Category Of C-Motives Over Finite Fields

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Abstract

In this article we introduce and study a motivic category in the arithmetic of function fields set up. This category generalizes the previous construction due to Taelman and is more relevant for applications to the theory of G-Shtukas, such as formulating Langlands-Rapoport conjecture over function fields. We further discuss the analogy with the theory of motives over number fields and in particular we prove an analog of the Jannsen's semisimplicity result [Jan].

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0.1 Notation and Conventions

Throughout this article we denote by

$\mathbb{F}_q$	a finite field with q elements and characteristic p,
L	a ring containing $\mathbb{F}_q$,
$\mathbb{F}$	a finite field containing $\mathbb{F}_q$,
$\overline{\mathbb{F}}$	an algebraic closure of $\mathbb{F}$,
C	a smooth projective geometrically irreducible curve over $\mathbb{F}_q$,
η	the generic point of C,
ν	a closed point of C, also called a *place* of C,
$\underline{\nu}$	a set of n distinct places ν_i on C,
$Q := \kappa(\eta) = \mathbb{F}_q(C)$	the function field of C,
$\mathbb{F}_\nu := \kappa(\nu)$	the residue field at the place ν on C,
A	ring of regular functions outside $\underline{\nu}$,
A_L	the ring $A \otimes_{\mathbb{F}_q} L$,
Q_L	the ring of fractions $\operatorname{Frac}(A_L)$,
A_ν	the completion of the stalk $\mathcal{O}_{C,\nu}$ at $\nu \neq \nu_i$,
$Q_\nu := \operatorname{Frac}(A_\nu)$	its fraction field,
$\widehat{A} := \mathbb{F}[\![z]\!]$	the ring of formal power series in z with coefficients in $\mathbb{F}$,
$\widehat{Q} := \operatorname{Frac}(\widehat{A})$	its fraction field,
$\mathbb{A}^{\underline{\nu}}$	the ring of integral adeles of C outside $\underline{\nu}$,
$\mathbb{A}_Q^{\underline{\nu}} := \mathbb{A}^{\underline{\nu}} \otimes_{\mathcal{O}_C} Q$	the ring of adeles of C outside $\underline{\nu}$,
$\mathbb{D}_R := \operatorname{Spec} R[\![z]\!]$	the spectrum of the ring of formal power series in z with coefficients in an $\mathbb{F}$-algebra R,
$\hat{\mathbb{D}}_R := \operatorname{Spf} R[\![z]\!]$	the formal spectrum of $R[\![z]\!]$ with respect to the z-adic topology.

For a formal scheme $\widehat{S}$ we denote by $\mathcal{N}ilp_{\widehat{S}}$ the category of schemes over $\widehat{S}$ on which an ideal of definition of $\widehat{S}$ is locally nilpotent. We equip $\mathcal{N}ilp_{\widehat{S}}$ with the étale topology. We also denote by

$n \in \mathbb{N}_{>0}$	a positive integer,
$\underline{\nu} := (\nu_i)_{i=1\dots n}$	an n-tuple of closed points of C,
$A_{\underline{\nu}}$	the completion of the local ring $\mathcal{O}_{C^n,\underline{\nu}}$ of C^n at the closed point $\underline{\nu} = (\nu_i)$,
$\mathcal{N}ilp_{A_{\underline{\nu}}} := \mathcal{N}ilp_{\operatorname{Spf} A_{\underline{\nu}}}$	the category of schemes over C^n on which the ideal defining the closed point $\underline{\nu} \in C^n$ is locally nilpotent,
$\mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]} := \mathcal{N}ilp_{\hat{\mathbb{D}}}$	the category of $\mathbb{D}$-schemes S for which the image of z in $\mathcal{O}_S$ is locally nilpotent. We denote the image of z by ζ since we need to distinguish it from $z \in \mathcal{O}_{\mathbb{D}}$.
$\mathfrak{G}$	a smooth affine group scheme of finite type over C,
$\mathbb{P}_\nu := \mathfrak{G} \times_C \operatorname{Spec} A_\nu$,	the base change of $\mathfrak{G}$ to $\operatorname{Spec} A_\nu$,
$P_\nu := \mathfrak{G} \times_C \operatorname{Spec} Q_\nu$,	the generic fiber of $\mathbb{P}_\nu$ over $\operatorname{Spec} Q_\nu$,
$\mathbb{P}$	a smooth affine group scheme of finite type over $\mathbb{D} = \operatorname{Spec} \mathbb{F}[\![z]\!]$,
P	the generic fiber of $\mathbb{P}$ over $\operatorname{Spec} \mathbb{F}(\!(z)\!)$.

Let S be an $\mathbb{F}_q$-scheme and consider an n-tuple $\underline{s} := (s_i)_i \in C^n(S)$. We denote by $\Gamma_{\underline{s}}$ the union $\bigcup_i \Gamma_{s_i}$ of the graphs $\Gamma_{s_i} \subseteq C_S$.

For an affine closed subscheme Z of C_S with sheaf $\mathcal{I}_Z$ we denote by $\mathbb{D}_S(Z)$ the scheme obtained by taking completion along Z. Moreover we set $\dot{\mathbb{D}}_S(Z) := \mathbb{D}_S(Z) \times_{C_S} (C_S \smallsetminus Z)$.

We denote by $\sigma_S \colon S \to S$ the $\mathbb{F}_q$-Frobenius endomorphism which acts as the identity on the points of S and as the q-power map on the structure sheaf. We set

$C_S := C \times_{\operatorname{Spec} \mathbb{F}_q} S$, and

$\sigma := \mathrm{id}_C \times \sigma_S$.

Likewise we let $\hat{\sigma}_S : \mathcal{O}_S[\![z]\!] \to \mathcal{O}_S[\![z]\!]$ to be the morphism which is σ_S on $\mathcal{O}_S$ and maps z to itself. We drop the subscript S and simply write $\hat{\sigma}$, when it is precise from the context.

$Ch_{\sim,d}(-,\mathbb{Q})$	the group of cycles of dimension d modulo the equivalence relation $\sim$, with coefficients in $\mathbb{Q}$. Here $\sim$ stands for an adequate equivalence relation, e.g. rational, algebraic, homological and numerical equivalence relations.

Definition 0.1. (a) Let R be a ring and $X \subseteq R$ be a subset. We denote by $C_R(X)$ the centralizer of X in R, i.e. $C_R(X) := \{a \in R; a \cdot x = x \cdot a \forall x \in X\}$. The subring $C_R(C_R(X))$ is called *bicommutant* of X in R.

