left(M \otimes_{R \llbracket z \rrbracket} R \llbracket z \rrbracket\left[\frac{1}{z}\right]\right)$ inside $M \otimes_{R \llbracket z]} L((z))$, and likewise for $M^{\vee \vee}$. This shows that $M \rightarrow M^{\vee \vee}$ is an isomorphism and $M$ is free over $R \llbracket y_{i} \rrbracket$.

It remains to compute the rank. Let $r_{i, j}:=\operatorname{rk}_{\left.R \llbracket y_{i}\right]}\left(\hat{M} \otimes_{\mathcal{O}_{E_{v}, R}} \mathcal{O}_{E_{v, i}, R} / \mathfrak{b}_{i, j}\right)$ for all $i=1, \ldots, s$ and all $j \in \mathbb{Z} / f_{i} \mathbb{Z}$. Then $\sum_{i, j} r_{i, j} \cdot e_{i}=\operatorname{rk} \underline{\hat{M}}$. We first prove that for a fixed $i$ all $r_{i, j}$ are equal. Since $\left(\hat{\sigma}^{*} \hat{M}\right) \otimes_{\mathcal{O}_{E_{v}, R}} \mathcal{O}_{E_{v, i}, R} / \mathfrak{b}_{i, j}=\hat{\sigma}^{*}\left(\hat{M} \otimes_{\mathcal{O}_{E_{v}, R}} \mathcal{O}_{E_{v, i}, R} / \mathfrak{b}_{i, j-1}\right) \cong R \llbracket y_{i} \rrbracket^{r_{i, j-1}}$, we can write the isomorphism $\tau_{\hat{M}}: \hat{\sigma}^{*} \hat{M}\left[\frac{1}{z-\zeta}\right] \xrightarrow{\sim} \hat{M}\left[\frac{1}{z-\zeta}\right]$ in the form

$$
\prod_{j} R \llbracket y_{i} \rrbracket\left[\frac{1}{z-\zeta}\right]^{r_{i, j-1}} \xrightarrow{\sim} \prod_{j} R \llbracket y_{i} \rrbracket\left[\frac{1}{z-\zeta}\right]^{r_{i, j}}
$$

which gives us $r_{i, j-1}=r_{i, j}=: r_{i}$ for all $j$, and hence $\sum_{i} r_{i} f_{i} e_{i}=\operatorname{rk} \underline{\hat{M}}=\operatorname{dim}_{Q_{v}} E_{v}=\sum_{i} \operatorname{dim}_{Q_{v}} E_{v, i}=$ $\sum_{i} f_{i} e_{i}$. Thus if we prove that $r_{i} \neq 0$ then all $r_{i}$ must be 1 and so $\hat{M}$ is a free $\mathcal{O}_{E_{v}, R}$-module of rank 1 and $\operatorname{rk} \underline{\hat{M}}_{i}=f_{i} e_{i}=\left[E_{v, i}: Q_{v}\right]$. Now $r_{i}=0$ means that $\hat{M} \otimes_{\mathcal{O}_{E_{v}}} \mathcal{O}_{E_{v, i}}=(0)$, and hence $E_{v, i}$ acts as zero on $\underline{\hat{M}}$ in contradiction to $E_{v} \subset \operatorname{QEnd}_{R}(\underline{\hat{M}})$. This finishes the proof.

Proposition 4.9. If $\underline{\hat{M}}$ has complex multiplication by a commutative semi-simple $Q_{v}$-algebra $E_{v}$ then $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, Q_{v}\right)$ is a free $E_{v}$-module of rank 1 and $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket)$ is a free $E_{v} \otimes_{Q_{v}} L \llbracket z-\zeta \rrbracket$-module of rank one, where the homomorphism $Q_{v}=\mathbb{F}_{v}((z)) \rightarrow L \llbracket z-\zeta \rrbracket$ is given by $z \mapsto z=\zeta+(z-\zeta)$. If we assume that $L \supset \psi\left(E_{v}\right)$ for all $\psi \in H_{E_{v}}$ then the decomposition (A.1) induces a decomposition

$$
\begin{equation*}
\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket)=\bigoplus_{\psi \in H_{E_{v}}} \mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right), \tag{4.3}
\end{equation*}
$$

where $\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)$ is free of rank 1 over $L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket$. In particular,

$$
\begin{equation*}
\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L)=\bigoplus_{\psi \in H_{E_{v}}} \mathrm{H}^{\psi}(\underline{\hat{M}}, R) \otimes_{R} L \tag{4.4}
\end{equation*}
$$

is the decomposition into generalized eigenspaces of the $E_{v}$-action. Here

$$
\left.\mathrm{H}^{\psi}(\underline{\hat{M}}, R):=\left\{\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R):\left([a]^{*}-\psi(a)\right)^{\left[E_{v, i}(\psi):\right.}: Q_{v}\right]_{\text {insep }} \cdot \omega=0 \quad \forall a \in E_{v} \cap \operatorname{End}_{R}(\underline{\hat{M}})\right\}
$$

is a free $R$-module of rank equal to the inseparability degree $\left[E_{v, i(\psi)}: Q_{v}\right]_{\text {insep }}$ of $E_{v, i(\psi)}$ over $Q_{v}$.
Proof. By the faithfulness of the functor $\underline{\hat{M}} \rightarrow \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, Q_{v}\right)$ we have $E_{v} \subset \operatorname{End}_{Q_{v}} \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, Q_{v}\right)$. So the first statement follows from [BH09, Lemma 7.2].

Since $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket)$ is an isogeny invariant, we may by Proposition 4.2 assume that $\mathcal{O}_{E_{v}} \subset$ $\operatorname{End}_{R}(\underline{\hat{M}})$ and then $\hat{M}$ is free of rank 1 over $\mathcal{O}_{E_{v}, R}$ by Proposition 4.8. It follows that $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket):=$ $\hat{\sigma}^{*} \hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z-\zeta \rrbracket \cong E_{v} \otimes_{Q_{v}} L \llbracket z-\zeta \rrbracket$. Now we use Lemma A.3. In particular, (4.4) and the statement about $\mathrm{H}^{\psi}(\underline{\hat{M}}, R)$ follow from (A.2) and the equation $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L)=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R) \otimes_{R} L$.

The proposition allows us to make two definitions.
Definition 4.10. Let $\underline{\hat{M}}$ have complex multiplication by $\mathcal{O}_{E_{v}}$ and assume that $L \supset \psi\left(E_{v}\right)$ for all $\psi \in H_{E_{v}}$. Fix a $\psi \in H_{E_{v}}$ and let $i:=i(\psi)$. Let $\omega_{\psi}^{\circ} \in \mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$ be an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$ generator whose reduction $\omega_{\psi}^{\circ} \bmod \left(y_{i}-\psi\left(y_{i}\right)\right) \in \mathrm{H}^{\psi}(\underline{\hat{M}}, L) /\left(y_{i}-\psi\left(y_{i}\right)\right) \mathrm{H}^{\psi}(\underline{\hat{M}}, L)$ is a generator of the free $R$-module of rank one $\mathrm{H}^{\psi}(\underline{\hat{M}}, R) /\left(y_{i}-\psi\left(y_{i}\right)\right) \mathrm{H}^{\psi}(\underline{\hat{M}}, R)$. Such a $\omega_{\psi}^{\circ}$ is uniquely determined up to multiplication by an element of $R^{\times}+\left(y_{i}-\psi\left(y_{i}\right)\right) L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$. Note that if $E_{v, i}$ is separable over $Q_{v}$ then $y_{i}-\psi\left(y_{i}\right)$ acts trivially on $\mathrm{H}^{\psi}(\underline{M}, L)$ and $\mathrm{H}^{\psi}(\underline{\hat{M}}, R)$ is a free $R$-module of rank 1 . Also $L \llbracket y_{i}-\varphi\left(y_{i}\right) \rrbracket=L \llbracket z-\zeta \rrbracket$.

If $\omega_{\psi} \in \mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$ is any generator, there is an element $x \in L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket^{\times}$with $\omega_{\psi}=x \omega_{\psi}^{\circ}$. We define the valuation of $\omega_{\psi}$ as $v\left(\omega_{\psi}\right):=v\left(x \bmod y_{i}-\psi\left(y_{i}\right)\right)$. (NOTE THAT THIS DEFINITION IS WRONG; SEE ERRATUM (B.2) It only depends on the image of $\omega_{\psi}$ in $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L)$ and is also independent of the choice of $\omega_{\psi}^{\circ}$.

Note that if $\underline{M}=\left(M, \tau_{M}\right)$ is an $A$-motive over $L$ with good model $\underline{\mathcal{M}}$ over $R$, and $\underline{\hat{M}}=\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ is the local shtuka at $v$ associated with $\underline{\mathcal{M}}$ as in Example [3.2, then for an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator $\omega_{\psi} \in \mathrm{H}^{\psi}\left(\underline{M}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)=\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$ the present definition of $v\left(\omega_{\psi}\right)$ coincides with the definition of $v\left(\omega_{\psi}\right)$ from (1.13).

Definition 4.11. A local CM-type at $v$ is a pair $\left(E_{v}, \Phi\right)$ with $E_{v}$ a semisimple commutative $Q_{v}$-algebra and $\Phi=\left(d_{\psi}\right)_{\psi \in H_{E_{v}}}$ a tuple of integers $d_{\psi} \in \mathbb{Z}$.

If $\underline{\hat{M}}$ is a local shtuka with complex multiplication by a commutative semi-simple $Q_{v}$-algebra $E_{v}$ and if $L \supset \psi\left(E_{v}\right)$ for all $\psi \in H_{E_{v}}$ then the Hodge-Pink lattice $\mathfrak{q} \underline{\hat{M}}=\tau_{\hat{M}}^{-1}\left(\hat{M} \otimes_{R \llbracket z \rrbracket} L \llbracket z-\zeta \rrbracket\right)$ of $\underline{\hat{M}}$ satisfies $\mathfrak{q}^{\underline{\hat{M}}}=\prod_{\psi \in H_{E_{v}}}\left(y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)^{-d_{\psi}} \mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)$ for integers $d_{\psi}$ under the decomposition (4.3). We call $\Phi=\left(d_{\psi}\right)_{\psi \in H_{E_{v}}}$ the local CM-type of $\underline{\underline{M}}$.

Note that if $\underline{M}=\left(M, \tau_{M}\right)$ is an $A$-motive over $L$ with good model $\underline{\mathcal{M}}$ over $R$, and $\underline{\hat{M}}=\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ is the local shtuka at $v$ associated with $\underline{\mathcal{M}}$ as in Example [3.2, and $E_{v}:=E \otimes_{Q} Q_{v}$, then we see from the isomorphism $\mathrm{H}^{\psi}\left(\underline{M}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)=\mathrm{H}^{\psi}\left(\underline{\underline{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)$ that the local CM-type of $\underline{\underline{M}}$ is equal to the CM-type of $\underline{M}$ under the identification $H_{E} \xrightarrow{\sim} H_{E_{v}}$, which extends $\psi: E \rightarrow Q^{\text {alg }} \subset Q_{v}^{\text {alg }}$ to the completion $\psi: E_{v} \rightarrow Q_{v}^{\text {alg }}$.

## 5 Periods of Local Shtukas with Complex Multiplication

5.1. In this section we let $\underline{\hat{M}}$ be a local $\hat{\sigma}$-shtuka over $R$ with complex multiplication by $\mathcal{O}_{E_{v}}$ where $E_{v}$ is a commutative semi-simple $Q_{v}$-algebra as in the preceding section. From Theorem 5.13 on we assume that the factors $E_{v, i}$ of $E_{v}$ are separable field extensions of $Q_{v}$. Throughout we assume that $L \supset \psi\left(E_{v}\right)$ for all $\psi \in H_{E_{v}}$. Using Proposition 4.8 we may choose a basis of $\underline{\hat{M}}$ and write it under the decomposition (4.1) as

$$
\underline{\hat{M}} \cong \prod_{i} \prod_{j \in \mathbb{Z} / f_{i} \mathbb{Z}}\left(R \llbracket y_{i} \rrbracket, \tau_{i, j}\right) \quad \text { with } \quad \tau_{i, j} \in R \llbracket y_{i} \rrbracket\left[\frac{1}{z-\zeta}\right]^{\times} .
$$

Let $c \in \mathbb{N}_{0}$ be such that $(z-\zeta)^{c} \tau_{i j},(z-\zeta)^{c} \tau_{i j}^{-1} \in R \llbracket y_{i} \rrbracket$. Since the $y_{i}-\varphi\left(y_{i}\right)$ for $\varphi \in H_{E_{v}}$ with $(i, j)(\varphi)=$ $(i, j)$ are prime elements in the factorial ring $R \llbracket y_{i} \rrbracket$, Lemma4.6 applied to $(z-\zeta)^{c} \tau_{i j} \cdot(z-\zeta)^{c} \tau_{i j}^{-1}=(z-\zeta)^{2 c}$ shows that

$$
\begin{equation*}
\tau_{i, j}=\varepsilon_{i, j} \cdot \prod_{\varphi \in H_{E_{v}}:(i, j)(\varphi)=(i, j)}\left(y_{i}-\varphi\left(y_{i}\right)\right)^{d_{\varphi}} \tag{5.1}
\end{equation*}
$$

for a unit $\varepsilon_{i, j} \in R \llbracket y_{i} \rrbracket^{\times}$and integers $d_{\varphi} \in \mathbb{Z}$. By Definition4.11 the tuple $\Phi=\left(d_{\varphi}\right)_{\varphi}$ is the local CM-type of $\underline{\hat{M}}$.

Note that we can view $\underline{\hat{M}}$ as the tensor product $\underline{\hat{M}}_{E_{v}, 0} \otimes \otimes_{\varphi} \underline{\hat{M}}_{E_{v}, \varphi} \otimes d_{\varphi}$ over $\mathcal{O}_{E_{v}, R}$ of $\underline{\hat{M}}_{E_{v}, 0}:=$ $\left(\mathcal{O}_{E_{v}, R}, \tau_{0}=\prod_{i, j} \varepsilon_{i, j}\right)$ and all the $d_{\varphi}$-th powers of $\hat{\underline{M}}_{E_{v}, \varphi}:=\left(\mathcal{O}_{E_{v}, R}, \prod_{i, j} \tau_{\varphi, i, j}\right)$ where

$$
\tau_{\varphi, i, j}= \begin{cases}1 & \text { if }(i, j) \neq(i, j)(\varphi) \\ y_{i}-\varphi\left(y_{i}\right) & \text { if }(i, j)=(i, j)(\varphi)\end{cases}
$$

Likewise the cohomology realizations decompose as tensor products

$$
\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \cong \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}_{E_{v}, 0}, A_{v}\right) \otimes \bigotimes_{\varphi \in H_{E_{v}}} \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, A_{v}\right)^{\otimes d_{\varphi}},
$$

where the tensor product is over $\mathcal{O}_{E_{v}}$, and

$$
\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L \llbracket z-\zeta \rrbracket) \cong \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, 0}, L \llbracket z-\zeta \rrbracket\right) \otimes \bigotimes_{\varphi \in H_{E_{v}}} \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L \llbracket z-\zeta \rrbracket\right)^{\otimes d_{\varphi}},
$$

where the tensor product is over $E_{v} \otimes_{Q_{v}} L \llbracket z-\zeta \rrbracket$. For the purpose of computing the period isomorphism $h_{v, \mathrm{dR}}$, we may therefore treat all factors of $\tau_{i, j}$ separately; see 5.4 below.
5.2. We first treat the case of $\underline{\hat{M}}_{E_{v}, 0}=\left(\mathcal{O}_{E_{v}, R}, \tau_{0}=\left(\varepsilon_{i, j}\right)_{i, j}\right)$, where $\varepsilon_{i, j} \in R \llbracket y_{i} \rrbracket^{\times}$. We compute the $\tau$-invariants $\mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, 0}, A_{v}\right)$ as the set of tuples $\left(c_{i, j}\right)_{i, j}$ with $c_{i, j}:=\sum_{n=0}^{\infty} c_{i, j, n} y_{i}^{n} \in L^{\text {sep }} \llbracket y_{i} \rrbracket$ subject to the condition

$$
\left(c_{i, j}\right)_{i, j}=\tau_{0} \circ \hat{\sigma}\left(\left(c_{i, j}\right)_{i, j}\right), \quad \text { that is } \quad c_{i, j}=\varepsilon_{i, j} \cdot \hat{\sigma}\left(c_{i, j-1}\right) \quad \text { for all } i, j .
$$