(b) Let M be an R-module. Set $R_M := \operatorname{im}(R \to \operatorname{End}(M,+), a \mapsto (a : m \mapsto a.m))$

(c) We denote by $\mathcal{Z}(R)$ the centralizer of R in R, i.e. $\mathcal{Z}(R) := C_R(R)$.

(d) We denote by $Bicom_R(M) := C_{\operatorname{End}(M,+)}(\operatorname{End}_R(M))$

Remark 0.2. (a) Let K be a field and R and R' be two K-algebra. Let $S \subseteq R$ and $S' \subseteq R'$ be sub-K-algebras. Then we have

$$C_{R\otimes_K R'}(S \otimes_K S') = C_R(S) \otimes_K C_{R'}(S')$$

in particular $\mathcal{Z}(R \otimes_K R') = \mathcal{Z}(R) \otimes_K \mathcal{Z}(R')$.

(b) M be a semi-simple R-module which is finitely generated $\mathrm{End}_R(M)$-module. Then $Bicom_R(M) = R_M$.

1 Review of the theory motives and their analogues over function fields

Grothendieck's category of pure motives and generalizations Seeking a universal cohomology theory, Grothendieck constructed the category of effective Chow motives over a field k. According to his method, to produce the relevant category, one must modify the notion of "map" between schemes. Namely, he considers the category $Cor_\sim(k, \mathbb{Q})$ whose objects are smooth projective schemes over k and morphisms are given by

$$\mathrm{Hom}(X, Y) = \oplus_{X_i} Ch_{\sim,\dim X_i}(X_i \times_k Y, \mathbb{Q}).$$

Here X_i denote the connected components of X and $Ch_{\sim,d}(-, \mathbb{Q})$ denotes the group of cycles of dimension d modulo the equivalence relation $\sim$, with coefficients in $\mathbb{Q}$.
Then he adds kernel and cokernel of the projectors to the category, and considers the following sequence of functors

$$Smproj(k) \longrightarrow Cor_\sim(k, \mathbb{Q}) \longrightarrow Ch^{eff}_\sim(k, \mathbb{Q}) \xrightarrow{\mathbb{L}^{-1}} Ch_\sim(k, \mathbb{Q}).$$

Here $Smproj(k)$ is the category of smooth projective schemes and $Ch^{eff}_\sim(k)$ is the pseudo-abelian envelop of $Cor_\sim(k, \mathbb{Q})$. The category $Ch_\sim(k, \mathbb{Q})$ is obtained by inverting *Lefschetz motive* $\mathbb{L}$. Where the Lefschetz motive $\mathbb{L}$ comes from canonical decomposition $[\mathbb{P}^1_k] = [\mathrm{Spec}\, k] \oplus \mathbb{L}$ in $Ch^{eff}_\sim(k)$.

The standard conjectures was proposed by Grothendieck, to prove that the above construction of the category of pure motives, leads to a semisimple abelian category. Although the standard conjectures remain open problems, but nevertheless over finite fields the following result is known according to an elegant and (unexpectedly) elementary proof of Jannsen [Jan].

Theorem 1.1 (Jannsen)**.** *Let $\sim$ be the numerical equivalence, then $Ch_\sim(k, \mathbb{Q})$ is semi-simple abelian category.*

The category of motives $Ch_{\sim}(k,\mathbb{Q})$ possesses various realization functors. In particular it possesses ℓ-adic and crystalline realizations.

Remark 1.2 (Tate Conjecture)**.** Let k be a field of characteristic p and fix a prime $\ell \neq p$. Let X be a smooth geometrically irreducible projective variety over k of dimension d. Let us denote by $\overline{X}$ the base change of X to the algebraic closure k^{alg}. Let $Z^r(\overline{X})$ denote the free abelian group generated by algebraic cycles of codimension r. The cycle class map $c^r : Z^r(\overline{X}) \to H^{2r}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(r))$, induces $Ch^r_{rat}(\overline{X},\mathbb{Q}) \to H^{2r}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(r))$, which further induces

$$c^r : Ch^r_{rat}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \to H^{2r}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(k^{\mathrm{sep}}/k)}.$$

The Tate conjecture predicts that it is a bijection. This further implies that the functor from the category of pure motives (defined using the numerical equivalence) over k to the category of ℓ-adic $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-representations, given by taking ℓ-adic étale cohomology, is fully faithful.

The known cases of the conjecture includes the case of abelian varieties, over finite fields which was proved by Tate himself [Tat66], also called Tate's isogeny theorem. Zarhin [Zar] proved the case of function fields of positive characteristic, and the number field case was proved by Faltings [Fal83] throughout the course of the proof of the Mordell conjecture.

Remark 1.3 (Honda-Tate theory)**.** Assume that k is finite. The *Honda-Tate theory* establishes a bijection between the set $\Sigma Ch_{num}(k,\mathbb{Q})$ of the simple objects of $Ch_{num}(k,\mathbb{Q})$ and the conjugacy classes in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})\backslash W(q)$, where $W(q)$ is the subgroup of Weil q-numbers in $(\mathbb{Q}^{\mathrm{alg}})^\times$ and $q := \sharp k$. This consequently provides a morphism

$$K_0(Ch_{num}(k,\mathbb{Q})) \to \mathbb{Z}(\mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q})\backslash W(q))$$

.

Motives After Voevodsky The process of constructing a reasonable motivic category in such a way that it further allows to assign a motive to objects of the category $SmSch(k)$ of smooth schemes over k (or more generally category $Sch(k)$ of schemes of finite type over k), which also extends the above functor $[-] : SmProj/k \to Ch^{eff}_{\sim}(k,\mathbb{Q})$, is much more involved. This problem has been considered by several mathematicians for some decades. Among those, one should mention the remarkable results of Vladimir Voevodsky (and et al). He proceeds in the following way. He first considers the category $\sharp SmCor$ of finite smooth correspondences. For a pair X, Y of smooth schemes over k denote by c(X,Y) the free abelian group generated by integral closed subschemes Z in $X \times Y$ which are finite over X and surjective over a connected component of X. An element of $c(X,Y)$ is called a finite correspondence from X to Y. Define the category $\sharp SmCor(k)$ as the category whose objects are smooth schemes of finite type over k and morphisms are finite correspondences. The compositions of morphisms are compositions of correspondences. We denote the object in $\sharp SmCor(k)$ corresponding to a smooth scheme X by $[X]$. Now his construction can roughly be demonstrated as follows