The latter implies $c_{i, j}=\varepsilon_{i, j} \cdot \hat{\sigma}\left(\varepsilon_{i, j-1}\right) \cdot \ldots \cdot \hat{\sigma}^{j-1}\left(\varepsilon_{i, 1}\right) \cdot \hat{\sigma}^{j}\left(c_{i, 0}\right)$ and $c_{i, 0}=\varepsilon_{i} \cdot \hat{\sigma}^{f_{i}}\left(c_{i, 0}\right)$, where we set $\varepsilon_{i}:=\varepsilon_{i, 0} \cdot \hat{\sigma}\left(\varepsilon_{i, f_{i}-1}\right) \cdot \ldots \cdot \hat{\sigma}^{f_{i}-1}\left(\varepsilon_{i, 1}\right)=\sum_{n=0}^{\infty} b_{i, n} y_{i}^{n} \in R \llbracket y_{i} \rrbracket^{\times}$. In particular $b_{i, 0} \in R^{\times}$. The resulting formulas for the coefficients

$$
c_{i, 0,0}=b_{i, 0} \cdot c_{i, 0,0}^{\tilde{q}_{i}} \quad \text { and } \quad c_{i, 0, n}-b_{i, 0} \cdot c_{i, 0, n}^{\tilde{q}_{i}}=\sum_{\ell=1}^{n} b_{i, \ell} \cdot c_{i, 0, n-\ell}^{\tilde{q}_{i}},
$$

where $\tilde{q}_{i}=q_{v}^{f_{i}}$, lead to the formulas

$$
c_{i, 0,0}^{\tilde{q}_{i}-1}=b_{i, 0}^{-1} \quad \text { and } \quad \frac{c_{i, 0, n}}{c_{i, 0,0}}-\left(\frac{c_{i, 0, n}}{c_{i, 0,0}}\right)^{\tilde{q}_{i}}=\sum_{\ell=1}^{n} \frac{b_{i, \ell}}{b_{i, 0}} \cdot\left(\frac{c_{i, 0, n-\ell}}{c_{i, 0,0}}\right)^{\tilde{q}_{i}},
$$

which have solutions $c_{i, 0, n} \in \mathcal{O}_{L^{\text {sep }}}$ with $c_{i, 0,0} \in \mathcal{O}_{L^{\text {sep }}}^{\times}$. In particular, the field extension of $L$ generated by the $c_{i, j, n}$ is unramified. Then $\left(c_{i, j}\right)_{i, j}$ is an $\mathcal{O}_{E_{v}}$-basis of $\mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, 0}, A_{v}\right)$. Under the period isomorphism $h_{v, \mathrm{dR}}$ it is mapped to

$$
\left(\varepsilon_{i, j}^{-1} c_{i, j}\right)_{i, j} \in\left(\mathcal{O}_{E_{v}} \otimes_{A_{v}} \mathcal{O}_{\mathbb{C}_{v}} \llbracket z \rrbracket\right)^{\times} \subset E_{v} \otimes_{Q_{v}} \mathbb{C}_{v} \llbracket z-\zeta \rrbracket=\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, 0}, \mathbb{C}_{v} \llbracket z-\zeta \rrbracket\right) .
$$

5.3. Next we compute the period isomorphism for the local shtuka $\underline{\hat{M}}_{E_{v, \varphi}}$ from above. For an element $0 \neq \xi \in\left(\pi_{L}\right) \subset R$ we consider the equation

$$
\begin{equation*}
\hat{\sigma}^{f_{i}}\left(\ell_{y_{i}, \xi}^{+}\right)=\left(y_{i}-\xi\right) \cdot \ell_{y_{i}, \xi}^{+} \quad \text { for } \quad \ell_{y_{i}, \xi}^{+}:=\sum_{n=0}^{\infty} \ell_{n} y_{i}^{n} \in L^{\text {sep }} \llbracket y_{i} \rrbracket . \tag{5.2}
\end{equation*}
$$

The equation can be solved by taking $\ell_{n} \in L^{\text {sep }}$ with $\ell_{0}^{\tilde{q}_{i}-1}=-\xi$ and $\ell_{n}^{\tilde{q}_{i}}+\xi \ell_{n}=\ell_{n-1}$. This implies that $\left|\ell_{n}\right|=|\xi|_{\tilde{q_{i}}}^{\tilde{\tau}^{n}} /\left(\tilde{q}_{i}-1\right)<1$ and $\ell_{n} \in \mathcal{O}_{L^{\text {sep }}}$. Note that this solution is not unique, but that every other solution $\tilde{\ell}_{y_{i}, \xi}^{+}$of (5.2) is obtained by multiplying $\ell_{y_{i}, \xi}^{+}$by an element of $\mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket=\mathcal{O}_{E_{v, i}}$, because $\hat{\sigma}^{f_{i}}\left(\frac{\tilde{\ell}_{i_{2}}^{+}, \xi}{\ell_{y_{i}, \xi}^{+}}\right)=\frac{\left(y_{i}-\xi\right) \cdot \tilde{\ell}_{i, \xi}^{+}}{\left(y_{i}-\xi\right) \cdot \ell_{y_{i}, \xi}^{+}}=\frac{\tilde{\ell}_{i_{2}}^{+}, \xi}{\ell_{y_{i}, \xi}^{+}} \in L^{\text {sep }} \llbracket y_{i} \rrbracket$ is invariant under $\hat{\sigma}^{f_{i}}$ and hence lies in $\mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket$.

According to the decomposition (4.1) the $\tau$-invariants $\check{u} \in \mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, \varphi}, A_{v}\right)$ of $\underline{\hat{M}}_{E_{v}, \varphi}$ have the form $\check{u}=\left(\check{u}_{i, j}\right)_{i, j} \in \prod_{i, j} L^{\text {sep }} \llbracket y_{i} \rrbracket$ with $\check{u}=\tau_{\varphi} \cdot \hat{\sigma}(\check{u})$, that is

$$
\check{u}_{i, j}= \begin{cases}\hat{\sigma}\left(\check{u}_{i, j-1}\right) & \text { if }(i, j) \neq(i, j)(\varphi), \\ \left(y_{i}-\varphi\left(y_{i}\right)\right) \cdot \hat{\sigma}\left(\check{u}_{i, j-1}\right) & \text { if }(i, j)=(i, j)(\varphi) .\end{cases}
$$

For $j, j^{\prime} \in \mathbb{Z} / f_{i} \mathbb{Z}$ we denote by $\left(j, j^{\prime}\right)$ the representative of $j-j^{\prime}$ in $\left\{0, \ldots, f_{i}-1\right\}$. This implies that $\check{u}_{(i, j)(\varphi)}=\left(y_{i(\varphi)}-\varphi\left(y_{i(\varphi)}\right)\right) \cdot \hat{\sigma}^{f_{i(\varphi)}}\left(\check{u}_{(i, j)(\varphi)}\right)$, and $\check{u}_{i(\varphi), j}=\hat{\sigma}^{(j, j(\varphi))}\left(\check{u}_{(i, j)(\varphi)}\right)$, as well as $\check{u}_{i, j} \in \mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket$ for all $i \neq i(\varphi)$ and all $j$. In particular an $\mathcal{O}_{E_{v}}$-basis of $\mathrm{H}_{v}^{1}\left(\underline{\underline{\hat{M}}}_{E_{v}, \varphi}, A_{v}\right)$ is given by

$$
\begin{equation*}
\check{u}=\left(\check{u}_{i, j}\right)_{i, j} \quad \text { with } \quad \check{u}_{i, j}=\hat{\sigma}^{(j, j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)^{-\delta_{i, i(\varphi)}}=\left(\ell_{y_{i}, \varphi\left(y_{i}\right)^{+}}^{q_{v}^{(j, j(\varphi))}}\right)^{-\delta_{i, i(\varphi)}} \tag{5.3}
\end{equation*}
$$

where $\delta_{i, i(\varphi)}$ is the Kronecker $\delta$. The comparison isomorphism $h_{v, \mathrm{dR}}$ sends this $\check{u}$ to the element

$$
\begin{equation*}
\tau_{\varphi}^{-1} \cdot \check{u}=\left(\left(\left(y_{i}-\varphi\left(y_{i}\right)\right)^{\delta_{j, j(\varphi)}} \cdot \hat{\sigma}^{(j, j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right)^{-\delta_{i, i(\varphi)}}\right)_{i, j} \tag{5.4}
\end{equation*}
$$

of $E_{v} \otimes_{Q_{v}} \mathbb{C}_{v}((z-\zeta))=\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, \mathbb{C}_{v}((z-\zeta))\right)$.
5.4. Putting everything together we see that our $\underline{\hat{M}} \cong\left(\mathcal{O}_{E_{v}, R}, \prod_{i, j} \tau_{i, j}\right)$ with $\tau_{i, j}$ from (5.1) has

$$
\check{u}=\left(\check{u}_{i, j}\right)_{i, j}=\left(c_{i, j} \cdot \prod_{\varphi \in H_{E_{v, i}}} \hat{\sigma}^{(j, j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)^{-d_{\varphi}}\right)_{i, j}
$$

as an $\mathcal{O}_{E_{v}}$-basis of $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right) \cong \mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, 0}, A_{v}\right) \otimes \bigotimes_{\varphi \in H_{E_{v}}} \mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, \varphi}, A_{v}\right)^{\otimes d \varphi}$, where the tensor product is over $\mathcal{O}_{E_{v}}$. Under $h_{v, \mathrm{dR}}$ this $\check{u}$ is mapped to the element

$$
\begin{gather*}
\tau_{\hat{M}}^{-1} \cdot \check{u}=\left(\varepsilon_{i, j}^{-1} c_{i, j} \cdot \prod_{\varphi \in H_{E_{v, i}}}\left(\left(y_{i}-\varphi\left(y_{i}\right)\right)^{\delta_{j, j(\varphi)}} \cdot \hat{\sigma}^{(j, j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right)^{-d_{\varphi}}\right)_{i, j}  \tag{5.5}\\
\text { of } E_{v} \otimes Q_{v} \mathbb{C}_{v}((z-\zeta)) \cong \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}, \mathbb{C}_{v}((z-\zeta))\right) \cong \mathrm{H}_{\mathrm{dR}}^{1}\left(\hat{\underline{M}}_{E_{v}, 0}, \mathbb{C}_{v} \llbracket z-\zeta \rrbracket\right) \otimes \otimes_{\varphi \in H_{E_{v}}} \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, \mathbb{C}_{v} \llbracket z-\zeta \rrbracket\right)^{\otimes d_{\varphi}},
\end{gather*}
$$ where the tensor product is over $E_{v} \otimes_{Q_{v}} \mathbb{C}_{v} \llbracket z-\zeta \rrbracket$.

Remark 5.5. Note that $\underline{\underline{M}}_{E_{v}, \varphi} \otimes_{\mathcal{O}_{E_{v}}} \mathcal{O}_{E_{v, i}}$ with $i=i(\varphi)$ is the local $\hat{\sigma}$-shtuka associated with a Lubin-Tate formal group, and so our treatment is analogous to Colmez's Col93, § I.2]. Namely, let $\hat{G}=\hat{\mathbb{G}}_{a, R}=\operatorname{Spf} R \llbracket X \rrbracket$ be the formal additive group over $R$ with an action of $\mathcal{O}_{E_{v, i}}=\mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket$ given by

$$
\begin{aligned}
{[\lambda]: X } & \longmapsto \varphi(\lambda) \cdot X=\lambda^{q_{v}^{(\varphi)}} \cdot X \quad \text { for } \lambda \in \mathbb{F}_{\tilde{v}_{i}} \\
{\left[y_{i}\right]: X } & \longmapsto X^{\tilde{q}_{i}}+\varphi\left(y_{i}\right) \cdot X .
\end{aligned}
$$

Then $\hat{G}$ is the Lubin-Tate formal group over $R$ associated with $\mathcal{O}_{\varphi\left(E_{v, i}\right)}$; see LT65. It is a $z$-divisible local Anderson module in the sense of HS15, Definition 7.1]. For an element $a \in \mathcal{O}_{E_{v, i}}$ let $\hat{G}[a]:=\operatorname{ker}[a]$. Under the anti-equivalence between $z$-divisible local Anderson modules and effective local $\hat{\sigma}$-shtukas over $S=\operatorname{Spec} R$ from HS15, Theorem 8.3] the associated local shtuka is
with $\tau^{0}:=\mathrm{id}: \hat{G} \xrightarrow{\sim} \hat{\mathbb{G}}_{a, R}$ and $\tau^{k}:=\operatorname{Frob}_{q_{v}, \hat{\mathbb{G}}_{a, R}} \circ \tau^{0}: X \mapsto X^{q_{v}^{k}}$. It is an $\mathcal{O}_{E_{v, i}, R}$-module via the $\mathcal{O}_{E_{v, i}}$-action on $\hat{G}\left[z^{n}\right]$ and the $R$-action on $\mathbb{G}_{a, R}$, and is equipped with the Frobenius $\tau_{\hat{M}}: \hat{\sigma}^{*} \hat{M} \rightarrow \hat{M}$ given by $\hat{\sigma}_{\hat{M}}^{*} m \mapsto \operatorname{Frob}_{q_{v}, \mathbb{G}_{a, R}} \circ m$ for $m \in \hat{M}$. We set $\underline{\hat{M}}(\hat{G}):=\left(\hat{M}, \tau_{\hat{M}}\right)$. In particular, we see that $\lambda \in \mathbb{F}_{\tilde{v}_{i}}$ acts on $R \llbracket y_{i} \rrbracket \tau^{k}$ as $\lambda^{q_{v}^{k+j(\varphi)}}$ and so $\hat{M} / \mathfrak{b}_{i, j} \hat{M}=R \llbracket y_{i} \rrbracket \tau^{(j, j(\varphi))}$ under the decomposition (4.1). Since $\tau_{\hat{M}}\left(\hat{\sigma}_{\hat{M}}^{*} \tau^{f_{i}-1}\right)=\tau^{f_{i}}=\left[y_{i}\right]-\varphi\left(y_{i}\right): X \mapsto X^{\tilde{q}_{i}}=\left(\left[y_{i}\right]-\varphi\left(y_{i}\right)\right)(X)$, we see that

$$
\tau_{\hat{M}}=\left(\tau_{\hat{M}, j}\right)_{j} \quad \text { with } \quad \tau_{\hat{M}, j}= \begin{cases}1 & \text { if } j \neq j(\varphi) \\ y_{i}-\varphi\left(y_{i}\right) & \text { if } j=j(\varphi)\end{cases}
$$

that is, $\underline{\hat{M}}(\hat{G})=\underline{\hat{M}}_{E_{v}, \varphi} \otimes_{\mathcal{O}_{E_{v}}} \mathcal{O}_{E_{v, i}}$.
Moreover, if we want to also consider the other components of $\underline{\hat{M}}_{E_{v}, \varphi}$ for $i \neq i(\varphi)$ we take the divisible local Anderson module $\hat{G}_{E_{v}, \varphi}:=\hat{G} \times \prod_{i \neq i(\varphi)}\left(E_{v, i} / \mathcal{O}_{E_{v, i}}\right)_{R}$. It has local shtuka $\underline{\hat{M}}\left(\hat{G}_{E_{v}, \varphi}\right)=$ $\underline{\hat{M}}(\hat{G}) \oplus \bigoplus_{i \neq i(\varphi)} \underline{\hat{M}}\left(\left(E_{v, i} / \mathcal{O}_{E_{v, i}}\right)_{R}\right)=\underline{\hat{M}}_{E_{v}, \varphi}$, because $\underline{\hat{M}}\left(\left(E_{v, i} / \mathcal{O}_{E_{v, i}}\right)_{R}\right)=\left(\mathcal{O}_{E_{v, i}, R}, \tau=1\right)$.
5.6. We want to describe the Galois action of $\mathscr{G}_{L}$ on $H_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, \varphi}, A_{v}\right)$. Recall from HK15, Definition 4.8, Proposition 4.9 and Remark 4.10] that the Tate module of $\hat{G}$ is defined as $T_{v} \hat{G}:=\operatorname{Hom}_{A_{v}}\left(Q_{v} / A_{v}, \hat{G}\left(L^{\text {sep }}\right)\right)$ and that there is a perfect pairing of $A_{v}$-modules

$$
\begin{equation*}
T_{v} \hat{G} \times \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}(\hat{G}), A_{v}\right) \longrightarrow \operatorname{Hom}_{\mathbb{F}_{v}}\left(Q_{v} / A_{v}, \mathbb{F}_{v}\right), \quad(f, m) \longmapsto m \circ f, \tag{5.6}
\end{equation*}
$$

which is equivariant for the actions of $\mathscr{G}_{L}$ and $\operatorname{End}_{R}(\underline{\hat{M}}(\hat{G}))=\operatorname{End}_{R}(\hat{G})^{\text {op }}$. Here the $A_{v}$-module $\operatorname{Hom}_{\mathbb{F}_{v}}\left(Q_{v} / A_{v}, \mathbb{F}_{v}\right) \cong \widehat{\Omega}_{A_{v} / \mathbb{F}_{v}}^{1} \cong \mathbb{F}_{v} \llbracket z \rrbracket d z$ is free of rank one; see HK15, Equation (4.5) before Proposition 4.9]. We have already computed $\mathrm{H}_{v}^{1}\left(\underline{\underline{\hat{M}}}_{E_{v}, \varphi}, A_{v}\right)=E_{v, i} \cdot\left(\check{u}_{i, j}\right)_{i, j}$ in (5.3). We will now compute
$T_{v} \hat{G}_{E_{v}, \varphi}$ and the action of $\mathscr{G}_{L}$ on both $T_{v} \hat{G}_{E_{v}, \varphi}$ and $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, A_{v}\right)$. Let again $i=i(\varphi)$. Since $\mathcal{O}_{E_{v, i}}$ acts on $\hat{G}\left(L^{\text {sep }}\right)$ we have

$$
\begin{aligned}
T_{v} \hat{G} & =\operatorname{Hom}_{\mathcal{O}_{E_{v, i}}}\left(\mathcal{O}_{E_{v, i}} \otimes_{A_{v}}\left(Q_{v} / A_{v}\right), \hat{G}\left(L^{\text {sep }}\right)\right) & & \\
& =\operatorname{Hom}_{\mathcal{O}_{E_{v, i}}}\left(E_{v, i} / \mathcal{O}_{E_{v, i}} \hat{G}\left(L^{\text {sep }}\right)\right) & & \ni f \\
& =\left\{\left(P_{n}\right)_{n} \in \prod_{n \in \mathbb{N}_{0}} \hat{G}\left[y_{i}^{n}\right]\left(L^{\text {sep }}\right):\left[y_{i}\right]\left(P_{n}\right)=P_{n-1}\right\} & & \ni\left(P_{n}\right)_{n}:=\left(f\left(y_{i}^{-n}\right)\right)_{n},
\end{aligned}
$$

where $f$ is reconstructed from $\left(P_{n}\right)_{n}$ as $f\left(a y_{i}^{-n}\right):=[a]\left(P_{n}\right)$ for $a \in \mathcal{O}_{E_{v, i}}^{\times}$. From equation (5.2) we see that $\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}=\sum_{n=0}^{\infty} \ell_{n} y_{i}^{n}$ satisfies

$$
\left[y_{i}\right]\left(\ell_{0}\right)=\ell_{0}^{\tilde{q}_{i}}+\varphi\left(y_{i}\right) \ell_{0}=0 \quad \text { and } \quad\left[y_{i}\right]\left(\ell_{n}\right)=\ell_{n}^{\tilde{q}_{i}}+\varphi\left(y_{i}\right) \ell_{n}=\ell_{n-1}
$$