$$Sm(k) \longrightarrow \sharp SmCor(k) \longrightarrow \mathcal{H}^b(\sharp SmCor(k), \mathbb{Q}) \xrightarrow{Q} DM_{gm}^{eff}(k, \mathbb{Q}) \xrightarrow{\mathbb{Z}(1)} DM_{gm}(k, \mathbb{Q}),$$

where $\mathcal{H}^b(\sharp SmCor(k), \mathbb{Q})$ is the homotopy category of bounded cochain complexes on (the additive symmetric monoidal category $\sharp SmCor(k)$). Here Q is the quotient map defined by modding out $\mathbb{A}^1$-homotopy invariance

$$[X \times_k \mathbb{A}^1] \to [X]$$

and Mayer-Vitorius triangle

$$[X] \longrightarrow [U] \oplus [V] \longrightarrow [U \cap V] \longrightarrow [X] \xrightarrow{[1]}$$

for a open covering $X = U \cup V$. Finally, the category $DM_{gm}(k, \mathbb{Q})$ is defined by inverting $\mathbb{Z}(1) = [\mathbb{G}_m][1]$, where $[\mathbb{G}_m]$ denotes the class associated to the multiplicative group $\mathbb{G}_m$ in $DM_{gm}^{eff}(k, \mathbb{Q})$. We use the notation $M_{gm}(-)$ for the functor

$$SmSch(k) \to DM_{gm}(k, \mathbb{Q}).$$

Remark 1.4. The above constructions can also be worked out with coefficient in a general ring A instead of $\mathbb{Q}$. We denote the resulting motivic categories by $Ch_\sim(k, A)$ and $DM_{gm}(k, A)$. On the other hand this construction can be generalized to

$$S \mapsto DM_{gm}(S, \mathbb{Q}),$$

when S is normal, according to Cisinski and Déglise [CD].

It can be observed that when $\sim$ is rational equivalence, the above construction generalizes the Grothendieck's category of Chow motives, in the sense that there is an embedding $Ch_{rat}(k, \mathbb{Z}) \to DM_{gm}(k, \mathbb{Z})$ in a compatible way

$$\begin{array}{ccc} SmProj/k & \longrightarrow & Sch_k \\ {\scriptstyle [-]}\downarrow & & \downarrow {\scriptstyle M_{gm}(-)} \\ Ch_{rat}(k, \mathbb{Z}) & \longrightarrow & DM_{gm}(k, \mathbb{Z}). \end{array}$$

One can recover the Chow groups associated with X from the corresponding motive $M_{gm}(X)$, according to the following theorem [VSF, Corollary 4.2.5].

Theorem 1.5. *Let X be a smooth scheme over k. There is a canonical isomorphism*

$$Ch^i_{rat}(X) \cong Hom_{DM_{gm}^{eff}}(M(X), \mathbb{Z}(i)[2i]).$$

Remark 1.6. The essential idea of algebraic geometry, which is to study geometry via functions, can be applied to the above geometric construction of $DM_{gm}(k,\mathbb{Q})$. Let us explain it a bit further. Let $\mathbf{PreShv}(\sharp SmCor(k))$ denote the category of *presheaf with transfers* on $Sm(k)$ (i.e. additive contravariant functors from the category $\sharp SmCor(k)$ to the category of abelian groups). Consider the category $D^-(\mathbf{PreShv}(\sharp SmCor(k)))$ of complexes of presheaves with transfers bounded from the above. This category can be viewed as an ambient category for various motivic categories in the following sense. Namely, one has the following sequence of functors

$$D^-(\mathbf{Shv_{Nis}}(\sharp SmCor(k))) \to D^-(\mathbf{Shv_{\acute{e}t}}(\sharp SmCor(k))) \to D^-(\mathbf{PreShv}(\sharp SmCor(k)))$$

induced by the obvious functors

$$\mathbf{Shv_{Nis}}(\sharp SmCor(k)) \to \mathbf{Shv_{\acute{e}t}}(\sharp SmCor(k)) \to \mathbf{PreShv}(\sharp SmCor(k)).$$

Here $\mathbf{Shv_{Nis}}(\sharp SmCor(k))$ (resp. $\mathbf{Shv_{\acute{e}t}}(\sharp SmCor(k)))$ denotes the category of Nisnevich (resp. étale) sheaves with transfers. Recall that a presheaf with transfers $\mathcal{F}$ is a Nisnevich (resp. étale) sheaf with transfers if its underlying presheaf is a Nisnevich (étale) sheaf on $SmSch/k$. We denote by $DM^{eff,-}(k)$ (resp. $DM^{eff,-}_{\acute{e}t}(k)$) the full subcategory of $D^-(\mathbf{Shv_{Nis}}(\sharp SmCor(k)))$ (resp. $D^-(\mathbf{Shv_{\acute{e}t}})$ which consists of complexes with *homotopy invariant cohomology sheaves.* A presheaf with transfers $\mathcal{F}$ is called homotopy invariant if for any smooth scheme X over k the projection $X \times \mathbb{A}^1 \to X$ induces isomorphism $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$. Now consider the full subcategory $DM^{eff,-}(k)$ of $D^-(\mathbf{Shv_{Nis}}(\sharp SmCor(k)))$ consisting of complexes with homotopy invariant cohomology sheaves. It can be shown that $DM^{eff,-}(k)$ is a triangulated subcategory; see Proposition 3.1.13. Note that after passing to rational coefficients the triangulated categories $DM^{eff,-}(k)$ and $DM^{eff,-}_{\acute{e}t}(k)$ are equivalent; see [VSF][Proposition 3.3.2].

It can be shown that the category $DM^{eff,-}_{\acute{e}t}(k)$ is the same as the derived category of $Gal(k^{\mathrm{sep}}/k)$-modules [MVW, Theorem 9.35] when we pass to coefficients in a $\mathbb{Z}/m\mathbb{Z}$ for $(m, \mathrm{char}\, k)$ = 1. This fact follows from Suslin's Rigidity Theorem; see [MVW][Theorem 7.20]. Also compare Remark 1.2 and Theorem 2.14.

Theorem 1.7. *Assume that m is an integer prime to* char k. *Let $D^-(\Gamma,\mathbb{Z}/m)$ denote the (bounded above) derived category of the complexes on the category $Mod(\Gamma,\mathbb{Z}/m)$ of discrete $\mathbb{Z}/m$-modules over the Galois group $\Gamma := Gal(k^{\mathrm{sep}}/k)$. There is an equivalence of tensor triangulated categories*

$$DM^{eff,-}_{\acute{e}t}(k,\mathbb{Z}/m) \to D^-(\Gamma,\mathbb{Z}/m).$$

Remark 1.8 (Zeta function of a motive)**.** Let $\mathcal{M}$ be a rigid F-linear tensor category. To $M \in \mathcal{M}$ and $f \in \mathrm{End}_{\mathcal{M}}(1)$, one associates the trace $tr(f)$. This is defined as the element of $\mathrm{End}_{\mathcal{M}}(1)$ given by

$$1 \xrightarrow{\eta} M \otimes \check{M} \xrightarrow{1\otimes f} \check{M} \otimes M \xrightarrow{\sigma} M \otimes \check{M} \xrightarrow{\varepsilon} 1.$$

where σ is the switch and η and ε are duality structures. Consequently one may define

$$Z(M, f, t) = \exp(\sum_{n\geq 1} tr(f^n)t^n/n \in F[\![t]\!].$$

Note that for a field k the category $DM_{gm}(k, \mathbb{Q})$ is rigid according to de Jong's theorem [deJ]. Furthermore when $k = \mathbb{F}_q$ every M in $DM_{gm}(k, \mathbb{Q})$ comes with a Frobenius endomorphism Fr_M. Set $Z(X, t) = Z(M_{gm}(X), Fr_X^{-1}, t)$. We let $\zeta(M, s) = Z(M, q^{-s})$. It can be verified that the zeta function is multiplicative on exact triangles, i.e. for $M' \to M \to M'' \to M'[1]$ we have $\zeta(M, s) = \zeta(M', s) \cdot \zeta(M'', s)$. Therefore it induces a morphism

$$K_0(DM_{gm}(k, \mathbb{Q})) \to \mathbb{Q}(q^{-s})^\times$$

Note that $K_0(DM_{gm}(k, \mathbb{Q})) \cong K_0(Ch_{rat}(k, \mathbb{Q}))$ see [Bon], and that $\zeta(-, s) : K_0(Ch_{rat}(k, \mathbb{Q})) \to \mathbb{Q}(q^{-s})^\times$ factors through the K-group of the semi-simple category of numerical motives

$$\begin{array}{ccccc} K_0(Ch_{rat}(k, \mathbb{Q})) & \xrightarrow{\sim} & K_0(DM_{gm}(k, \mathbb{Q})) & \longrightarrow & \mathbb{Q}(q^{-s})^\times \\ & \searrow & & \nearrow & \\ & & K_0(Ch_{num}(k, \mathbb{Q})) & & \end{array}$$

Note further that it can be shown that the functional equation and standard identities holds for $\zeta(M, s)$. Let us finally mention that for a motive M in $DM_{gm}(S, \mathbb{Q})$, over a general base S of finite type over $\mathbb{Z}$, one defines

$$\zeta(M, s) = \prod_p \zeta(i_p^* M, s)$$

where p runs over set of closed points $i_p : \operatorname{Spec} \kappa(p) \to S$ of S. This defines a Dirichlet series, absolutely convergent for $Re(s) \gg 0$. For a detail account we refer the interested reader to [Kah].

Motives in Function Fields Let us now move to the function fields set up. In [Tae] Taelman proposes several categories that serve the analogues role (of the above motivic categories) over function fields. He constructs these categories in the following way. He first considers the Anderson category of A-motives. Let us recall the definition.

Definition 1.9. Let $A := \mathbb{F}_q[z]$, and let L be an A-field under $A \to L,\ t \mapsto \theta$. An effective A-motive of rank r over L is a pair $M = (M, \tau)$ consisting of a free and finitely generated A_L-module M of rank r, and a map $\tau : \sigma^* M \to M$ such that a power $(z - \theta)^d$ of $(z - \theta)$ annihilates the cokernel of τ.

This category possesses a special A-motive $\underline{\mathbf{C}}$, which mirrors the motive associated with the (dual of) $M(\mathbb{G}_m)$ over number fields. The definition of Carlitz motive is as follows.

Example 1.10. Let $A = \mathbb{F}_q[z]$ and L be an A-field $A \to L$, $z \mapsto \theta$. The pair $\underline{\mathbf{C}} = (Ae, \tau)$ with $\tau(fe) = \sigma(f)(z - \theta)e$ is called the Carlitz (A-)motive.

As an evidence of the above analogy, one may notice that the zeta-function of the above A-motive $\underline{\mathbf{C}}$ is of the following form

$$\frac{1 - (z - \theta)t}{1 - t} \in A[\![t]\!];$$

see Section 6 for the definition of zeta function associated with an A-motive. Note further that $M(\mathbb{G}_m)$ is obtained by splitting off $M(\mathbb{A}^1) \to M(\operatorname{Spec} k)$, with respect to the specified rational point $0 \in \mathbb{A}^1(k)$. Here this role is played by the place given by the ideal $z - \theta$.
To pass from the category of effective motives to the category of (mixed) motives, one extends the coefficients to rational coefficients (by tensoring up the Hom-sets with $Q = \operatorname{Frac}(A)$) and then formally inverting (tensor powers of) Carlitz module $\underline{\mathbf{C}}$. Taelman denotes the resulting category by $t\mathcal{M}^\circ$.

Remark 1.11 (realizations)**.** The category $t\mathcal{M}^\circ$ is a rigid abelian $\operatorname{Frac}(A) =: Q$-linear tensor category, which inherits étale and crystalline realization functors. Note however that objects of this category are not pure, in the sense that the eigenvalue of Frobenius endomorphism operating on the cohomology groups might have different absolute value (unless one imposes purity conditions); also compare this with [Gos][Theorem 5.6.10]. We discuss this in a slightly modified situation in chapter 2. Furthermore, one can set $Ch^i(-) := \operatorname{Hom}(-, \underline{C}^{\otimes i})$ and $Ch_i(-) := \operatorname{Hom}(\underline{C}^{\otimes i}.-)$. Consider the following functor

$$Ch(-) := \oplus_i Ch^i(-) : t\mathcal{M}^\circ \to Q - \text{Vect. Sp.}$$

to the category of finite dimensional vector spaces. Compare Theorem 1.5. Note the Hom-set $\operatorname{Hom}(\underline{M}, \underline{M}') \otimes Q_\nu$ equals $\operatorname{Hom}(\omega_\nu(M), \omega_\nu(M'))$ for the associated iso-crystals $\omega_\nu(\underline{M})$ and $\omega_\nu(\underline{M}')$ at the place $\nu = \infty$. The iso-crystals admit weight decomposition and since non-trivial morphisms between them only exist when the weights are equal, therefore the Hom-sets are non-zero only for finitely many i. Hence $Ch(\underline{M})$ is a finite vector space. Note in addition that $Ch(\underline{M})$ has a natural graded Q-algebra structure.