Thus $\ell_{n-1} \in \hat{G}\left[y_{i}^{n}\right]\left(L^{\text {sep }}\right)$ and $T_{v} \hat{G}=\mathcal{O}_{E_{v, i}} \cdot\left(\ell_{n-1}\right)_{n}$. To compute the $\mathscr{G}_{L^{-}}$action on $T_{v} \hat{G}$ we need the following

Proposition 5.7. Let $\mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)_{\infty}:=\mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)\left(\ell_{n}: n \in \mathbb{N}_{0}\right)$. Then there is an isomorphism of topological groups

$$
\chi: \operatorname{Gal}\left(\mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)_{\infty} / \mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)\right) \xrightarrow{\longrightarrow} \mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket^{\times}=\mathcal{O}_{E_{v, i}}^{\times}
$$

satisfying $g\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right):=\sum_{n=0}^{\infty} g\left(\ell_{n}\right) y_{i}^{n}=\chi(g) \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}$in $\mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)_{\infty} \llbracket y_{i} \rrbracket$ for $g$ in the Galois group. The isomorphism $\chi$ is independent of the choice of $\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}$and is called the cyclotomic character of the field $E_{v, i}=\mathbb{F}_{\tilde{v}_{i}}\left(\left(y_{i}\right)\right)$.

Proof. The existence of $\chi$ follows from the equation $\hat{\sigma}^{f_{i}}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)=\left(y_{i}-\varphi\left(y_{i}\right)\right) \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}$, which implies that $\chi(g):=\frac{g\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)}{\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}}$is $\hat{\sigma}^{f_{i} \text { invariant, that is } \chi(g) \in \mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket^{\times} \text {. Furthermore, } \chi \text { is an isomorphism }{ }^{\text {a }} \text {. }}$ because $\ell_{n-1}$ is a uniformizing parameter of $\mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)\left(\ell_{0}, \ldots, \ell_{n-1}\right)$ and so the equations defining the $\ell_{n}$ are irreducible by Eisenstein. Every other solution of (5.2) is of the form $a \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}$ with $a \in \mathbb{F}_{\tilde{v}_{i}} \llbracket y_{i} \rrbracket$ and so $g\left(a \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)=a \cdot g\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)=\chi(g) \cdot a \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}$. This shows that $\chi(g)$ does not depend on the solution $\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}$.

Let $\mathcal{I}_{L} \subset \mathscr{G}_{L}$ be the inertia subgroup and similarly for other fields. By local class field theory, see Lubin and Tate LT65, Corollary on p. 386], the image of $g \in \mathcal{I}_{\varphi\left(E_{v, i}\right)}$ in $\mathscr{G}_{\varphi\left(E_{v, i}\right)}^{\text {ab }}$, equals the norm residue symbol $\left(\left.\chi(g)^{-1}\right|_{y_{i}=\varphi\left(y_{i}\right)}, \varphi\left(E_{v, i}\right)^{\text {ab }} / \varphi\left(E_{v, i}\right)\right)$ where $\varphi\left(E_{v, i}\right)^{\text {ab }}$ is the maximal abelian extension of $\varphi\left(E_{v, i}\right)$ in $Q_{v}^{\text {sep }}$. In general, the homomorphism $\chi_{L}: \mathcal{I}_{L} \rightarrow \mathcal{O}_{L}^{\times}$with $\left.g\right|_{L^{\mathrm{ab}}}=\left(\chi_{L}(g)^{-1}, L^{\mathrm{ab}} / L\right)$ is sometimes called the character of local class field theory of the field $L$. So we see that $\left.\chi(g)\right|_{y_{i}=\varphi\left(y_{i}\right)}=\chi_{\varphi\left(E_{v, i}\right)}(g)$. If $L$ is separable over $\varphi\left(E_{v, i}\right)$ these characters are compatible for $g \in \mathcal{I}_{L}$ in the sense that $\chi_{\varphi\left(E_{v, i}\right)}(g)=$ $N_{L / \varphi\left(E_{v, i}\right)}\left(\chi_{L}(g)\right)$.
5.8. From $T_{v} \hat{G}=\mathcal{O}_{E_{v, i}} \cdot\left(\ell_{n-1}\right)_{n}$ it follows that $g$ acts on $T_{v} \hat{G}$ in the same way as an endomorphism in $\mathcal{O}_{E_{v, i}}^{\times}$. Let us compute this endomorphism. We write $\chi(g)=\sum_{k=0}^{\infty} a_{k} y_{i}^{k}$ with $a_{k} \in \mathbb{F}_{\tilde{v}_{i}}$. Then the expansion $g\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)=\chi(g) \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}=\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{k} \ell_{n-k} y_{i}^{n}$ implies that $g\left(\ell_{n}\right)=\sum_{k=0}^{n} a_{k} \ell_{n-k}=$ $\sum_{k=0}^{n} \varphi\left(a_{k}^{q_{v}^{-j(\varphi)}}\right)\left[y_{i}^{k}\right]\left(\ell_{n}\right)$. Thus every element $g \in \mathcal{I}_{L}$ acts on $T_{v} \hat{G}$ as the endomorphism $\sum_{k=0}^{\infty} a_{k}^{q_{v}^{-j(\varphi)}} y_{i}^{k}=$ $\hat{\sigma}^{-j(\varphi)}(\chi(g))=\varphi^{-1}\left(\left.\chi(g)\right|_{y_{i}=\varphi\left(y_{i}\right)}\right)=\varphi^{-1} \circ \chi_{\varphi\left(E_{v, i}\right)}(g) \in \mathcal{O}_{E_{v, i}}^{\times}$and on $T_{v} \hat{G}_{E_{v}, \varphi}$ as the endomorphism $\left(\hat{\sigma}^{-j(\varphi)}(\chi(g))^{\delta_{i, i(\varphi)}}\right)_{i} \in \mathcal{O}_{E_{v}}^{\times}$.
Definition 5.9. We define the character $\chi_{E_{v}, \varphi}:=\left(\left(\varphi^{-1} \circ \chi_{\varphi\left(E_{v, i}\right)}\right)^{\delta_{i, i(\varphi)}}\right)_{i}: \mathcal{I}_{L} \rightarrow \mathcal{O}_{E_{v}}^{\times}$by mapping $g \mapsto\left(\hat{\sigma}^{-j(\varphi)}(\chi(g))^{\delta_{i, i(\varphi)}}\right)_{i}=\left(1, \ldots, 1, \varphi^{-1} \circ \chi_{\varphi\left(E_{v, i}\right)}(g), 1, \ldots, 1\right)$.
5.10. Due to the equivariance of the pairing (5.6) under $\mathscr{G}_{L}$ and $\operatorname{End}_{R}\left(\hat{G}_{E_{v}, \varphi}\right)$ the action of $g \in \mathscr{G}_{L}$ on $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}\left(\hat{G}_{E_{v}, \varphi}\right), A_{v}\right)$ is given by the endomorphism $\chi_{E_{v}, \varphi}(g)^{-1}$. We can also compute this action directly as follows. It factors through the restriction of $g$ to $\left.\operatorname{Gal}\left(\mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)\right)_{\infty} / \mathbb{F}_{\tilde{v}_{i}}\left(\left(\varphi\left(y_{i}\right)\right)\right)\right)$ which we denote again by $g$. Then on the basis $\left(\check{u}_{i, j}\right)_{j}$ of $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}(\hat{G}), A_{v}\right)$ from (5.3) we compute

$$
\begin{aligned}
g\left(\check{u}_{i, j}\right)_{j} & =g\left(\hat{\sigma}^{(j, j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)^{-\delta_{i, i(\varphi)}}\right)_{j} \\
& =\left(\hat{\sigma}^{(j, j(\varphi))}\left(\chi(g) \cdot \ell_{y_{i, \varphi},\left(y_{i}\right)}^{+}\right)^{-\delta_{i, i(\varphi)}}\right)_{j} \\
& =\left(\hat{\sigma}^{j-j(\varphi)}\left(\chi(g)^{-1}\right)^{\delta_{i, i(\varphi)}} \cdot \hat{\sigma}^{(j, j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)^{-\delta_{i, i(\varphi)}}\right)_{j} \\
& =\left(\hat{\sigma}^{-j(\varphi)}\left(\chi(g)^{-1}\right) \otimes 1\right)^{\delta_{i, i(\varphi)}} \cdot\left(\check{u}_{i, j}\right)_{j}
\end{aligned}
$$

for the element $\hat{\sigma}^{-j(\varphi)}\left(\chi(g)^{-1}\right) \otimes 1 \in \mathcal{O}_{E_{v, i}} \otimes_{A_{v}} R \llbracket z \rrbracket$. That is, the action of $g \in \mathscr{G}_{L}$ on $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}(\hat{G}), A_{v}\right)$ coincides with the endomorphism $\hat{\sigma}^{-j(\varphi)}\left(\chi(g)^{-1}\right)$ and the action of $g \in \mathcal{I}_{L}$ on $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}\left(\hat{G}_{E_{v}, \varphi}\right), A_{v}\right)=$ $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, A_{v}\right)$ coincides with the endomorphism $\chi_{E_{v}, \varphi}(g)^{-1} \in \mathcal{O}_{E_{v}}^{\times}$.
Proposition 5.11. Let $\underline{\hat{M}}$ have complex multiplication by a commutative, semi-simple $Q_{v}$-algebra $E_{v}$ with local CM-type $\Phi=\left(d_{\varphi}\right)_{\varphi \in H_{E_{v}}}$. Then the action of $g \in \mathcal{I}_{L}$ on $\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right)$ coincides with the endomorphism $\prod_{\varphi \in H_{E_{v}}} \chi_{E_{v}, \varphi}(g)^{-d_{\varphi}} \in \mathcal{O}_{E_{v}}^{\times}$.
Proof. This follows from the computations in 5.45 .10 and 5.2 by observing that $\mathcal{I}_{L}$ acts trivially on $\mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, 0}, A_{v}\right)$, because its generator $\left(c_{i, j}\right)_{i, j}$ is defined over the maximal unramified extension of $L$.
5.12. To compute the absolute value $\left|\int_{u} \omega\right|_{v}$ we again treat each factor $\hat{\underline{M}}_{E_{v}, 0}$ and $\underline{\underline{M}}_{E_{v}, \varphi}$ of $\underline{\hat{M}}$ separately. We begin with $\underline{\hat{M}}_{E_{v}, \varphi}$ and set $i:=i(\varphi)$. Let $\omega_{\psi}^{\circ}:=1 \in \mathrm{H}^{\psi}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$. It is a generator as $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-module as in Definition 4.10, and is mapped under the period isomorphism of $\underline{\hat{M}}_{E_{v}, \varphi}$ from (5.4) to

$$
\begin{equation*}
h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}^{\circ}\right)=\left(0, \ldots,\left(\left(y_{i(\psi)}-\varphi\left(y_{i(\psi)}\right)\right)^{\delta_{j(\psi), j(\varphi)}} \cdot \hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i(\psi)}, \varphi\left(y_{i(\psi)}\right)}\right)\right)^{\delta_{i(\psi), i(\varphi)}}, \ldots, 0\right) \cdot \check{u}, \tag{5.7}
\end{equation*}
$$

where the non-zero entry is in component $\psi$. We denote this entry by $\Omega\left(E_{v}, \varphi, \psi\right)$. It is analogous to Colmez's Col93, Théorème I.2.1] element of $\mathbf{B}_{\mathrm{dR}}$ with the same name. It satisfies the following

Theorem 5.13. Let $\varphi, \psi \in H_{E_{v}}$ satisfy $i(\varphi)=i(\psi)=: i$ and assume that $E_{v, i}$ is separable over $Q_{v}$. Then the element

$$
\Omega\left(E_{v}, \varphi, \psi\right):=\left(y_{i}-\varphi\left(y_{i}\right)\right)^{\delta_{j(\psi), j(\varphi)}} \cdot \hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right) \in \mathbb{C}_{v}\left(\left(y_{i}-\psi\left(y_{i}\right)\right)\right)=\mathbb{C}_{v}((z-\zeta))
$$

satisfies
(a) $\hat{v}\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=1$ if $\varphi=\psi$ and $\hat{v}\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=0$ if $\varphi \neq \psi$.
(b)

$$
v\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)= \begin{cases}\frac{1}{e_{i}\left(\tilde{q}_{i}-1\right)}-v\left(\mathfrak{D}_{\left.\psi\left(E_{v, i}\right) / Q_{v}\right)}\right. & \text { if } \varphi=\psi, \\ \frac{1}{e_{i}\left(\tilde{q}_{i}-1\right)}+v\left(\psi\left(y_{i}\right)-\varphi\left(y_{i}\right)\right) & \text { if } \varphi \neq \psi \text { and } j(\varphi)=j(\psi), \\ \frac{q_{v}^{(j(\psi), j(\varphi))}}{e_{i}\left(\tilde{( }_{i}-1\right)} & \text { if } j(\varphi) \neq j(\psi),\end{cases}
$$

where $\mathfrak{D}_{\psi\left(E_{v, i}\right) / Q_{v}}$ is the different of $\psi\left(E_{v, i}\right)$ over $Q_{v}$.
(c) If $g \in \mathcal{I}_{L}$, then $g\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=\psi\left(\chi_{E_{v}, \varphi}(g)\right) \cdot \Omega\left(E_{v}, \varphi, \psi\right)$. Note that if $L$ is separable over $Q_{v}$ then $\psi\left(\chi_{E_{v}, \varphi}(g)\right)=\psi\left(\varphi^{-1}\left(N_{L / \varphi\left(E_{v, i}\right)} \chi_{L}(g)\right)\right)$.
(d) Let $u \in \mathrm{H}_{1, v}\left(\underline{\underline{\hat{M}}}_{E_{v}, \varphi}, A_{v}\right):=\operatorname{Hom}_{A_{v}}\left(\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, A_{v}\right), A_{v}\right)$ be a generator as $\mathcal{O}_{E_{v}}$-module and let $\omega_{\psi}^{\circ}$ be an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator of $\mathrm{H}^{\psi}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$ subject to the conditions in Definition 4.10, that is subject to $\omega_{\psi}^{\circ} \bmod y_{i}-\varphi\left(y_{i}\right) \in \mathrm{H}_{v}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L\right)$ being an $R$-generator of the free $R$-module of rank one $\mathrm{H}^{\psi}(\underline{\hat{M}}, R) /\left(y_{i}-\psi\left(y_{i}\right)\right) \mathrm{H}^{\psi}(\underline{\hat{M}}, R)$. Moreover, let $D_{\psi}$ be a generator as $\psi\left(\mathcal{O}_{E_{v, i}}\right)$-module of the different $\mathfrak{D}_{\psi\left(E_{v, i}\right) / Q_{v}}$. Then

$$
\int_{u} \omega_{\psi}^{\circ}:=u \otimes \operatorname{id}_{\mathbb{C}_{v}((z-\zeta))}\left(h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}^{\circ}\right)\right) \in \mathbb{C}_{v}((z-\zeta))
$$

equals $\Omega\left(E_{v}, \varphi, \psi\right) \cdot D_{\psi}^{-1}$ up to multiplication by an element of $R^{\times}+(z-\zeta) \cdot L \llbracket z-\zeta \rrbracket$.
Remark 5.14. Note that in contrast to the number field case Col93, Théorème I.2.1] the element $\Omega\left(E_{v}, \varphi, \psi\right) \in \mathbb{C}_{v}((z-\zeta))$ is by (a), (b) and (c) uniquely determined only up to multiplication by an element of $\mathcal{O}_{\widetilde{L}}^{\times}+(z-\zeta) \cdot \widetilde{L} \llbracket z-\zeta \rrbracket$, where $\widetilde{L}$ is the completion of the compositum of $\mathbb{F}_{q}^{\text {alg }}$ with the perfect closure of $L$ in $Q_{v}^{\text {alg }}$, because the fixed field of $\mathcal{I}_{L}$ in $\mathbb{C}_{v}((z-\zeta))$ equals $\widetilde{L}((z-\zeta))$ by the Ax-Sen-Tate theorem Ax70.

Proof of Theorem 5.13. In 5.3 we have seen that the coefficients of the series $\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}=\sum_{n=0}^{\infty} \ell_{n} y_{i}^{n}$ satisfy $v\left(\ell_{n}\right)=v\left(\varphi\left(y_{i}\right)\right) \cdot \tilde{q}_{i}^{-n} /\left(\tilde{q}_{i}-1\right)$. From $v\left(\psi\left(y_{i}\right)\right)=1 / e_{i}=v\left(\varphi\left(y_{i}\right)\right)$ it follows that the evaluation of $\left.\hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right|_{y_{i}=\psi\left(y_{i}\right)}=\sum_{n=0}^{\infty} \ell_{n}^{q_{v}^{(j(\psi), j(\varphi))}} \psi\left(y_{i}\right)^{n}$ at $y_{i}=\psi\left(y_{i}\right)$ satisfies

$$
\begin{equation*}
v\left(\ell_{n}^{q_{0}^{(j(\psi), j(\varphi))}} \psi\left(y_{i}\right)^{n}\right)=\frac{1}{e_{i}} \cdot\left(n+\frac{q_{v}^{(j(\psi), j(\varphi))}}{\tilde{q}_{i}^{n}\left(\tilde{q}_{i}-1\right)}\right) . \tag{5.8}
\end{equation*}
$$

Since $0 \leq(j(\psi), j(\varphi)) \leq f_{i}-1$ the second fraction in the parenthesis is strictly smaller than 1 , and so the valuations in (5.8) are strictly increasing with $n$ and attain their minimum $\frac{q^{(j(\psi)), j(\varphi))}}{e_{i}\left(\tilde{q}_{i}-1\right)}$ for $n=0$. This shows that $\left.\hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right|_{y_{i}=\psi\left(y_{i}\right)}$ is non-zero in $L$ and

$$
v\left(\left.\hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right|_{y_{i}=\psi\left(y_{i}\right)}\right)=\frac{q_{v}^{(j(\psi), j(\varphi))}}{e_{i}\left(\tilde{q}_{i}-1\right)} .
$$

In particular the valuation $\hat{v}\left(\hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right)=0$.
(a) Lemma A. 1 implies that in the $\psi$-component of $E_{v} \otimes_{Q_{v}} L \llbracket z-\zeta \rrbracket$ we have $\operatorname{ord}_{y_{i}-\psi\left(y_{i}\right)}=\operatorname{ord}_{z-\zeta}$. If $j(\varphi)=j(\psi)$, that is $\left.\varphi\right|_{\mathbb{F}_{\tilde{v}_{i}}}=\left.\psi\right|_{\mathbb{v}_{\tilde{v}_{i}}}$ then $\varphi \neq \psi$ implies $\psi\left(y_{i}\right)-\varphi\left(y_{i}\right) \neq 0$ in $L$, because $E_{v, i}=\mathbb{F}_{\tilde{v}_{i}}\left(\left(y_{i}\right)\right)$. Therefore the valuation $\hat{v}$ of $y_{i}-\varphi\left(y_{i}\right)=\left(\psi\left(y_{i}\right)-\varphi\left(y_{i}\right)\right)+\left(y_{i}-\psi\left(y_{i}\right)\right)$ equals zero for $\varphi \neq \psi$ and $j(\varphi)=j(\psi)$. This implies (a).
(b) We will calculate $v\left(\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)\right.$ in three different cases separately as follows.

Case1: $\psi=\varphi$. In this case $\hat{v}\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=1$ and so

$$
\begin{aligned}
v\left(\Omega\left(E_{v}, \varphi, \psi\right)\right) & =v\left(\left.\left(\frac{y_{i}-\varphi\left(y_{i}\right)}{z-\zeta} \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right|_{y_{i}=\varphi\left(y_{i}\right)}\right) \\
& =v\left(\left.\frac{y_{i}-\varphi\left(y_{i}\right)}{z-\zeta}\right|_{y_{i}=\varphi\left(y_{i}\right)}\right)+v\left(\left.\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right|_{y_{i}=\psi\left(y_{i}\right)}\right) \\
& =-v\left(\mathfrak{D}_{\psi\left(E_{v, i}\right) / Q_{v}}\right)+\frac{1}{e_{i}\left(\tilde{q}_{i}-1\right)}
\end{aligned}
$$

by Corollary 4.7

Case 2: $\psi \neq \varphi$ and $j(\psi)=j(\varphi)$. In this case $\hat{v}\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=0$ and so

$$
\begin{aligned}
v\left(\Omega\left(E_{v}, \varphi, \psi\right)\right) & =v\left(\left.\left(\left(y_{i}-\varphi\left(y_{i}\right)\right) \cdot \ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right|_{y_{i}=\psi\left(y_{i}\right)}\right) \\
& =v\left(\psi\left(y_{i}\right)-\varphi\left(y_{i}\right)\right)+v\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+} \mid y_{i}=\psi\left(y_{i}\right)\right) \\
& =v\left(\psi\left(y_{i}\right)-\varphi\left(y_{i}\right)\right)+\frac{1}{e_{i}\left(\tilde{q}_{i}-1\right)} .
\end{aligned}
$$

Case 3: $j(\psi) \neq j(\varphi)$. In this case $\hat{v}\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=0$ and so

$$
v\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=v\left(\left.\hat{\sigma}^{(j(\psi), j(\varphi))}\left(\ell_{y_{i}, \varphi\left(y_{i}\right)}^{+}\right)\right|_{y_{i}=\psi\left(y_{i}\right)}\right)=\frac{q_{v}^{(j(\psi), j(\varphi))}}{e_{i}\left(\tilde{q}_{i}-1\right)} .
$$

(c) For the $\mathcal{O}_{E_{v}}$-basis $\check{u}$ of $\mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, \varphi}, A_{v}\right)$ from (5.3) we have seen in 5.10 that $g(\check{u})=\chi_{E_{v}, \varphi}(g)^{-1} \cdot \check{u}$ and $g\left(0, \ldots, \Omega\left(E_{v}, \varphi, \psi\right), \ldots, 0\right) \cdot g(\breve{u})=h_{v, \mathrm{dR}}^{-1}\left(g\left(\omega_{\psi}^{\circ}\right)\right)=h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}^{\circ}\right)=\left(0, \ldots, \Omega\left(E_{v}, \varphi, \psi\right), \ldots, 0\right) \cdot \breve{u}$. Thus $g$ acts on the coefficient $\left(0, \ldots, \Omega\left(E_{v}, \varphi, \psi\right), \ldots, 0\right)$ as multiplication with $\chi_{E_{v}, \varphi}(g)$ and on its $\psi$-component $\Omega\left(E_{v}, \varphi, \psi\right)$ by multiplication with $\psi\left(\chi_{E_{v}, \varphi}(g)\right)$.
(d) Again we consider the $\mathcal{O}_{E_{v}}$-basis $\check{u}$ of $\mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, \varphi}, A_{v}\right)$ from (5.3). Let $D=\left(D_{i}\right)_{i} \in \mathcal{O}_{E_{v}}=\prod_{i} \mathcal{O}_{E_{v, i}}$ be a generator of the different $\mathfrak{D}_{E_{v} / Q_{v}}=\prod_{i} \mathfrak{D}_{E_{v, i} / Q_{v}}=D \cdot \mathcal{O}_{E_{v}}$ and let $c=\left(c_{i}\right)_{i} \in \mathcal{O}_{E_{v}}^{\times}=\prod_{i} \mathcal{O}_{E_{v, i}}^{\times}$be the element(s) from Lemma 5.15 below for which the pairing $\langle.,\rangle:. \mathrm{H}_{1, v}\left(\hat{\underline{M}}_{E_{v}, \varphi}, A_{v}\right) \times \mathrm{H}_{v}^{1}\left(\underline{\underline{M}}_{E_{v}, \varphi}, A_{v}\right) \rightarrow A_{v}$ takes the value $\langle a u, b \check{u}\rangle=\operatorname{Tr}_{E_{v} / Q_{v}}\left(a b c D^{-1}\right)$ for $a, b \in \mathcal{O}_{E_{v}}$. If $\omega_{\psi}^{\circ}=1$ is the generator from 5.12 then

$$
\int_{u} \omega_{\psi}^{\circ}=\operatorname{Tr}_{E_{v} / Q_{v}}\left(0, \ldots, \Omega\left(E_{v}, \varphi, \psi\right) \cdot \psi\left(c D^{-1}\right), \ldots, 0\right)=\Omega\left(E_{v}, \varphi, \psi\right) \cdot \psi\left(c_{i} D_{i}^{-1}\right)
$$

Any other generator $\omega_{\psi}^{\circ}$ differs from $\omega_{\psi}^{\circ}=1$ by multiplication by an element

$$
x \in R^{\times}+\left(y_{i}-\varphi\left(y_{i}\right)\right) \cdot L \llbracket y_{i}-\varphi\left(y_{i}\right) \rrbracket \subset E_{v} \otimes_{Q_{v}} L \llbracket y_{i}-\varphi\left(y_{i}\right) \rrbracket .
$$

Under the pairing $\langle.,$.$\rangle this leads to \int_{u} \omega_{\psi}^{\circ}=\Omega\left(E_{v}, \varphi, \psi\right) \cdot \psi\left(c_{i} x D_{i}^{-1}\right)$ with $\psi\left(D_{i}\right)=D_{\psi}$ and

$$
\psi\left(c_{i} x\right) \in R^{\times}+\left(y_{i}-\varphi\left(y_{i}\right)\right) \cdot L \llbracket y_{i}-\varphi\left(y_{i}\right) \rrbracket=R^{\times}+(z-\zeta) \cdot L \llbracket z-\zeta \rrbracket .
$$

It remains to record the following well known
Lemma 5.15. If $E_{v, i} / Q_{v}$ is separable let $D_{i} \in \mathcal{O}_{E_{v, i}}$ be a generator of the different $\mathfrak{D}_{E_{v, i} / Q_{v}}=D_{i} \cdot \mathcal{O}_{E_{v, i}}$. Then for any perfect pairing $\langle.,\rangle:. \mathcal{O}_{E_{v, i}} \times \mathcal{O}_{E_{v, i}} \rightarrow A_{v}$ satisfying $\langle a, b\rangle=\langle a b, 1\rangle=\langle 1, a b\rangle$, there is an element $c_{i} \in \mathcal{O}_{E_{v, i}}^{\times}$with $\langle a, b\rangle=\operatorname{Tr}_{E_{v, i} / Q_{v}}\left(a b c_{i} D_{i}^{-1}\right)$.
Proof. The set of bilinear forms $\mathcal{O}_{E_{v, i}} \times \mathcal{O}_{E_{v, i}} \rightarrow A_{v}$ equals $\operatorname{Hom}_{A_{v}}\left(\mathcal{O}_{E_{v, i}} \otimes_{A_{v}} \mathcal{O}_{E_{v, i}}, A_{v}\right)$ and the condition $\langle a, b\rangle=\langle a b, 1\rangle=\langle 1, a b\rangle$ implies that $\langle.,$.$\rangle lies in \operatorname{Hom}_{A_{v}}\left(\mathcal{O}_{E_{v, i}} \otimes_{\mathcal{O}_{E_{v, i}}} \mathcal{O}_{E_{v, i}}, A_{v}\right)=\operatorname{Hom}_{A_{v}}\left(\mathcal{O}_{E_{v, i}}, A_{v}\right)$. The condition that $\langle.,$.$\rangle is perfect implies that$

$$
\begin{equation*}
\mathcal{O}_{E_{v, i}} \xrightarrow{\sim} \operatorname{Hom}_{A_{v}}\left(\mathcal{O}_{E_{v, i}}, A_{v}\right), \quad a \longmapsto[b \mapsto\langle a, b\rangle] \tag{5.9}
\end{equation*}
$$

is an isomorphism of $\mathcal{O}_{E_{v, i}}$-modules. On the other hand, by definition of the different in [Ser79, § III.3], there are also isomorphisms of $\mathcal{O}_{E_{v, i}}-$ modules

$$
\mathfrak{D}_{E_{v, i} / Q_{v}}^{-1}=D_{i}^{-1} \cdot \mathcal{O}_{E_{v, i}} \xrightarrow{\sim} \operatorname{Hom}_{A_{v}}\left(\mathcal{O}_{E_{v, i}}, A_{v}\right), \quad \tilde{a} D_{i}^{-1} \longmapsto\left[b \mapsto \operatorname{Tr}_{E_{v, i} / Q_{v}}\left(\tilde{a} b D_{i}^{-1}\right)\right]
$$

for $\tilde{a} \in \mathcal{O}_{E_{v, i}}$, and

$$
\begin{equation*}
\mathcal{O}_{E_{v, i}} \xrightarrow{\sim} \operatorname{Hom}_{A_{v}}\left(\mathcal{O}_{E_{v, i}}, A_{v}\right), \quad \tilde{a} \longmapsto\left[b \mapsto \operatorname{Tr}_{E_{v, i} / Q_{v}}\left(\tilde{a} b D_{i}^{-1}\right)\right] . \tag{5.10}
\end{equation*}
$$

Comparing (5.9) and (5.10) yields a unit $c_{i} \in \mathcal{O}_{E_{v, i}}^{\times}$with $\tilde{a}=c_{i} a$ and $\langle a, b\rangle=\operatorname{Tr}_{E_{v, i} / Q_{v}}\left(a b c_{i} D_{i}^{-1}\right)$.
5.16. Analogously to Col93, §I.2], we can give a uniform formula for $v\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)$ by introducing certain measures on $\mathscr{G}_{Q_{v}}$. Let $\mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ be the $\mathbb{Q}$-vector space of locally constant functions $a: \mathscr{G}_{Q_{v}} \rightarrow \mathbb{Q}$. If $K$ is a finite separable extension of $Q_{v}$, and $\varphi, \psi \in H_{K}$, let $a_{K, \varphi, \psi} \in \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ be the function given by

$$
a_{K, \varphi, \psi}(g):= \begin{cases}1 & \text { if } \quad g \varphi=\psi \\ 0 & \text { otherwise }\end{cases}
$$

Note that the $a_{K, \varphi, \psi} \operatorname{span} \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$.
If $L \subset Q_{v}^{\text {sep }}$ is a finite Galois extension of $Q_{v}$, let $\mu_{L} \in \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ be the function given by the formula

$$
\mu_{L}(g):= \begin{cases}0 & \text { if } g \notin \mathcal{I}_{Q_{v}}, \\ -v\left(g\left(\pi_{L}\right)-\pi_{L}\right) & \text { if } g \in \mathcal{I}_{Q_{v}} \text { and } g\left(\pi_{L}\right) \neq \pi_{L}, \\ v\left(\mathfrak{D}_{L / Q_{v}}\right) & \text { if } g \in \mathcal{I}_{Q_{v}} \text { and } g\left(\pi_{L}\right)=\pi_{L},\end{cases}
$$

where $\pi_{L}$ is a uniformizer of $L$ and $\mathfrak{D}_{L / Q_{v}}$ is the different of $L$ over $Q_{v}$.
Moreover, we let $e_{K}$ be the ramification index of $K$ over $Q_{v}$ and $f_{K}$ the degree of the residue field of $K$ over $\mathbb{F}_{v}$. We let $W_{L}^{n}:=\left\{g \in \mathscr{G}_{Q_{v}}: g(x) \equiv x^{q_{v}^{n}} \bmod \mathfrak{m}_{Q_{v}^{\text {alg }}} \forall x \in \mathcal{O}_{Q_{v}^{\text {alg }}}\right\} / \mathcal{I}_{L}$. It is in bijection with $\left\{g \in \operatorname{Gal}\left(L / Q_{v}\right): g(x) \equiv x^{q_{v}^{n}} \bmod \left(\pi_{L}\right)\right\}$ under the map $\mathscr{G}_{Q_{v}} \rightarrow \operatorname{Gal}\left(L / Q_{v}\right)$.
Lemma 5.17. Let $K, L \subset Q_{v}^{\text {sep }}$ be finite separable extensions of $Q_{v}$ with $L$ finite Galois over $Q_{v}$ containing all the conjugates of $K$, and let $\varphi, \psi \in H_{K}$. The function $a_{K, \varphi, \psi}$ is constant modulo $\mathscr{G}_{L}$ and hence can be considered as a function on $G_{L}:=\operatorname{Gal}\left(L / Q_{v}\right)$. Then

$$
\sum_{g \in G_{L}} a_{K, \varphi, \psi}(g) \cdot \mu_{L}(g)= \begin{cases}0 & \text { if } j(\varphi) \neq j(\psi) \\ v\left(\mathfrak{D}_{\psi(K) / Q_{v}}\right) & \text { if } \varphi=\psi, \\ -v\left(\psi\left(\pi_{K}\right)-\varphi\left(\pi_{K}\right)\right) & \text { if } j(\varphi)=j(\psi) \text { and } \varphi \neq \psi\end{cases}
$$

and

$$
\frac{1}{e_{L}} \sum_{n=1}^{\infty} \sum_{g \in W_{L}^{n}} \frac{a_{K, \varphi, \psi}(g)}{q_{v}^{n s}}=\frac{1}{e_{K}} \frac{q_{v}^{(j(\psi), j(\varphi)) s}}{q_{v}^{f_{K^{s}}}-1} .
$$