The category $t\mathcal{M}^\circ$ together with the obvious fiber functor $\omega : t\mathcal{M}^\circ \to Q - \textit{vector spaces}$ provides a tannakian category which is a candidate for the analogues motivic category over function fields. Still one may naturally want

- to consider motives which admit multiplications by a discrete valuation ring which is strictly bigger than $\mathbb{F}_q[z]$,

- to construct a category analogous to the category (mixed) motives over a general base, see Remark 1.4, and

- to geometrize this category. More precisely, one may think of Shimura varieties as a moduli for motives according to the Deligne's conception of Shimura varieties [Del71]. But the above category of motives over function fields do not behave well concerning moduli problems. This is for example because there are too much freedom at infinity of $\operatorname{Spec} A$.

To handle the first issue, we take A to be the ring $A := \Gamma(\dot{C}, \mathcal{O}_C)$ of functions on a smooth projective curve C over $\mathbb{F}_q$ which are regular on $\dot{C} := C \smallsetminus \{\infty\}$ for a place ∞ on C. The characteristic morphism $A \to L$ can be replaced by a section $s : \operatorname{Spec} L \to C$ and the ideal $\langle z - \theta \rangle$ by the sheaf of ideal corresponding to the graph of s. Furthermore one can construct a generalization of Carlitz motive in this category. This is called *Carlitz-Hayes motive*. Finally, to construct the desired category, one inverts the motive $\mathcal{H}$, as is proposed in [Ha-Ju]. As we will see bellow, it is straightforward to generalize our definition to treat a general base ring L. This allows to think of the corresponding moduli problem. To handle the third issue, which is slightly more involved, one must make the objects completed at the place infinity $\infty \in C$. Thus we replace the (locally) free A_L-module M by a locally free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_L}$-modules (or equivalently a vector bundle over C_L). Note that one should be a bit careful with extending τ over infinity. According to the definition of A-motives, one requires the morphism τ to have it's zeros along $V(J)$, where J is the ideal corresponding to the graph of a (characteristic) section $s_1 : S := \operatorname{Spec} L \to C$. Note that this in particular indicates that the morphism τ can not be defined over the whole (relative) curve C_S. This is because over a projective curve, the balance between order of zero's and poles of τ should be preserved. Therefore to provide an appropriate definition, we should allow further characteristic section(s) s_i. Note that aside from making the definition efficient, in fact, introducing several characteristics turns out to be an extremely useful tool in function fields set up, which is still absent over number fields. Note in addition that introducing further characteristic sections corresponds to inverting several Carlitz-Hayes motives in Taelman's construction. Regarding this discussion, we eventually come to the following definition. Let $\dot{C} := C \smallsetminus \{\nu_i\}$. Let L be a ring over $\mathbb{F}_q$. Then we define a C-motive $\underline{\mathcal{M}}$ with characteristic $\underline{\nu}$ over L to be a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$ consisting of

(a) a locally free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_L}$-modules of finite rank,

(b) an isomorphism $\tau_{\mathcal{M}} : \sigma^* \dot{\mathcal{M}} \to \dot{\mathcal{M}}$ where $\dot{\mathcal{M}}$ denotes the pullback of $\mathcal{M}$ under the inclusion $\dot{C}_L \to C_L$, and $\sigma = id \times \sigma_L$ where $\sigma_L : L \to L$ is the absolute Frobenius morphism over $\mathbb{F}_q$.

The C-motives together with quasi-isogenies as its set of morphisms form a categorry $\mathcal{M}ot_C^{\underline{\nu}}(L)$; see Definition 2.1. This category admits étale and crystalline realization functors. We discuss this in Section 2.1. One can therefore think of the analog of Tate conjecture for this category, see Theorem 2.14 for the statement and proof over finite fields.
Moreover, the above notion collides the notion of (G)-Shtukas; see Chapter 4. Our method for

studying this category is almost elementary and similar to the number fields set up. Namely, by looking at the endomorphism algebra associated with the objects, e.g. see Theorem 3.4. Note however that as we mentioned before, this category consists of mixed motives rather than pure ones, and to define the subcategory of pure objects one has to impose purity conditions. But nevertheless, as we will see in section 5 and 7 respectively, it can be shown that a modified version of the Honda-Tate theory, and a fact similar to the semisimplicity result of Jannsen, applies to this category; compare Remark 1.3 and Theorem 1.1. Note however that the latter only holds for $\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}}_q)$; See Theorem 7.1. The reason behind this difference comes from the following elementary observation. Namely, despite the characteristic zero case, a non semi-simple matrix may become semi-simple after raising to a relevant power. We also discuss the zeta function associated with C-motives in section 6.

2 Category of C-Motives And Realization Functors

In this section we present the basic definitions of the category of C-motives.
Let $A := \Gamma(\dot{C}, \mathcal{O}_C)$ be the coordinate ring of the open subscheme $\dot{C} := C \smallsetminus \{\nu_i\}$.

Definition 2.1. (a) Let S be a scheme over $\mathbb{F}_q$. A C-motive $\underline{\mathcal{M}}$ with characteristic $\underline{\nu}$ over S is a tuple $(\mathcal{M}, \tau_{\mathcal{M}})$ consisting of

(i) a locally free sheaf $\mathcal{M}$ of $\mathcal{O}_{C_S}$-modules of finite rank,

(ii) an isomorphism $\tau_{\mathcal{M}} : \sigma^*\dot{\mathcal{M}} \to \dot{\mathcal{M}}$ where $\dot{\mathcal{M}}$ denotes the pullback of $\mathcal{M}$ under the inclusion $\dot{C}_S \to C_S$, and $\sigma = id \times \sigma_S$ where $\sigma_S : S \to S$ is the absolute Frobenius morphism over $\mathbb{F}_q$.