In particular, the left hand side of both equations does not depend on the choice of $L$.
Proof. The proof follows in the same way as [Col93, Lemma I.2.4].
Since the $a_{K, \varphi, \psi}$ generate the vector space $\mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$, we get the following proposition.
Proposition 5.18. There exist $\mathbb{Q}$-linear homomorphisms $Z_{v}(., s): \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right) \rightarrow \mathbb{C}$ if $s \in \mathbb{C}$ and $\mu_{\mathrm{Art}, v}: \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right) \rightarrow \mathbb{Q}$ defined by the following formulas: if $a \in \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ and if $L \subset Q_{v}^{\text {sep }}$ is a finite Galois extension of $Q_{v}$ such that $a$ is constant modulo $\mathscr{G}_{L}$, then

$$
\mu_{\mathrm{Art}, v}(a)=\sum_{g \in G_{L}} a(g) \cdot \mu_{L}(g)
$$

with $G_{L}:=\operatorname{Gal}\left(L / Q_{v}\right)$, and $Z_{v}(a, s)$ is obtained by meromorphic extension from the following formula, valid for $\operatorname{Re}(s)>0$ :

$$
Z_{v}(a, s)=\frac{1}{e_{L}} \sum_{n=1}^{\infty} \sum_{g \in W_{L}^{n}} \frac{a(g)}{q_{v}^{n s}}
$$

Remark 5.19. If $V$ is a finite dimensional $\mathbb{C}$-vector space, $\rho: \mathscr{G}_{Q_{v}} \rightarrow \operatorname{Aut}_{\mathbb{C}}(V)$ is a continuous complex representation of $\mathscr{G}_{Q_{v}}$, and if $\chi \in \mathcal{C}^{0}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right) \otimes_{\mathbb{Q}} \mathbb{C}$ is the character of $\rho$, then $\mu_{\mathrm{Art}, v}(\chi)$ is nothing else than the degree at $v$ of the conductor $\mathfrak{f}_{\chi}$ of $\chi$; cf. [Ser79, Chapter VI, § 2], where $\mu_{\operatorname{Art}, v}(\chi)$ is denoted by $f(\chi)$. And if $W$ is the sub vector space of $V$ stable by $I_{Q_{v}}$, we have

$$
Z_{v}(\chi, a) \log q_{v}=-\frac{d}{d s} \log \left(\operatorname{det}\left(1-\left.q_{v}^{-s} \rho\left(\operatorname{Frob}_{L / Q_{v}}\right)\right|_{W}\right)^{-1}\right)
$$

by Tat84, Chapter 0, §4] or Ros02, Lemma 9.14]. So the linear maps $\mu_{\text {Art, } v}$ and $Z_{v}(., s)$ coincide with the maps with the same names in Definition 1.2 in the introduction.

As a direct consequence of Theorem 5.13, Proposition 5.18 and Lemma 5.17 we get the following
Theorem 5.20. If $\varphi, \psi \in H_{E_{v}}$ satisfy $i(\varphi)=i(\psi)=: i$ and $E_{v, i}$ is separable over $Q_{v}$ then

$$
v\left(\Omega\left(E_{v}, \varphi, \psi\right)\right)=Z_{v}\left(a_{E_{v}, \psi, \varphi}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E_{v}, \psi, \varphi}\right)
$$

where we set $a_{E_{v}, \psi, \varphi}:=a_{E_{v, i}, \psi, \varphi}: g \mapsto \delta_{g \psi, \varphi}$.
Definition 5.21. If $u \in \mathrm{H}_{1, v}\left(\underline{\hat{M}}, Q_{v}\right):=\operatorname{Hom}_{A_{v}}\left(\mathrm{H}_{v}^{1}\left(\underline{\hat{M}}, A_{v}\right), Q_{v}\right)$ is an $E_{v}$-generator there is an $a \in E_{v}^{\times}$, unique up to multiplication with an element of $\mathcal{O}_{E_{v}}^{\times}$, such that $a^{-1} u$ is an $\mathcal{O}_{E_{v}}$-generator of $\mathrm{H}_{1, v}\left(\underline{\hat{M}}, A_{v}\right)$. Then we define the valuation $v_{\psi}(u):=v(\psi(a)) \in \mathbb{Z}$. (NOTE THAT $v_{\psi}(u) \in \mathbb{Q}$ IN GENERAL; SEE ERRATUM B. 1 )

Note that if $\underline{M}=\left(M, \tau_{M}\right)$ is a uniformizable $A$-motive over $L$ with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}}=\underline{\hat{M}} \underline{V}_{v}(\underline{\mathcal{M}})$ is the local shtuka at $v$ associated with $\underline{\mathcal{M}}$ as in Example 3.2, then for an $E$-generator $u \in \mathrm{H}_{1, \operatorname{Betti}}(\underline{M}, Q)$ the present definition of $v_{\psi}\left(h_{\mathrm{Betti}, v}(u)\right)$ coincides with the definition of $v_{\psi}(u)$ from (1.12).

Corollary 5.22. Let $\varphi, \psi \in H_{E_{v}}$ with $i(\varphi)=i(\psi)=: i$ and assume that $E_{v, i}$ is separable over $Q_{v}$. Let $u \in \mathrm{H}_{1, v}\left(\underline{\hat{M}}_{E_{v}, \varphi}, Q_{v}\right)$ be a generator as $E_{v}$-module and let $\omega_{\psi}$ be an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator of $\mathrm{H}^{\psi}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$. Then $\int_{u} \omega_{\psi}:=u \otimes \operatorname{id}_{\mathbb{C}_{v}((z-\zeta))}\left(h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}\right)\right)$ has valuation

$$
v\left(\int_{u} \omega_{\psi}\right)=Z_{v}\left(a_{E_{v}, \psi, \varphi}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \psi, \varphi}\right)-v\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)+v\left(\omega_{\psi}\right)+v_{\psi}(u)
$$

where $v\left(\omega_{\psi}\right)$ and $v_{\psi}(u)$ were defined in Definitions 4.10 and 5.21, and $\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}$ is the different.
Proof. Let $a \in E_{v}$ be such that $u^{\circ}:=a^{-1} u$ is an $\mathcal{O}_{E_{v}}$-generator of $\mathrm{H}_{1, v}\left(\underline{\hat{M}}_{E_{v}, \varphi}, A_{v}\right)$ and let $\omega_{\psi}^{\circ}$ be an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator of $\mathrm{H}^{\psi}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$ such that $\omega_{\psi}^{\circ} \bmod y_{i}-\varphi\left(y_{i}\right) \in \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L\right)$ is an $R$-generator of $\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{E_{v}, \varphi}, R\right)$. Let $x \in L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket^{\times}$such that $\omega_{\psi}=x \cdot \omega_{\psi}^{\circ}$. Moreover, let $D_{\psi}$ be a generator as $\psi\left(\mathcal{O}_{E_{v}}\right)$-module of the different $\mathfrak{D}_{\psi\left(E_{v, i}\right) / Q_{v}}$. Then

$$
\int_{u} \omega_{\psi}=(a \otimes 1) x \cdot \int_{u^{\circ}} \omega_{\psi}^{\circ}=(a \otimes 1) x \cdot \Omega\left(E_{v}, \varphi, \psi\right) \cdot D_{\psi}^{-1} \in \mathbb{C}_{v}((z-\zeta))
$$

up to multiplication by an element of $R^{\times}+(z-\zeta) \cdot L \llbracket z-\zeta \rrbracket$ by Theorem 5.13)(d). The element $(a \otimes 1) x \in E_{v} \otimes_{Q_{v}} L \llbracket z-\zeta \rrbracket$ lies in the $\psi$-component $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$ of the product decomposition (A.1), and in that component $a \otimes 1$ is congruent to $\psi(a)$ modulo $y_{i}-\psi\left(y_{i}\right)$. Therefore
$v\left(\int_{u} \omega_{\psi}\right)=v\left(\psi(a) x \cdot \Omega\left(E_{v}, \varphi, \psi\right) \cdot D_{\psi}^{-1}\right)=Z_{v}\left(a_{E_{v}, \psi, \varphi}, 1\right)-\mu_{\text {Art }, v}\left(a_{E_{v}, \psi, \varphi}\right)-v\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)+v\left(\omega_{\psi}\right)+v_{\psi}(u)$.
by Theorem 5.20 .
To compute $v\left(\int_{u} \omega_{\psi}\right)$ for general $\underline{\hat{M}}$ we need the following

Definition 5.23. Let $E_{v}$ be separable over $Q_{v}$ and let $\Phi=\left(d_{\varphi}\right)_{\varphi \in H_{E_{v}}}$ be a local CM-type. For $\psi \in H_{E_{v}}$ let $a_{E_{v}, \psi, \Phi} \in \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ and $a_{E_{v}, \psi, \Phi}^{0} \in \mathcal{C}^{0}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ be given by the formulas

$$
\begin{align*}
& a_{E_{v}, \psi, \Phi}(g):=\sum_{\varphi \in H_{E_{v}}} d_{\varphi} \cdot a_{E_{v}, \psi, \varphi}(g)=d_{g \psi} \quad \text { and }  \tag{5.11}\\
& a_{E_{v}, \psi, \Phi}^{0}(g):=\frac{1}{\# H_{L}} \sum_{\eta \in H_{L}} d_{\eta^{-1} g \eta \psi} . \tag{5.12}
\end{align*}
$$

Note that $a_{E_{v}, \psi, \Phi}$ and $a_{E_{v}, \psi, \Phi}^{0}$ factor through $\operatorname{Gal}\left(E_{v}^{\text {nor }} / Q_{v}\right)$ where $E_{v}^{\text {nor }}$ is the Galois closure of $\psi\left(E_{v}\right)$ in $Q_{v}^{\text {alg }}$. In particular, $a_{E_{v}, \psi, \Phi}^{0}$ does not depend on the field $L$ provided $\psi\left(E_{v}\right) \subset L$ for all $\psi \in H_{E_{v}}$.

These functions are the local counterparts to the functions $a_{E, \psi, \Phi} \in \mathcal{C}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ and $a_{E, \psi, \Phi}^{0} \in \mathcal{C}^{0}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ which were defined in (1.10) and (1.11). The membership in $\mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$, respectively $\mathcal{C}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$ is indicated by the index which gives reference to the $Q_{v}$-algebra $E_{v}$, respectively the $Q$-algebra $E$. In fact, if $E_{v}=E \otimes_{Q} Q_{v}$ and hence $H_{E_{v}}=H_{E}$, then $a_{E_{v}, \psi, \Phi}$ is equal to the image of $a_{E, \psi, \Phi}$ under the map $\mathcal{C}\left(\mathscr{G}_{Q}, \mathbb{Q}\right) \rightarrow \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ from Definition [1.2, However, this is in general not true for $a_{E_{v}, \psi, \Phi}^{0}$ and $a_{E, \psi, \Phi}^{0}$, because if $L$ is the closure of $K$ in $Q_{v}^{\text {alg }}$, then $H_{L}$ is in general strictly contained in $H_{K}$.

For general $\underline{\hat{M}}$ we can now prove the following
Theorem 5.24. Let $\underline{\hat{M}}$ be a local shtuka over $R$ with complex multiplication by the ring of integers $\mathcal{O}_{E_{v}}$ in a commutative, semi-simple, separable $Q_{v}$-algebra $E_{v}$ with local CM-type $\Phi$, and assume that $\psi\left(E_{v}\right) \subset L$ for all $\psi \in H_{E_{v}}$ and that $L$ is separable over $Q_{v}$. Let $u \in \mathrm{H}_{1, v}\left(\underline{\hat{M}}, Q_{v}\right)$ be an $E_{v}$-generator and let $\omega_{\psi}$ be an $L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket$-generator of $\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)$. Then the period $\int_{u} \omega_{\psi}:=$ $u \otimes \operatorname{id}_{\mathbb{C}_{v}((z-\zeta))}\left(h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}\right)\right)$ has valuation

$$
v\left(\int_{u} \omega_{\psi}\right)=Z_{v}\left(a_{E_{v}, \psi, \Phi}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \psi, \Phi}\right)-v\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)+v\left(\omega_{\psi}\right)+v_{\psi}(u),
$$

where $v\left(\omega_{\psi}\right)$ and $v_{\psi}(u)$ were defined in Definitions 4.10 and 5.21, and $\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}$ is the different.
Proof. As in 5.1 the local shtuka $\underline{\hat{M}}$ is isomorphic to the tensor product $\underline{\hat{M}}_{E_{v}, 0} \otimes \bigotimes_{\varphi} \underline{\hat{M}}_{E_{v}, \varphi} \otimes d_{\varphi}$ over $\mathcal{O}_{E_{v}, R}$. Let $i:=i(\psi)$ and $j:=j(\psi)$. For every $\underline{\hat{M}}_{E_{v}, \varphi}$ we fix the $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator $\omega_{\psi, \varphi}^{\circ}:=1 \in$ $\mathrm{H}^{\psi}\left(\underline{\underline{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$. In addition, we let $\omega_{\psi, 0}^{\circ}:=1 \in \mathrm{H}^{\psi}\left(\underline{\underline{\hat{M}}}_{E_{v}, 0}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$. Then we can take the tensor product $\omega_{\psi}^{\circ}:=\omega_{\psi, 0}^{\circ} \otimes \underset{\varphi \in H_{E_{v}}}{\bigotimes}\left(\omega_{\psi, \varphi}^{\circ}\right)^{\otimes d_{\varphi}}$ in

$$
\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right) \cong \mathrm{H}^{\psi}\left(\underline{\hat{M}}_{E_{v}, 0}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right) \otimes \bigotimes_{\varphi \in H_{E_{v}}} \mathrm{H}^{\psi}\left(\underline{\hat{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)^{\otimes d_{\varphi}} .
$$

It is an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator as in Definition 4.10. Let $x \in L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket^{\times}$be such that $\omega_{\psi}=x \cdot \omega_{\psi}^{\circ}$, and let further $a \in E_{v}$ be such that $u^{\circ}:=a^{-1} u$ is an $\mathcal{O}_{E_{v}}$-generator of $\mathrm{H}_{1, v}\left(\underline{\underline{M}}, A_{v}\right)$. Moreover, let $D_{\psi}$ be a generator as $\psi\left(\mathcal{O}_{E_{v}}\right)$-module of the different $\mathfrak{D}_{\psi\left(E_{v, i}\right) / Q_{v}}$. Then (5.5), (5.7) and Lemma 5.15imply that

$$
\int_{u^{\circ}} \omega_{\psi}^{\circ}=D_{\psi}^{-1} \cdot \varepsilon_{i, j} c_{i, j}^{-1} \prod_{\varphi \in H_{E_{v, i}}} \Omega\left(E_{v}, \varphi, \psi\right)^{d_{\varphi}}
$$

up to multiplication by an element of $R^{\times}+(z-\zeta) \cdot L \llbracket z-\zeta \rrbracket$. Since $\varepsilon_{i, j} c_{i, j}^{-1} \in\left(\mathcal{O}_{E_{v}} \otimes_{A_{v}} \mathcal{O}_{\mathbb{C}_{v}} \llbracket z \rrbracket\right)^{\times}$we conclude as in the proof of Corollary 5.22 that $\int_{u} \omega_{\psi}=(a \otimes 1) x \cdot \int_{u^{\circ}} \omega_{\psi}^{\circ}$ and

$$
\begin{aligned}
v\left(\int_{u} \omega_{\psi}\right) & =v\left(D_{\psi}^{-1} \cdot \psi(a) x \cdot \varepsilon_{i, j} c_{i, j}^{-1} \prod_{\varphi \in H_{E_{v, i}}} \Omega\left(E_{v}, \varphi, \psi\right)^{d_{\varphi}}\right) \\
& =v\left(\omega_{\psi}\right)+v_{\psi}(u)-v\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)+\sum_{\varphi \in H_{E_{v, i}}}\left(Z_{v}\left(a_{E_{v}, \psi, \varphi}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \psi, \varphi}\right)\right) \cdot d_{\varphi} \\
& =Z_{v}\left(a_{E_{v}, \psi, \Phi}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E_{v}, \psi, \Phi}\right)-v\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)+v\left(\omega_{\psi}\right)+v_{\psi}(u),
\end{aligned}
$$

because $i(\varphi) \neq i(\psi)$ implies that $a_{E_{v}, \psi, \varphi}(g)=\delta_{g \psi, \varphi}=0$ for all $g \in \mathscr{G}_{Q_{v}}$.
Corollary 5.25. Keep the situation of Theorem 5.24. For every $\eta \in H_{L}$ note that $i(\eta \psi)=i(\psi)$, let $\underline{\underline{M}}^{\eta}$ and $\omega_{\psi}^{\eta} \in \mathrm{H}^{\eta \psi}\left(\underline{\hat{M}}^{\eta}, L \llbracket y_{i(\psi)}-\eta \psi\left(y_{i(\psi)}\right) \rrbracket\right)$ be obtained by extension of scalars via $\eta$, and choose an $E_{v}$-generator $u_{\eta} \in \mathrm{H}_{1, v}\left(\underline{\underline{M}}^{\eta}, Q_{v}\right)$. Then

$$
\frac{1}{\# H_{L}} \sum_{\eta \in H_{L}} v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E_{v}, \psi, \Phi}^{0}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \psi, \Phi}^{0}\right)-\frac{v\left(\mathfrak{d}_{\psi\left(E_{v}\right) / Q_{v}}\right)}{\left[\psi\left(E_{v}\right): Q_{v}\right]}+\frac{1}{\# H_{L}} \sum_{\eta \in H_{L}}\left(v\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right)\right),
$$

where $\mathfrak{d}_{\psi\left(E_{v}\right) / Q_{v}} \subset A_{v}$ is the discriminant of the field extension $\psi\left(E_{v}\right) / Q_{v}$.
Proof. Since $\underline{\hat{M}}^{\eta}$ has complex multiplication by $\mathcal{O}_{E_{v}}$ with local CM-type $\eta \Phi:=\left(d_{\varphi}^{\prime}\right)_{\varphi \in H_{E_{v}}}$ with $d_{\varphi}^{\prime}=$ $d_{\eta^{-1} \varphi}$, Theorem 5.24 implies

$$
\begin{equation*}
v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E_{v}, \eta \psi, \eta \Phi}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \eta \psi, \eta \Phi}\right)-v\left(\mathfrak{D}_{\eta \psi\left(E_{v}\right) / Q_{v}}\right)+v\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right) . \tag{5.13}
\end{equation*}
$$