A morphism $\underline{\mathcal{M}} \to \underline{\mathcal{N}}$ is a commutative diagram

$$\begin{array}{ccc} \sigma^*\dot{\mathcal{M}} & \xrightarrow{\tau_{\mathcal{M}}|_{\dot{C}_S}} & \dot{\mathcal{M}} \\ \sigma^* f \downarrow & & \downarrow f \\ \sigma^*\dot{\mathcal{N}} & \xrightarrow{\tau_{\mathcal{N}}|_{\dot{C}_S}} & \dot{\mathcal{N}}. \end{array}$$

Similarly the set of quasi-morphisms $Q\operatorname{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ is given by the following commutative diagrams

$$\begin{array}{ccc} \sigma^*\mathcal{M}_\eta & \xrightarrow{\tau_{\mathcal{M},\eta}} & M_\eta \\ \sigma^* f \downarrow & & \downarrow f \\ \sigma^*\mathcal{N}_\eta & \xrightarrow{\tau_{\mathcal{N},\eta}} & \mathcal{N}_\eta. \end{array}$$

Here $\mathcal{M}_\eta$ denotes the pull back of $\mathcal{M}$ under $\eta \times_{\mathbb{F}_q} S \to C_S$. One can equivalently say that $Q\operatorname{Hom}_L(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ consists of the equivalence classes of the commutative diagrams

$$\begin{array}{ccc} \sigma^*\dot{\mathcal{M}} & \xrightarrow{\tau_{\mathcal{M}}} & \dot{\mathcal{M}} \\ {\scriptstyle \sigma^* f}\downarrow & & \downarrow{\scriptstyle f} \\ \sigma^*\dot{\mathcal{N}} \otimes \mathcal{O}(D_S) & \xrightarrow{\tau_{\mathcal{N}}} & \dot{\mathcal{N}} \otimes \mathcal{O}(D_S), \end{array}$$

where D is a divisor on C and $D_S := D \times_{\mathbb{F}_q} S$, and two such diagrams for divisors D and D' are called equivalent provided that the corresponding diagrams agree when we tensor with $\mathcal{O}(D_L + D'_L)$. We further sometimes use $\operatorname{Hom}_L(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ to denote the set consisting of the true morphisms $\dot{\mathcal{M}} \to \dot{\mathcal{N}}$ which makes a similar diagram (without twisting by $\mathcal{O}(D_S)$) commutative. We also sometimes drop the subscript L from the Hom-sets when it is clear from the context.

(b) A quasi-isogeny between $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ is a morphism in $Q\operatorname{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ which admits an inverse.

(c) We denote by $\mathcal{M}ot_C^{\underline{\nu}}(S)$ the Q-linear category whose objects are C-motives of characteristic $\underline{\nu}$ as above, with quasi-morphisms (resp. morphisms) as its morphisms. We further denote by $\mathcal{M}ot_C^{\underline{\nu}}(S)^\circ$ the category obtained by restricting the set of morphisms to quasi-isogenies. When $S = \operatorname{Spec} L$ we simply use the notation $\mathcal{M}ot_C^{\underline{\nu}}(L)$.

Proposition 2.2. *Let L be a field over $\mathbb{F}_q$. The category $\mathcal{M}ot_C^{\underline{\nu}}(L)$ is a Q-linear rigid abelian tensor category. It further admits a fiber functor over $Q_L := \operatorname{Frac}(A_L)$.*

Proof. Let $f : \mathcal{M} \to \mathcal{N}(D)$ be a representative for a morphism in $Q\operatorname{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$. Then $\underline{\ker} f := (\ker f, \tau_{\mathcal{M}}|_{\sigma^*\ker f})$ and $\underline{\operatorname{im}} f := (\operatorname{im} f, \tau_{\mathcal{N}}|_{\sigma^*\operatorname{im} f})$. Consider the cokernel $\mathcal{F} := \operatorname{coker} f : \mathcal{M} \to \mathcal{N}$ as a coherent sheaf of $\mathcal{O}_{C_L}$-modules. The torsion subsheaf $\mathcal{T}$ has finite support and $\mathcal{F}/\mathcal{T}$ is a torsion free sheaf. The morphism $\tau_{\mathcal{N}}$ induces a morphism $\tau_{\mathcal{F}} : \sigma^*\mathcal{F} \to \mathcal{F}$. We define $\underline{\operatorname{coker}} f := (\mathcal{F}/\mathcal{T}, \tau_{\mathcal{F}/\mathcal{T}})$ and $\underline{\operatorname{coim}} f := \underline{\ker}(\underline{\mathcal{N}} \to \underline{\operatorname{coker}} f)$. Moreover this is clear that the natural morphism $\underline{\operatorname{im}} f \to \underline{\operatorname{coim}} f$ is a quasi-isogeny and therefore an isomorphism in $\mathcal{M}ot_C^{\underline{\nu}}(L)$.
The tensor product of two C-motives $\underline{\mathcal{M}} := (\mathcal{M}, \tau_{\mathcal{M}})$ and $\underline{\mathcal{N}} := (\mathcal{N}, \tau_{\mathcal{N}})$ is the C-motive $\underline{\mathcal{M}} \otimes \underline{\mathcal{N}}$ consisting of the locally free sheaf of $\mathcal{O}_{C_L}$-modules $\mathcal{M} \otimes_{\mathcal{O}_{C_L}} \mathcal{N}$ and the isomorphism $\tau_{\underline{\mathcal{M}} \otimes \underline{\mathcal{N}}} := \tau_{\mathcal{M}} \otimes \tau_{\mathcal{N}}$. The unit object for the tensor product is $\underline{\mathbb{1}} := (\mathcal{O}_{C_L}, \mathrm{id})$, and precisely we have $Q\operatorname{End}(\underline{\mathbb{1}}) = Q$. One can easily see that this category has an internal $\underline{\operatorname{Hom}}$ object. Namely we define $\underline{\mathcal{H}} := \underline{\operatorname{Hom}}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ to be the object with $\mathcal{H} := \operatorname{Hom}_{\mathcal{O}_{C_L}}(\mathcal{M}, \mathcal{N})$ as the underlying locally free sheaf and $\tau_{\mathcal{H}}$ is given by sending $h \in \mathcal{H}$ to $\tau_{\mathcal{N}} \circ h \circ \tau_{\mathcal{M}}^{-1}$. This further defines the functor

$$\check{(-)} := \underline{\operatorname{Hom}}(-, \underline{\mathbb{1}}) : \mathcal{M}ot_C^{\underline{\nu}}(L) \to \mathcal{M}ot_C^{\underline{\nu}}(L),$$

which sends $\underline{\mathcal{M}}$ to its dual $\check{\underline{\mathcal{M}}}$. Finally sending a C-motive $\underline{\mathcal{M}} := (\mathcal{M}, \tau_{\mathcal{M}})$ to the generic fiber $\mathcal{M}_\eta$ of the underlying locally free sheaf $\mathcal{M}$ equips the category with a fiber functor

$$\omega(-) : \mathcal{M}ot_C^{\nu}(L) \to Q_L - \text{Vector spaces}.$$

□

Proposition 2.3. *Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{N}}$ be C-motives, over a field L. Assume that L is finite over $\mathbb{F}_q$ of degree s. The following are equivalent*

(a) $f \in Q\,\mathrm{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ is an quasi-isogeny.