We sum over all $\eta \in H_{L}$, divide by $\# H_{L}=\left[L: Q_{v}\right]$, and observe that $a_{E_{v}, \eta \psi, \eta \Phi}(g)=d_{g \eta \psi}^{\prime}=d_{\eta^{-1} g \eta \psi}$ and $\mathfrak{D}_{\eta \psi\left(E_{v}\right) / Q_{v}}=\eta\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)$, and hence

$$
\begin{aligned}
\sum_{\eta \in H_{L}} v\left(\mathfrak{D}_{\eta \psi\left(E_{v}\right) / Q_{v}}\right) & =v\left(\prod_{\eta \in H_{L}} \eta\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)\right)=v\left(N_{L / Q_{v}}\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)\right) \\
& =v\left(N_{\psi\left(E_{v}\right) / Q_{v}}\left(N_{L / \psi\left(E_{v}\right)}\left(\mathfrak{D}_{\left.\psi\left(E_{v}\right) / Q_{v}\right)}\right)\right)=\left[L: \psi\left(E_{v}\right)\right] \cdot v\left(\mathfrak{d}_{\psi\left(E_{v}\right) / Q_{v}}\right)\right.
\end{aligned}
$$

This proves the corollary.
Finally we are ready to give the
Proof of Theorem 1.3. The proof proceeds like the one of the previous corollary applied to $\underline{\hat{M}}:=\hat{\hat{M}}_{v}(\underline{\mathcal{M}})$ for a model $\underline{\mathcal{M}}$ of $\underline{M}$ with good reduction. Setting $E_{v}:=E \otimes_{Q} Q_{v}$, still $\underline{\hat{M}}^{\eta}:=\underline{\hat{M}}_{v}\left(\underline{\mathcal{M}}^{\eta}\right)$ has complex multiplication by $\mathcal{O}_{E_{v}}$ with local CM-type $\eta \Phi:=\left(d_{\varphi}^{\prime}\right)_{\varphi \in H_{E}}$ with $d_{\varphi}^{\prime}=d_{\eta^{-1} \varphi}$, where $\eta$ runs over all elements in $H_{K}$. To translate the global situation to the local one, we also use the elements $h_{\text {Betti, } v}\left(u_{\eta}\right) \in$ $\mathrm{H}_{1, v}\left(\underline{\hat{M}}^{\eta}, Q_{v}\right)$, and $\omega_{\psi}^{\eta} \otimes_{K} L \in \mathrm{H}^{\eta \psi}\left(\underline{M}, L \llbracket y_{i(\eta \psi)}-\eta \psi\left(y_{i(\eta \psi)}\right) \rrbracket\right)=\mathrm{H}^{\eta \psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\eta \psi)}-\eta \psi\left(y_{i(\eta \psi)}\right) \rrbracket\right)$. Then $a_{E_{v}, \psi, \Phi} \in \mathcal{C}\left(\mathscr{G}_{Q_{v}}, \mathbb{Q}\right)$ is the image of $a_{E, \psi, \Phi} \in \mathcal{C}\left(\mathscr{G}_{Q}, \mathbb{Q}\right)$, and (5.13) takes the form

$$
\begin{equation*}
v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E, \eta \psi, \eta \Phi}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E, \eta \psi, \eta \Phi}\right)-v\left(\mathfrak{D}_{\eta \psi\left(E_{v}\right) / Q_{v}}\right)+v\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right), \tag{5.14}
\end{equation*}
$$

because the value of $v_{\eta \psi}\left(h_{\mathrm{Betti}, v}\left(u_{\eta}\right)\right)$ from Definition 5.21 used in (5.13), coincides with the value of $v_{\eta \psi}\left(u_{\eta}\right)$ from (1.12). This time we sum over all $\eta \in H_{K}$, divide by $\# H_{K}=[K: Q]$, and observe that $a_{E, \eta \psi, \eta \Phi}(g)=d_{g \eta \psi}^{\prime}=d_{\eta^{-1} g \eta \psi}$, and hence

$$
\begin{equation*}
\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}} Z_{v}\left(a_{E, \eta \psi, \eta \Phi}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E, \eta \psi, \eta \Phi}\right)=Z_{v}\left(a_{E, \psi, \Phi}^{0}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E, \psi, \Phi}^{0}\right) . \tag{5.15}
\end{equation*}
$$

For every place $w$ of the field $\psi(E)$ above $v$ let $\psi(E)_{w}$ be the completion. Then $\psi(E) \otimes_{Q} Q_{v}=$ $\prod_{w \mid v} \psi(E)_{w}$. Via the fixed inclusion $Q^{\text {alg }} \subset Q_{v}^{\text {alg }}$ we consider every $\eta \in H_{K}$ as a morphism $\eta: K \rightarrow$ $Q^{\text {alg }} \subset Q_{v}^{\text {alg }}$. The induced morphism $\eta \otimes \operatorname{id}_{Q_{v}}: K \otimes_{Q} Q_{v} \rightarrow Q_{v}^{\text {alg }}$, when restricted to a morphism $\psi(E) \otimes_{Q} Q_{v} \rightarrow Q_{v}^{\text {alg }}$ factors over $\psi(E)_{w}$ for a unique $w$ which we denote by $w(\eta)$. We set $H_{K, w}:=\{\eta \in$ $\left.H_{K}: w(\eta)=w\right\}$ and consider the map

$$
\begin{equation*}
H_{K, w} \longrightarrow H_{\psi(E)_{w}},\left.\quad \eta \longmapsto\left(\eta \otimes \operatorname{id}_{Q_{v}}\right)\right|_{\psi(E)_{w}} \tag{5.16}
\end{equation*}
$$

The map is surjective, because every element of $H_{\psi(E)_{w}}$ can be restricted to a $Q$-homomorphism $\psi(E) \rightarrow$ $Q^{\text {alg }} \subset Q_{v}^{\text {alg }}$, which extends to a $Q$-homomorphism $\left(\eta: K \rightarrow Q^{\text {alg }}\right) \in H_{K}$ that automatically lies in $H_{K, w}$.

We claim that two elements $\eta, \tilde{\eta} \in H_{K, w}$ have the same image under the map (5.16) if and only if $\tilde{\eta}=\eta \circ \alpha$ for an element $\alpha \in \operatorname{Gal}(K / \psi(E))$. Indeed the latter condition is sufficient, because $\alpha \otimes \operatorname{id}_{Q_{v}}$ induces the identity on $\psi(E)_{w}$. To see that it is necessary let $\eta, \tilde{\eta} \in H_{K, w} \subset H_{K}=\operatorname{Gal}(K / Q)$ have the same image. Then their restrictions to $\psi(E) \subset \psi(E)_{w}$ coincide, and hence $\alpha:=\eta^{-1} \circ \tilde{\eta} \in \operatorname{Gal}(K / Q)$ lies in $\operatorname{Gal}(K / \psi(E))$ as claimed. For every $\eta \in H_{K, w}$ the different $\mathfrak{D}_{\eta \psi\left(E_{v}\right) / Q_{v}}$ equals $\left(\eta \otimes \operatorname{id}_{Q_{v}}\right)\left(\mathfrak{D}_{\psi(E)_{w} / Q_{v}}\right)$ and only depends on the image of $\eta$ under the map (5.16). Therefore, we compute

$$
\begin{aligned}
\sum_{\eta \in H_{K, w}} v\left(\mathfrak{D}_{\eta \psi\left(E_{v}\right) / Q_{v}}\right) & =v\left(\prod_{\eta \in H_{K, w}}\left(\eta \otimes \operatorname{id}_{Q_{v}}\right)\left(\mathfrak{D}_{\psi(E)_{w} / Q_{v}}\right)\right) \\
& =v\left(N_{\psi(E)_{w} / Q_{v}}\left(\mathfrak{D}_{\psi(E)_{w} / Q_{v}}\right)^{\# \operatorname{Gal}(K / \psi(E))}\right)=[K: \psi(E)] \cdot v\left(\mathfrak{d}_{\psi(E)_{w} / Q_{v}}\right) .
\end{aligned}
$$

Summing over all $w \mid v$ and using that $\sum_{w \mid v} v\left(\mathfrak{d}_{\psi(E)_{w} / Q_{v}}\right)=v\left(\mathfrak{d}_{\psi(E) / Q}\right)$ by Ser79, §III.4, Corollary to Proposition 10] we obtain from (5.14) and (5.15)

$$
\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}} v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E, \psi, \Phi}^{0}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E, \psi, \Phi}^{0}\right)-\frac{v\left(\mathfrak{d}_{\psi(E) / Q}\right)}{[\psi(E): Q]}+\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}}\left(v\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right)\right) .
$$

This proves Theorem [1.3,

## A Appendix: Product Decompositions of Certain Rings

In this appendix we establish certain product decompositions for the rings used in this article. We begin with the following

Lemma A.1. Let $k$ be a field and let $z=\sum_{n=0}^{\infty} b_{n} y^{n} \in k \llbracket y \rrbracket$. Let $\psi: k \llbracket y \rrbracket \rightarrow R$ be a ring homomorphism into a $k$-algebra $R$. Then in $k \llbracket y \rrbracket \widehat{\otimes}_{k, \psi} R:=\underset{\underset{\leftarrow}{\lim }}{\lim } \llbracket y \rrbracket /\left(y^{n}\right) \otimes_{k, \psi} R \cong R \llbracket y \rrbracket$ the fraction $\frac{z \otimes 1-1 \otimes \psi(z)}{y \otimes 1-1 \otimes \psi(y)}$ exists and is congruent to $1 \otimes \psi\left(\frac{d z}{d y}\right)$ modulo $y \otimes 1-1 \otimes \psi(y)$.

Proof. The lemma follows from the computation

$$
\begin{aligned}
z \otimes 1-1 \otimes \psi(z) & =\sum_{n=0}^{\infty}\left(b_{n} y^{n} \otimes 1-1 \otimes \psi\left(b_{n}\right) \psi(y)^{n}\right) \\
& =\sum_{n=1}^{\infty}\left(1 \otimes \psi\left(b_{n}\right)\right) \cdot \sum_{\nu=0}^{n-1}\left(y^{\nu} \otimes \psi(y)^{n-1-\nu}\right) \cdot(y \otimes 1-1 \otimes \psi(y)) \\
& =(y \otimes 1-1 \otimes \psi(y)) \cdot \sum_{\nu=0}^{\infty} y^{\nu} \otimes \psi\left(\sum_{n=\nu+1}^{\infty} b_{n} y^{n-1-\nu}\right)
\end{aligned}
$$

where the second factor converges in $k \llbracket y \rrbracket \widehat{ब}_{k, \psi} R$. Modulo $y \otimes 1-1 \otimes \psi(y)$ this factor equals

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left(1 \otimes \psi\left(b_{n}\right)\right) \cdot \sum_{\nu=0}^{n-1}\left(y^{\nu} \otimes \psi(y)^{n-1-\nu}\right) & =\sum_{n=1}^{\infty}\left(1 \otimes \psi\left(b_{n}\right)\right) \cdot n\left(1 \otimes \psi(y)^{n-1}\right) \\
& =1 \otimes \psi\left(\sum_{n=1}^{\infty} n b_{n} y^{n-1}\right) \\
& =1 \otimes \psi\left(\frac{d z}{d y}\right)
\end{aligned}
$$

We need the following well known fact from field theory. For the convenience of the reader we include a proof.

Lemma A.2. Let $E$ be a finite field extension of $Q$ (or of $Q_{v}$ ) of inseparability degree $p^{m}$. Then the separable closure $E^{\prime}$ of $Q$ (resp. of $Q_{v}$ ) in $E$ equals $E^{p^{m}}:=\left\{x^{p^{m}}: x \in E\right\}$. If y is a uniformizing parameter at a place $\tilde{v}$ of $E$ then $y^{\prime}:=y^{p^{m}}$ is a uniformizing parameter at the place $\tilde{v}^{\prime}$ of $E^{\prime}$ below $\tilde{v}$ and $E=E^{\prime}(y)=E^{\prime}[X] /\left(X^{p^{m}}-y^{\prime}\right)$.

Proof. This is due to the fact that $Q$ has transcendence degree one over $\mathbb{F}_{q}$, respectively that $Q_{v}$ is a discretely valued field. Namely, consider the case for $Q_{v}$. Then $E=k((y))$ where $k$ is the finite residue field of $E$. Clearly $E^{p^{m}}=k\left(\left(y^{\prime}\right)\right)$ and $E=E^{p^{m}}(y)=E^{p^{m}}[X] /\left(X^{p^{m}}-y^{\prime}\right)$, because $X^{p^{m}}-y^{\prime}$ is irreducible in $E^{p^{m}}[X]$ by Eisenstein. In particular $\left[E: E^{p^{m}}\right]=p^{m}$. On the other hand, the minimal polynomial $f(X)$ of $y$ over $E^{\prime}$ is of the form $g\left(X^{p^{m^{\prime}}}\right)$ for a separable, irreducible polynomial $g$ over $E^{\prime}$ and an integer $m^{\prime} \geq 0$. Therefore the minimal polynomial of $y^{p^{m^{\prime}}}$ over $E^{\prime}$ is $g$ and $y^{p^{m^{\prime}}}$ is separable over $E^{\prime}$. This implies $y^{p^{m^{\prime}}} \in E^{\prime}$ and $\operatorname{deg} g=1$, whence $\operatorname{deg} f=p^{m^{\prime}} \leq p^{m}$. Therefore $m^{\prime} \leq m$ and $y^{p^{m}} \in E^{\prime}$, and hence $E^{p^{m}} \subset E^{\prime}$. Since $\left[E: E^{\prime}\right]=p^{m}=\left[E: E^{p^{m}}\right]$ it follows that $E^{\prime}=E^{p^{m}}$. This proves the lemma for $Q_{v}$.

For $Q$ a proof for the equality $E^{\prime}=E^{p^{m}}$ can be found for example in Sil86, Chapter II, Corollary 2.12]. If $\mathcal{O}_{E, \tilde{v}}$ is the valuation ring of $E$ at $\tilde{v}$ then $\left(\mathcal{O}_{E, \tilde{v}}\right)^{p^{m}}$ equals the valuation ring $\mathcal{O}_{E^{\prime}, \tilde{v}^{\prime}}$ of $E^{\prime}$ at $\tilde{v}^{\prime}$ and so $y^{\prime}$ is a uniformizing parameter of $\mathcal{O}_{E^{\prime}, \tilde{v}^{\prime}}$. The last equality follows from the fact that the polynomial $X^{p^{m}}-y^{\prime} \in \mathcal{O}_{E^{\prime}, \tilde{v^{\prime}}}[X]$ is irreducible by Eisenstein.

In the next lemma we consider the embeddings $Q \hookrightarrow K \llbracket z_{v}-\zeta_{v} \rrbracket$ and $Q_{v} \hookrightarrow L \llbracket z_{v}-\zeta_{v} \rrbracket$ given by $z_{v} \mapsto z_{v}=\zeta_{v}+\left(z_{v}-\zeta_{v}\right)$.

Lemma A.3. Let $E=E_{1} \times \ldots \times E_{s}$ be a product of finite field extensions of $Q$ and let $K \subset Q^{\text {alg }}$ be a field extension of $Q$ with $\psi(E) \subset K$ for all $\psi \in H_{E}:=\operatorname{Hom}_{Q}\left(E, Q^{\text {alg }}\right)$. Let $i(\psi)$ be such that $\psi$ factors through $E \rightarrow E_{i(\psi)}$ and let $y_{i(\psi)} \in E_{i(\psi)}$ be a uniformizing parameter at a place of $E_{i(\psi)}$ above $v$. Then

$$
\begin{aligned}
& E \otimes_{Q} K \llbracket z_{v}-\zeta_{v} \rrbracket=\prod_{\psi \in H_{E}} K \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket \quad \text { and } \\
& E \otimes_{Q} K=\prod_{\psi \in H_{E}} K \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket /\left(y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)^{\left[E_{i(\psi)}: Q\right]_{\text {insep }}},
\end{aligned}
$$

where $\left[E_{i(\psi)}: Q\right]_{\text {insep }}$ is the inseparability degree of $E_{i(\psi)}$ over $Q$.
Likewise, let $E_{v}=E_{v, 1} \times \ldots \times E_{v, s}$ be a product of finite field extensions of $Q_{v}$ and let $L \subset Q_{v}^{\text {alg }}$ be a field extension of $Q_{v}$ with $\psi\left(E_{v}\right) \subset L$ for all $\psi \in H_{E_{v}}:=\operatorname{Hom}_{Q_{v}}\left(E_{v}, Q_{v}^{\text {alg }}\right)$. Let $i(\psi)$ be such that $\psi$ factors through $E \rightarrow E_{v, i(\psi)}$ and let $y_{i(\psi)} \in E_{v, i(\psi)}$ be a uniformizing parameter. Then

$$
\begin{align*}
& E_{v} \otimes Q_{Q_{v}} L \llbracket z_{v}-\zeta_{v} \rrbracket=\prod_{\psi \in H_{E_{v}}} L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket \quad \text { and }  \tag{A.1}\\
& E_{v} \otimes Q_{v} L=\prod_{\psi \in H_{E_{v}}} L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket /\left(y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)^{\left[E_{v, i(\psi)}: Q_{v}\right]_{\text {insep }}}, \tag{A.2}
\end{align*}
$$

where $\left[E_{v, i(\psi)}: Q_{v}\right]_{\text {insep }}$ is the inseparability degree of $E_{v, i(\psi)}$ over $Q_{v}$.
Proof. Fix a $\psi$, set $i:=i(\psi)$ and let $E_{i}^{\prime}$, respectively $E_{v, i}^{\prime}$ be the separable closure of $Q$ in $E_{i}$, respectively of $Q_{v}$ in $E_{v, i}$. Then $H_{E_{i}}=H_{E_{i}^{\prime}}$, respectively $H_{E_{v, i}}=H_{E_{v, i}^{\prime}}$, and

$$
\begin{equation*}
E_{i}^{\prime} \otimes_{Q} K \xrightarrow{\sim} \prod_{\psi \in H_{E_{i}}} K, \quad \text { respectively } \quad E_{v, i}^{\prime} \otimes_{Q_{v}} L \xrightarrow{\sim} \prod_{\psi \in H_{E_{v, i}}} L \tag{A.3}
\end{equation*}
$$

Let $p^{m}:=\left[E_{i}: Q\right]_{\text {insep }}=\left[E_{i}: E_{i}^{\prime}\right]$, respectively $p^{m}:=\left[E_{v, i}: Q_{v}\right]_{\text {insep }}=\left[E_{v, i}: E_{v, i}^{\prime}\right]$, and let $y_{i}^{\prime}:=y_{i}^{p^{m}}$. Then Lemma A. 2 implies that $y_{i}^{\prime} \in E_{i}^{\prime}$ is a uniformizing parameter at a place above $v$. By Hensel's lemma the decompositions (A.3) extend to decompositions

$$
\begin{aligned}
E_{i}^{\prime} \otimes_{Q} K \llbracket z_{v}-\zeta_{v} \rrbracket & \sim \prod_{\psi \in H_{E_{i}}} K \llbracket z_{v}-\zeta_{v} \rrbracket \\
E_{v, i}^{\prime} \otimes_{Q_{v}} L \llbracket z_{v}-\zeta_{v} \rrbracket & \xrightarrow{\sim} \prod_{\psi \in H_{E_{i}}} K \llbracket y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right) \rrbracket, \quad \text { respectively } \\
& H_{E_{v, i}}
\end{aligned}
$$

Here the last identifications in each line follow from [HJ16, Lemma 1.2 and 1.3] which states that both $K \llbracket z_{v}-\zeta_{v} \rrbracket$ and $K \llbracket y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right) \rrbracket$ are canonically isomorphic to the completion of the ring $\mathcal{O}_{E_{i}^{\prime}} \otimes_{\mathbb{F}_{q}} K$ at the ideal $\left(a \otimes 1-1 \otimes \psi(a): a \in \mathcal{O}_{E_{i}^{\prime}}\right)$. The identification in the second line also follows from Lemma A. 1 by observing that the derivative $\frac{d z_{v}}{d y_{i}^{\prime}}$ equals $-\frac{\partial}{\partial Y_{i}^{\prime}} m\left(z_{v}, Y_{i}^{\prime}\right) /\left.\frac{\partial}{\partial z_{v}} m\left(z_{v}, Y_{i}^{\prime}\right)\right|_{Y_{i}^{\prime}=y_{i}^{\prime}}$ where $m\left(z_{v}, Y_{i}^{\prime}\right) \in \mathbb{F}_{v} \llbracket z_{v} \rrbracket\left[Y_{i}^{\prime}\right]$ is the minimal polynomial of $y_{i}^{\prime}$ over $Q_{v}$, and hence $\psi\left(\frac{d z_{v}}{d y_{i}^{\prime}}\right)$ is non-zero by the separability of $y_{i}^{\prime}$ over $Q_{v}$, and the injectivity of $\psi$ on $E_{v, i}$. Now $E_{i}=E_{i}^{\prime}\left(y_{i}\right)$, respectively $E_{v, i}=E_{v, i}^{\prime}\left(y_{i}\right)$, and hence $E_{i} \otimes_{E_{i}^{\prime}}$ $K \llbracket y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right) \rrbracket=K \llbracket y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right) \rrbracket\left[y_{i}-\psi\left(y_{i}\right)\right]=K \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$, respectively $E_{v, i} \otimes_{E_{v, i}^{\prime}} L \llbracket y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right) \rrbracket=$ $L \llbracket y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right) \rrbracket\left[y_{i}-\psi\left(y_{i}\right)\right]=L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$ with $\left(y_{i}-\psi\left(y_{i}\right)\right)^{p^{m}}=y_{i}^{\prime}-\psi\left(y_{i}^{\prime}\right)$. Since $H_{E}=\bigcup_{i} H_{E_{i}}$, respectively $H_{E_{v}}=\bigcup_{i} H_{E_{v, i}}$, the lemma follows.

## B Erratum

## B. 1 First Error

In (1.13) and (1.12) and Definition 5.21 it is claimed that $v\left(\omega_{\psi}\right)$ and $v_{\psi}(u)$ are integers. However, in general they only lie in the rational numbers $\mathbb{Q}$, because the valuation $v$ is normalized to be an isomorphism $v: Q_{v}^{\times} / A_{v}^{\times} \xrightarrow{\sim} \mathbb{Z}$, but the arguments of $v$ in both formulas lie in $Q_{v}^{\text {alg }}$ instead of $Q_{v}$.

This error is harmless, as the integrality of $v_{\psi}(u)$ and $v\left(\omega_{\psi}\right)$ is nowhere used.

## B. 2 Second Error

In Formula (1.13) and Definition 4.10 there is an error in the definition of $v\left(\omega_{\psi}\right)$.
As in most of the main text we fix a finite separable semi-simple $Q$-algebra $E$. That is, $E$ is a product of finite separable field extensions of $Q$. We fix a finite place $v$ of $Q$ and consider the decomposition of the separable $Q_{v}$-algebra $E_{v}:=E \otimes_{Q} Q_{v}=E_{v, 1} \times \cdots \times E_{v, s}$ into a product of finite field extensions $E_{v, i}$ of $Q_{v}$ as after Definition 4.1. We fix a finite Galois extension $K \subset Q^{\text {alg }}$ of $Q$ and we let $L:=K_{v} \subset Q_{v}^{\text {alg }}$ be the closure of $K$. It is a finite Galois extension of $Q_{v}$. We fix a $\psi \in H_{E}$. The canonical extension $\psi \otimes \operatorname{id}_{Q_{v}}: E_{v} \rightarrow L$ will be denoted again by $\psi$ and factors through the quotient $E_{v, i(\psi)}$ of $E_{v}$; see Definition 4.5,

Let $\underline{\hat{M}}$ be a local shtuka over $R:=\mathcal{O}_{L}$ with complex multiplication by $\mathcal{O}_{E_{v}}$ as in Definition 4.3, It may arise from a good model $\underline{\mathcal{M}}$ of an $A$-motive over $R$ as in Example 3.2. We consider the onedimensional $L$-vector space

$$
\begin{align*}
\mathrm{H}^{\psi}(\underline{\hat{M}}, L) & :=\left\{\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L):[a]^{*} \omega=\psi(a) \cdot \omega \forall a \in \mathcal{O}_{E_{v}}\right\} \\
& \xrightarrow{\longrightarrow} \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L) /\left([a]^{*}-\psi(a): a \in \mathcal{O}_{E_{v}}\right) \cdot \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, L), \tag{B.1}
\end{align*}
$$

where the isomorphism comes from Proposition 4.9 using that $E$ is separable over $Q$.
In Formula (1.13) and Definition4.10 there is an error in the definition of $v\left(\omega_{\psi}\right)$ for $L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket$ generators $\omega_{\psi}$ of $\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)$. Namely, there as reference integral structure on the $L$-vector space $\mathrm{H}^{\psi}(\underline{\hat{M}}, L)=\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right) /\left(y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)$ the $R$-module

$$
\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R):=\left\{\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R):[a]^{*} \omega=\psi(a) \cdot \omega \forall a \in \mathcal{O}_{E_{v}}\right\}
$$

was used (which was denoted without the ${ }^{\sim}$ on $\widetilde{H}$ in (1.13) and Definition 4.10). Then $v\left(\omega_{\psi}\right)$ was defined to be

$$
\begin{equation*}
v^{\sim}\left(\omega_{\psi}\right):=v(\tilde{x}) \in \mathbb{Q} \tag{B.2}
\end{equation*}
$$

where $\tilde{x} \in L^{\times}$satisfies that $\tilde{x}^{-1}\left(\omega_{\psi} \bmod y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)$ is an $R$-generator of $\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R)$. (To clarify the error we write $v^{\sim}\left(\omega_{\psi}\right)$ instead of $v\left(\omega_{\psi}\right)$ in this erratum.)

However, in the rest of the main text the $R$-submodule

$$
\mathrm{H}^{\psi}(\underline{\hat{M}}, R):=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R) /\left([a]^{*}-\psi(a): a \in \mathcal{O}_{E_{v}}\right) \cdot \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R) \subset \mathrm{H}^{\psi}(\underline{\hat{M}}, L)
$$

is used as reference integral structure on $\mathrm{H}^{\psi}(\underline{\hat{M}}, L)$. (See Lemma B. 1 below for why the latter is an inclusion, and how the two integral structures can be compared.) Correspondingly, the following definition for $v\left(\omega_{\psi}\right)$ is used in the main text.

$$
\begin{equation*}
v\left(\omega_{\psi}\right):=v(x) \in \mathbb{Q}, \tag{B.3}
\end{equation*}
$$

where $x \in L^{\times}$satisfies that $x^{-1}\left(\omega_{\psi} \bmod y_{i(\psi)}-\psi\left(y_{i(\psi)}\right)\right)$ is an $R$-generator of $H^{\psi}(\underline{\hat{M}}, R)$. Indeed, in 5.12 the generator $\omega_{\psi}^{\circ}:=1$ of $\mathrm{H}^{\psi}(\underline{\underline{M}}, R)$ is used, which might not lie in $\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R)$. Afterwards, any other generator $\omega_{\psi}$ is compared to the generator $\omega_{\psi}^{\circ}$. This error occurs in Theorems 1.3 and 5.24 and in Corollaries 5.22 and 5.25. In terms of the valuation $v^{\sim}\left(\omega_{\psi}\right)$ from (B.2), all these theorems and corollaries have to be reformulated as explained below. However, with the definition of $v\left(\omega_{\psi}\right)$ in (B.3) above, all these theorems and corollaries are correct.

Note that if $\underline{\hat{M}}=\underline{\hat{M}}_{v}(\underline{\mathcal{M}})$ arises from a good model $\underline{\mathcal{M}}$ of an $A$-motive over $R$ as in Example 3.2, then $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R)=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{M}}, R):=\sigma^{*} \mathcal{M} \otimes_{A_{R}, \gamma \otimes \mathrm{id}{ }_{R}} R$, and hence

$$
\begin{aligned}
& \widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R)=\widetilde{\mathrm{H}}^{\psi}(\underline{\mathcal{M}}, R):=\left\{\omega \in \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{M}}, R):[a]^{*} \omega=\psi(a) \cdot \omega \forall a \in \mathcal{O}_{E}\right\}, \\
& \mathrm{H}^{\psi}(\underline{\hat{M}}, R)=\mathrm{H}^{\psi}(\underline{\mathcal{M}}, R):=\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{M}}, R) /\left([a]^{*}-\psi(a): a \in \mathcal{O}_{E}\right) \cdot \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\mathcal{M}}, R) \\
& \text { inside } \quad \mathrm{H}^{\psi}(\underline{\hat{M}}, L)=\mathrm{H}^{\psi}(\underline{\mathcal{M}}, L)=\widetilde{\mathrm{H}}^{\psi}(\underline{\mathcal{M}}, R) \otimes_{R} L=\mathrm{H}^{\psi}(\underline{\mathcal{M}}, R) \otimes_{R} L .
\end{aligned}
$$

We next show how the two integral structures can be compared.
Lemma B.1. The integral structures $\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R)$ and $\mathrm{H}^{\psi}(\underline{\hat{M}}, R)$ are free $R$-modules of rank one and contained in the L-vector space $\mathrm{H}^{\psi}(\underline{\hat{M}}, L)$. The natural $R$-morphism

$$
\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R) \longleftrightarrow \mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R) \longrightarrow \mathrm{H}^{\psi}(\underline{\hat{M}}, R)
$$

is injective with cokernel isomorphic to $R / R \cdot \psi\left(\mathfrak{D}_{E_{v} / Q_{v}}\right)$, where $\mathfrak{D}_{E_{v} / Q_{v}}$ is the different of $E_{v}=\prod_{i=1}^{s} E_{v, i}$ over $Q_{v}$.
Proof. The morphism fits into the following diagram

in which the lower isomorphism was described in (B.1), the lower triangle is the tensor product of the upper row with $L$, and the injectivity of the right vertical arrow still has to be proved. Note that the argument will not use the specific situation of de Rham cohomology of local shtukas. It will only use
the isomorphism (B.1) comming from Proposition 4.9 and the freeness of the $R$-module $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{M}, R)$ over $\mathcal{O}_{E, v} \otimes_{A_{v}} R$, see below.