(b) There is a non-zero element $a \in Q$ and $\check{f} \in Q\,\mathrm{Hom}(\underline{\mathcal{N}}, \underline{\mathcal{M}})$ with $\check{f} \circ f = a.\,\mathrm{id}_{\mathcal{M}}$ and $f \circ \check{f} = a.\,\mathrm{id}_{\mathcal{M}'}$.

Proof. After replacing f with $\alpha.f$, for a relevant $\alpha \in Q$, we may assume that f can be represented by a morphism of locally free sheaves $\mathcal{M} \to \mathcal{N}$ with a finitely supported cokernel. This is because f is a quasi-isogeny, and therefore a generic isomorphism. We denote it again by f. Let $h \in A_L$ be a non-zero element which annihilates the cokernel $h \cdot \operatorname{coker} f = 0$. Now set

$$a := h.\sigma(h). \dots .\sigma^{s-1}(h)$$

Precisely $\sigma(a) := \sigma(h). \dots .\sigma^{s-1}(h).\sigma^s(h) = a$ and thus $a \in A$. As a also annihilates the coker f we obtain the morphism $\check{f}$

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{f} & \mathcal{N} & \longrightarrow & \operatorname{coker} f \\ \downarrow{\scriptstyle a} & \check{f} & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a=0} \\ \mathcal{M} & \xrightarrow[f]{} & \mathcal{N} & \longrightarrow & \operatorname{coker} f\ . \end{array}$$

b) implies a) is obvious.

□

Remark 2.4. One can prove that $f \in Q\,\mathrm{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{N}})$ is a quasi-isogeny if and only if there is a non-zero element $a \in Q$ and $\check{f} \in Q\,\mathrm{Hom}(\underline{\mathcal{N}}, \underline{\mathcal{M}})$ with $\check{f} \circ f = a.\,\mathrm{id}_{\mathcal{M}}$ and $f \circ \check{f} = a.\,\mathrm{id}_{\mathcal{N}}$.

2.1 Crystalline and Étale realization functors and Tate conjecture

In analogy with the theory of motives, the category of C-motives also admits crystalline and étale realizations. In this section we recall the definition of the corresponding realization categories and functors. We further prove the analog of the Tate conjecture for the category C-motives over a finite field.

2.1.1 Category Of (Iso-)Crystals

Definition 2.5. Set $\hat{A} := \mathbb{F}[\![z]\!]$ and $\widehat{Q} := \mathrm{Frac}(\hat{A})$.

(a) A *$\hat{\sigma}$-crystal* $\underline{\hat{M}}$ (resp. *$\hat{\sigma}$-iso-crystal*) over $L/\mathbb{F}_q$ of rank r is a tuple $(\hat{M}, \hat{\tau})$ (resp. $(\dot{\hat{M}}, \hat{\tau})$) consisting of the following data

  (i) a free $\hat{A}_L := \hat{A}\widehat{\otimes}_{\mathbb{F}} L$-modulue (resp. $\hat{Q}_L$-module) $\hat{M}$ (resp. $\dot{\hat{M}}$) of rank r,

  (ii) an isomorphism $\hat{\tau} : \hat{\sigma}^*\hat{M}[1/z] \to \hat{M}[1/z]$ (resp. $\hat{\tau} : \hat{\sigma}^*\dot{\hat{M}} \to \dot{\hat{M}}$), where $\hat{M}[1/z] := \hat{M} \otimes_{L[\![z]\!]} L(\!(z)\!)$.

(b) A quasi-morphism between $\underline{\hat{M}} := (\hat{M}, \hat{\tau})$ to $\underline{\hat{M}}' := (\hat{M}', \hat{\tau}')$ is a morphism $f : \hat{M}[1/z] \to \hat{M}'[1/z]$ which makes the following diagram

$$\begin{array}{ccc} \hat{\sigma}^*\hat{M}[1/z] & \xrightarrow{\hat{\tau}} & \hat{M}[1/z] \\ {\scriptstyle \sigma_L^* f}\downarrow & & \downarrow{\scriptstyle f} \\ \sigma^*\hat{M}'[1/z] & \xrightarrow{\hat{\tau}'} & \hat{M}'[1/z] \end{array}$$

commutative. We denote the resulting category by $\hat{\sigma}$-$\mathbf{Cryst}_{\mathbb{F}}(L)$.

(c) A $\hat{\sigma}$-crystal $\underline{\hat{M}} := (\hat{M}, \hat{\tau})$ is called *étale* if $\hat{\tau}$ lifts to an isomorphism $\hat{\tau} : \hat{\sigma}^*\hat{M} \to \hat{M}$. We denote the corresponding category by **Ét** $\hat{\sigma}$-$\mathbf{Cryst}_{\mathbb{F}}(L)$.

Definition 2.6. The first étale cohomology realization of an étale $\hat{\sigma}$-crystal $\underline{\hat{M}}$ at ν is the $\Gamma_L := Gal(L^{sep}/L)$-module of $\hat{\tau}$-invariants

$$\mathrm{H}^1_{\text{ét}}(-, \hat{A}) : \underline{\hat{M}} \mapsto \left(\hat{M} \otimes_{\hat{A}_L} \hat{A}_{L^{\text{sep}}}\right)^{\hat{\tau}}.$$

We set $\mathrm{H}^i_{\text{ét}}(\underline{\hat{M}}, \hat{A}) := \wedge^i \mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}, \hat{A})$ and $\mathrm{H}^i_{\text{ét}}(\underline{\hat{M}}, \hat{Q}) := \mathrm{H}^i_{\text{ét}}(\underline{\hat{M}}, \hat{A}) \otimes_{\hat{A}} \hat{Q}$.