The $Q_{v}$-algebra $E_{v}$ acts on $\mathrm{H}^{\psi}(\underline{\hat{M}}, L)$ through the character $\psi: E_{v} \rightarrow E_{v, i(\psi)} \hookrightarrow L$. By [Ser79, § III.6, Proposition 12] there exists an element $y \in \mathcal{O}_{E_{v, i(\psi)}}$ such that $\mathcal{O}_{E_{v, i(\psi)}}=A_{v}[y]=A_{v}[Y] /(m)$ where $m \in A_{v}[Y]$ is the minimal polynomial of $y$ over $A_{v}$. The image $\gamma(m)$ under the map $\gamma: A_{v}[Y] \hookrightarrow R[Y]$ has $\psi(y)$ as a zero and correspondingly factors as

$$
\gamma(m)=(Y-\psi(y)) \cdot g(Y)
$$

for a monic polynomial $g(Y) \in R[Y]$. The derivative $m^{\prime}:=\frac{d m}{d Y} \in A_{v}[Y]$ satisfies

$$
\begin{equation*}
\psi\left(m^{\prime}(y)\right)=\gamma(m)^{\prime}(\psi(y))=g(\psi(y)) \tag{B.5}
\end{equation*}
$$

Recall that $A_{v, R}$ is the $v$-adic completion of $A_{R}$. By Proposition 4.8 we can decompose $\underline{\hat{M}}=\bigoplus_{i=1}^{s} \underline{\hat{M}}_{i}$ into local shtukas $\underline{M}_{i}$ over $R$ with complex multiplication by $\mathcal{O}_{E_{v, i}}$. In particular,

$$
\begin{aligned}
\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R) & :=\left\{\omega \in \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{i(\psi)}, R\right):[a]^{*} \omega=\psi(a) \cdot \omega \forall a \in \mathcal{O}_{E_{v, i}}\right\} \quad \text { and } \\
\mathrm{H}^{\psi}(\underline{\hat{M}}, R) & :=\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{i(\psi)}, R\right) /\left([a]^{*}-\psi(a): a \in \mathcal{O}_{E_{v, i}}\right) \cdot \mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{i(\psi)}, R\right)
\end{aligned}
$$

can be computed from

$$
\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{i(\psi)}, R\right):=\sigma^{*} \hat{M}_{i(\psi)} \otimes_{A_{v, R}, \gamma \otimes \mathrm{id}_{R}} R
$$

instead of $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R)$. Moreover, by loc. cit. $\underline{\hat{M}}$ is free over $\mathcal{O}_{E_{v}, R}:=\mathcal{O}_{E_{v}} \otimes_{A_{v}} R \llbracket z \rrbracket=\mathcal{O}_{E_{v}} \widehat{\otimes}_{\mathbb{F}_{v}} R$ of rank one, and we may choose a generator of $\hat{M}$. This generator provides an isomorphism

$$
\mathrm{H}_{\mathrm{dR}}^{1}\left(\underline{\hat{M}}_{i(\psi)}, R\right) \cong\left(\mathcal{O}_{E_{v, i(\psi)}} \widehat{\otimes}_{\mathbb{F}_{q}} R\right) \underset{A_{v, R}, \gamma \otimes \mathrm{id}_{R}}{\otimes} R=\mathcal{O}_{E_{v, i(\psi)}} \otimes_{A_{v}, \gamma} R=R[Y] /(\gamma(m))
$$

Since $[a]^{*}-\psi(a)=[a]^{*}-\gamma(a)$ for $a \in A_{v}$ already annihilates $\mathrm{H}_{\mathrm{dR}}^{1}(\underline{\hat{M}}, R)$, this yields the upper vertical isomorphisms in the following diagram


The injectivity of the horizontal maps follows from diagram (B.4). The lower left equality holds because $R[Y]$ has no $(Y-\psi(y))$-torsion. Next, $\mathrm{H}^{\psi}(\underline{\hat{M}}, R) \cong R[Y] /(Y-\psi(y)) \cong R$ is free, and hence contained in $\mathrm{H}^{\psi}(\underline{\hat{M}}, R) \otimes_{R} L=\mathrm{H}^{\psi}(\underline{\hat{M}}, L)$. Finally, the image of the lower horizontal map is the ideal

$$
R \cdot g(\psi(y))=R \cdot \psi\left(m^{\prime}(y)\right)=R \cdot \psi\left(\mathfrak{D}_{E_{v, i(\psi)} / Q_{v}}\right)=R \cdot \psi\left(\mathfrak{D}_{E_{v} / Q_{v}}\right)
$$

see [Ser79, § III.4, Proposition 10 and § III.6, Corollary 2].
Corollary B.2. For an $L$-generator $\omega_{\psi}$ of $\mathrm{H}^{\psi}(\underline{\hat{M}}, L)$, the two valuations ( (B.2) and (B.3) satisfy

$$
v\left(\omega_{\psi}\right)-v^{\sim}\left(\omega_{\psi}\right)=v\left(\mathfrak{D}_{\psi\left(E_{v}\right) / Q_{v}}\right)=v\left(\psi\left(\mathfrak{D}_{E_{v} / Q_{v}}\right)\right)=v\left(\psi\left(\mathfrak{D}_{E / Q}\right)\right)
$$

Proof. Let $x, \tilde{x} \in L^{\times}$be elements such that $x^{-1} \omega_{\psi}$ is an $R$-generator of $\mathrm{H}^{\psi}(\underline{\hat{M}}, R)$ and $\tilde{x}^{-1} \omega_{\psi}$ is an $R$-generator of $\widetilde{\mathrm{H}}^{\psi}(\underline{\hat{M}}, R)$. Then $x / \tilde{x}$ is an $R$-generator of $\psi\left(\mathfrak{D}_{E_{v} / Q_{v}}\right)$ by Lemma B. 1 and the corollary follows.

Now let $\underline{M}$ be an $A$-motive over a finite Galois extension $K \subset Q^{\text {alg }}$ of $Q$ with complex multiplication by a finite separable semi-simple $Q$-algebra $E$. Assume that $\psi(E) \subset K$ for all $\psi \in H_{E}$. Fix a $\psi \in H_{E}$ and let $\omega_{\psi}$ be a generator of the $K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket$-module $\mathrm{H}^{\psi}\left(\underline{M}^{\eta}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$. For every $\eta \in H_{K}$ let $\underline{M}^{\eta}$ and $\omega_{\psi}^{\eta} \in \mathrm{H}^{\eta \psi}\left(\underline{M}^{\eta}, K \llbracket y_{\eta \psi}-\eta \psi\left(y_{\eta \psi}\right) \rrbracket\right)$ be obtained by extension of scalars via $\eta$. With the corollary and the computation

$$
\begin{aligned}
\sum_{\eta \in H_{K}} v\left(\omega_{\psi}^{\eta}\right)-v^{\sim}\left(\omega_{\psi}^{\eta}\right) & =\sum_{\eta \in H_{K}} v\left(\eta \psi\left(\mathfrak{D}_{E / Q}\right)\right)=v\left(\prod_{\eta \in H_{K}} \eta \psi\left(\mathfrak{D}_{E / Q}\right)\right)=v\left(N_{K / Q}\left(\mathfrak{D}_{\psi(E) / Q}\right)\right) \\
& =v\left(N_{\psi(E) / Q}\left(N_{K / \psi(E)}\left(\mathfrak{D}_{\psi(E) / Q}\right)\right)\right)=[K: \psi(E)] \cdot v\left(\mathfrak{d}_{\psi\left(\mathcal{O}_{E}\right) / A}\right)
\end{aligned}
$$

we obtain a reformulation of Theorems 1.3 and 5.24 and Corollaries 5.22 and 5.25 in terms of $v^{\sim}\left(\omega_{\psi}\right)$, which is even more analogous to [Co193, Theorem II.1.1(i)].

Theorem 1.3'. Let $\omega_{\psi}$ be a generator of the $K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket$-module $\mathrm{H}^{\psi}\left(\underline{M}, K \llbracket y_{\psi}-\psi\left(y_{\psi}\right) \rrbracket\right)$. For every $\eta \in H_{K}$ let $\underline{M}^{\eta}$ and $\omega_{\psi}^{\eta} \in \mathrm{H}^{\eta \psi}\left(\underline{M}^{\eta}, K \llbracket y_{\eta \psi}-\eta \psi\left(y_{\eta \psi}\right) \rrbracket\right)$ be obtained by extension of scalars via $\eta$, and choose an E-generator $u_{\eta} \in \mathrm{H}_{1, \operatorname{Betti}}\left(\underline{M}^{\eta}, Q\right)$. Then for every place $v \neq \infty$ of $C$ we have

$$
\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}} v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E, \psi, \Phi}^{0}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E, \psi, \Phi}^{0}\right)+\frac{1}{\# H_{K}} \sum_{\eta \in H_{K}}\left(v^{\sim}\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right)\right) .
$$

Corollary 5.22'. Let $\varphi, \psi \in H_{E_{v}}$ with $i(\varphi)=i(\psi)=: i$ and assume that $E_{v, i}$ is separable over $Q_{v}$. Let $u \in \mathrm{H}_{1, v}\left(\underline{\hat{M}}_{E_{v}, \varphi}, Q_{v}\right)$ be a generator as $E_{v}$-module and let $\omega_{\psi}$ be an $L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket$-generator of $\mathrm{H}^{\psi}\left(\underline{\underline{M}}_{E_{v}, \varphi}, L \llbracket y_{i}-\psi\left(y_{i}\right) \rrbracket\right)$. Then $\int_{u} \omega_{\psi}:=u \otimes \operatorname{id}_{\mathbb{C}_{v}((z-\zeta))}\left(h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}\right)\right)$ has valuation

$$
v\left(\int_{u} \omega_{\psi}\right)=Z_{v}\left(a_{E_{v}, \psi, \varphi}, 1\right)-\mu_{\mathrm{Art}, v}\left(a_{E_{v}, \psi, \varphi}\right)+v^{\sim}\left(\omega_{\psi}\right)+v_{\psi}(u),
$$

where $v^{\sim}\left(\omega_{\psi}\right)$ and $v_{\psi}(u)$ were defined in (B.2) and Definition5.21, respectively.
Theorem 5.24'. Let $\underline{\hat{M}}$ be a local shtuka over $R$ with complex multiplication by the ring of integers $\mathcal{O}_{E_{v}}$ in a commutative, semi-simple, separable $Q_{v}$-algebra $E_{v}$ with local CM-type $\Phi$, and assume that $\psi\left(E_{v}\right) \subset L$ for all $\psi \in H_{E_{v}}$ and that $L$ is separable over $Q_{v}$. Let $u \in H_{1, v}\left(\underline{\hat{M}}, Q_{v}\right)$ be an $E_{v}$-generator and let $\omega_{\psi}$ be an $L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket$-generator of $\mathrm{H}^{\psi}\left(\underline{\hat{M}}, L \llbracket y_{i(\psi)}-\psi\left(y_{i(\psi)}\right) \rrbracket\right)$. Then the period $\int_{u} \omega_{\psi}:=$ $u \otimes \operatorname{id}_{\mathbb{C}_{v}((z-\zeta))}\left(h_{v, \mathrm{dR}}^{-1}\left(\omega_{\psi}\right)\right)$ has valuation

$$
v\left(\int_{u} \omega_{\psi}\right)=Z_{v}\left(a_{E_{v}, \psi, \Phi}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \psi, \Phi}\right)+v^{\sim}\left(\omega_{\psi}\right)+v_{\psi}(u),
$$

where $v^{\sim}\left(\omega_{\psi}\right)$ and $v_{\psi}(u)$ were defined in (B.2) and Definition5.21, respectively.
Corollary 5.25'. Keep the situation of Theorem 5.24'. For every $\eta \in H_{L}$ note that $i(\eta \psi)=i(\psi)$, let $\underline{\hat{M}}^{\eta}$ and $\omega_{\psi}^{\eta} \in \mathrm{H}^{\eta \psi}\left(\underline{\hat{M}}^{\eta}, L \llbracket y_{i(\psi)}-\eta \psi\left(y_{i(\psi)}\right) \rrbracket\right)$ be obtained by extension of scalars via $\eta$, and choose an $E_{v}$-generator $u_{\eta} \in \mathrm{H}_{1, v}\left(\underline{\underline{M}}^{\eta}, Q_{v}\right)$. Then

$$
\frac{1}{\# H_{L}} \sum_{\eta \in H_{L}} v\left(\int_{u_{\eta}} \omega_{\psi}^{\eta}\right)=Z_{v}\left(a_{E_{v}, \psi, \Phi}^{0}, 1\right)-\mu_{\operatorname{Art}, v}\left(a_{E_{v}, \psi, \Phi}^{0}\right)+\frac{1}{\# H_{L}} \sum_{\eta \in H_{L}}\left(v^{\sim}\left(\omega_{\psi}^{\eta}\right)+v_{\eta \psi}\left(u_{\eta}\right)\right)
$$

## References

[EGA] A. Grothendieck: Élements de Géométrie Algébrique, Publ. Math. IHES 4, 8, 11, 17, 20, 24, 28, 32, Bures-Sur-Yvette, 1960-1967; see also Grundlehren 166, Springer-Verlag, Berlin etc. 1971; also available at http://www.numdam.org/search/grothendieck-a.
[And86] G. Anderson: t-Motives, Duke Math. J. 53 (1986), 457-502.
[AGHM15] F. Andreatta, E. Goren, B. Howard, K. Madapusi Pera: Faltings heights of abelian varieties with complex multiplication, preprint 2015 on arXiv:math.NT/1508.00178.
[Ax70] J. Ax: Zero of polynomials over local fields - the Galois action, J. Algebra 15 (1970), 417-428.
[BSM16] A. Barquero-Sanchez, R. Masri: On the Colmez Conjecture for Non-Abelian CM Fields, preprint 2016 on arXiv:math.NT/1604.01057.
[BMS15] B. Bhatt, M. Morrow, P. Scholze: Integral p-adic Hodge theory - announcement, Math. Res. Lett. 22 (2015), no. 6, 1601-1612; also available as arXiv:math.AG/1507.08129.
[BMS16] B. Bhatt, M. Morrow, P. Scholze: Integral p-adic Hodge theory, preprint 2016 on arXiv:math.AG/1602.03148
[BH09] M. Bornhofen, U. Hartl: Pure Anderson Motives over Finite Fields, J. Number Th. 129, n. 2 (2009), 247-283; also available as arXiv:math.NT/0709.2815.
[BH11] M. Bornhofen, U. Hartl: Pure Anderson motives and abelian $\tau$-sheaves, Math. Z. 268 (2011), 67-100; also available as arXiv:math.NT/0709.2809.
[Car35] L. Carlitz: On certain functions connected with polynomials in a Galois field, Duke Math. J. 1 (1935), no. 2, 137-168.
[Col93] P. Colmez: Périodes des variétés abéliennes a multiplication complexe, Ann. of Math. (2) 138 (1993), no. 3, 625-683; available at http://www.math.jussieu.fr/~colmez.
[Dri76] V.G. Drinfeld: Elliptic Modules, Math. USSR-Sb. 23 (1976), 561-592.
[Eis95] D. Eisenbud: Commutative Algebra with a View Toward Algebraic Geometry, GTM 150, SpringerVerlag, Berlin etc. 1995.
[Gar03] F. Gardeyn: The structure of analytic $\tau$-sheaves, J. Number Th. 100 (2003), 332-362.
[Gos96] D. Goss: Basic Structures of Function Field Arithmetic, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 35, Springer-Verlag, Berlin-Heidelberg-New York 1996.
[Har09] U. Hartl: A Dictionary between Fontaine-Theory and its Analogue in Equal Characteristic, J. Number Th. 129 (2009), 1734-1757; also available as arXiv:math.NT/0607182.
[HJ16] U. Hartl, A.-K. Juschka: Pink's theory of Hodge structures and the Hodge conjecture over function fields, Proceedings of the conference on " $t$-motives: Hodge structures, transcendence and other motivic aspects", BIRS, Banff, Canada 2009, eds. G. Böckle, D. Goss, U. Hartl, M. Papanikolas, EMS 2017; also available as arXiv:math/1607.01412.
[HK15] U. Hartl, W. Kim: Local Shtukas, Hodge-Pink Structures and Galois Representations, to appear in Proceedings of the conference on " $t$-motives: Hodge structures, transcendence and other motivic aspects", BIRS, Banff, Canada 2009, eds. G. Böckle, D. Goss, U. Hartl, M. Papanikolas, EMS 2017; also available as arXiv:math/1512.05893.
[HS15] U. Hartl, R.K. Singh: Local Shtukas and Divisible Local Anderson Modules, preprint 2015 on arXiv:math/1511.03697.
[HS20] U. Hartl, R.K. Singh: Periods of Drinfeld modules and local shtukas with complex multiplication, J. Inst. Math. Jussieu 19, no. 1 (2020), 175-208; also available as arXiv:math/1603.03194
[LT65] J. Lubin, J. Tate: Formal complex multiplication in local fields, Ann. of Math. (2) 81 (1965), 380-387; available at http://www.jstor.org/stable/1970622.
[Obu13] A. Obus: On Colmez's product formula for periods of CM-abelian varieties, Math. Ann. 356 (2013), no. 2, 401-418; also available as arXiv:math.NT/1107.0684.
[Ros02] M. Rosen: Number theory in function fields, GTM 210, Springer-Verlag, New York, 2002.
[Sch09] A. Schindler: Anderson A-Motive mit komplexer Multiplikation, Diploma thesis, University of Muenster, 2009; available at http://www.math.uni-muenster.de/u/urs.hartl/Publikat/Schindler_Diplomarbeit.pdf.
[Sch16] P. Scholze: Canonical q-deformations in arithmetic geometry, preprint on arXiv:math.AG/1606.01796
[Ser58] J.-P. Serre: Classes de corps cyclotomiques, d'après K. Iwasawa, Séminaire Bourbaki, vol. 5, Exp. No. 174 (1958-1960), pp. 83-93, Soc. Math. France, Paris, 1995.
[Ser77] J.-P. Serre: Linear representations of finite groups, GTM 42, Springer-Verlag, New York-Heidelberg, 1977.
[Ser79] J.-P. Serre: Local fields, GTM 67, Springer-Verlag, New York-Berlin, 1979.
[Sil86] J. Silverman: The arithmetic of elliptic curves, GTM 106, Springer-Verlag, New York 1986.
[Tat84] J. Tate: Les conjectures de Stark sur les fonctions L d'Artin en $s=0$, Lecture notes edited by Dominique Bernardi and Norbert Schappacher, Progress in Mathematics 47, Birkhäuser Boston, Inc., Boston, MA, 1984.
[Wei48] A. Weil: Sur les courbes algébriques et les variétés qui s'en déduisent, Actualités Sci. Ind. $1041=$ Publ. Inst. Math. Univ. Strasbourg 7 (1945), Hermann et Cie., Paris, 1948.
[Yan13] T. Yang: Arithmetic intersection on a Hilbert modular surface and the Faltings height, Asian J. Math. 17 (2013), no. 2, 335-381; also available as arXiv:math.NT/1008.1854.
[YZ15] X. Yuan, S.-W. Zhang: On the Averaged Colmez Conjecture; preprint 2015 on arXiv:math.NT/1507.06903.

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[^0]:    * Both authors acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) in form of SFB 878 and Germany's Excellence Strategy EXC 2044-390685587"Mathematics Münster: Dynamics-Geometry-Structure". The first author was also supported the DFG in form of Project-ID 427320536 - SFB 1442.