Remark 2.7. The first étale cohomology functor $\mathrm{H}^1_{\text{ét}}(-, \hat{A})$ is a fully faithful embedding of the category of **Ét** $\hat{\sigma}$-$\mathbf{Cryst}_{\mathbb{F}}(L)$ into the category of $\hat{A}[\Gamma_L]$-modules. One can recover $\underline{\hat{M}}$ from its étale realization by taking Galois invariants

$$\underline{\hat{M}} := \left(\mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}, \hat{A}) \otimes_{\hat{A}} \hat{A}_{L^{\text{sep}}}\right)^{\Gamma_L}.$$

Proposition 2.8. *The first étale cohomology functor*

$$\mathrm{H}^1_{\acute{e}t}(-, \hat{A}) : \textit{\textbf{Ét}}\ \hat{\sigma}\textit{-\textbf{Cryst}}(L) \to \hat{A}[\Gamma_L] - \textit{modules}$$

(resp. $\mathrm{H}^1_{\acute{e}t}(-, \widehat{Q}) : \textit{\textbf{Ét}}\ \hat{\sigma}\textit{-\textbf{Cryst}}(L) \to \widehat{Q}[\Gamma_L] - \textit{modules}$*) is left exact (resp. exact).*

Proof. Let

$$0 \to \underline{\hat{M}}'' \to \underline{\hat{M}} \to \underline{\hat{M}}' \to 0$$

be an exact sequence of étale crystals over L. Now regarding the definition of the étale realization functor, Definition 2.6, we get the following diagram

$$\begin{array}{ccccccccc} & & \mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}'', \hat{A}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}, \hat{A}) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}', \hat{A}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{M}'' \otimes_L L^{\text{sep}} & \longrightarrow & \hat{M} \otimes_L L^{\text{sep}} & \longrightarrow & \hat{M}' \otimes_L L^{\text{sep}} & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle 1-\hat{\tau}_{\hat{M}''}} & & \downarrow {\scriptstyle 1-\hat{\tau}_{\hat{M}}} & & \downarrow {\scriptstyle 1-\hat{\tau}_{\hat{M}'}} & & \\ 0 & \longrightarrow & \hat{M}'' \otimes_L L^{\text{sep}} & \longrightarrow & \hat{M} \otimes_L L^{\text{sep}} & \longrightarrow & \hat{M}' \otimes_L L^{\text{sep}} & \longrightarrow & 0. \end{array}$$

Consequently $\mathrm{H}^1_{\text{ét}}(-, \hat{A})$ is left exact by snake lemma. According to Remark 2.7 and by comparing the dimensions of $\widehat{Q}$-vector spaces we argue that $\mathrm{H}^1_{\text{ét}}(-, \widehat{Q}) := \mathrm{H}^1_{\text{ét}}(-, \hat{A}) \otimes_{\hat{A}} \widehat{Q}$ is right exact.

□

2.1.2 Crystalline realization of C-Motives

The scheme $\mathrm{Spec}(A_\nu \hat{\otimes}_{\mathbb{F}_q} L)$ is the union of its connected components $\mathrm{Spec}\, A_\nu \hat{\otimes}_\ell L$, where ℓ is the intersection of $\mathbb{F}_\nu$ and L, i.e. those elements $\alpha \in L$ such that $\alpha^{q^{\deg \nu}} = \alpha$. The set of connected components is endowed with free $\mathbb{Z}/s := \mathrm{Gal}(\ell/\mathbb{F}_q)$-action, where $\sharp \ell = q^s$, and we write $\mathrm{Spec}\, A_\nu \hat{\otimes}_{\mathbb{F}_q} L = \amalg_{i \in \mathbb{Z}/s} V(\mathfrak{a}_{\tilde{\nu}_i})$. The connected component $V(\mathfrak{a}_{\tilde{\nu}_i})$ corresponds to the ideal $\mathfrak{a}_{\tilde{\nu}_i} = \left\langle a \otimes 1 - 1 \otimes a^{q^i} : a \in \ell \right\rangle$ and $\hat{\sigma}$ cyclically permutes these components and $\hat{\sigma}^s$ leaves each of the components $V(\mathfrak{a}_{\tilde{\nu}_i})$ stable. Here $\tilde{\nu}_i$ stands for the places of C_ℓ lying above $\nu \in C$.

Definition 2.9. For $\underline{\mathcal{M}} := (\mathcal{M}, \tau_\mathcal{M})$ in $\mathcal{M}ot_C^\nu(L)$ and a place $\nu \in C$, let $\hat{M}$ denote the pull back $\mathcal{M} \otimes_{\mathcal{O}_{C_L}} \hat{\mathcal{O}}_{C,\nu}$. Sending $\underline{\mathcal{M}}$ to $(\hat{M}, \hat{\tau}_{\hat{M}} := \tau_\mathcal{M} \otimes 1)$ defines a functor

$$\Gamma_\nu(-) : \mathcal{M}ot_C^\nu(L) \to \hat{\sigma}\text{-}\mathbf{Cryst}_{\mathbb{F}_\nu}(L).$$

Proposition 2.10. *Let L be a field and let $\ell = \mathbb{F}_{q^s}$, and $\tilde{\nu}_i$ be as above. The reduction modulo $\mathfrak{a}_{\tilde{\nu}_i}$ induces an equivalence of categories*

$$Red : \hat{\sigma}\text{-}\boldsymbol{Cryst}_{\mathbb{F}_\nu}(L) \xrightarrow{\sim} \hat{\sigma}^s\text{-}\boldsymbol{Cryst}_{\mathbb{F}_{\tilde{\nu}_i}}(L)$$

$$\underline{\hat{M}} := (\hat{M}, \hat{\tau}) \mapsto Red\ \underline{\hat{M}} := (\hat{M}/\mathfrak{a}_{\tilde{\nu}_i}, \hat{\tau}^s)$$

In addition when we restrict to the category of étale σ-crystals the above functor preserves the étale realization, i.e. $\mathrm{H}^1_{\acute{e}t}(\underline{\hat{M}}, \hat{A}_\nu) = \mathrm{H}^1_{\acute{e}t}(Red\ \underline{\hat{M}}, \hat{A}_{\tilde{\nu}_i})$

Proof. We construct a quasi-inverse functor to this functor. Let us set $\mathfrak{a}_i := \mathfrak{a}_{\tilde{\nu}_i}$. Let $\underline{\hat{M}}' := (\hat{M}', \hat{\tau}' : (\hat{\sigma}^s)^*\hat{M}' \to \hat{M}')$ be a local $\hat{\sigma}^s$-crystal at ν_i over L. The quasi-inverse functor sends $\underline{\hat{M}}'$ to the local $\hat{\sigma}$-crystal $\underline{\hat{M}} := (\oplus_{0\leq j<f}\hat{M}_i^j, \oplus_{0<j\leq f}\hat{\tau}_i^j)$ at ν over L, where $\hat{M}_i^j := (\hat{\sigma}^*)^j\hat{M}'$, $\hat{\tau}_i^j := \mathrm{id}_{\hat{M}_i^j} : \hat{\sigma}^*\hat{M}_i^{j-1} \to \hat{M}_i^j$ for $0 < j < s$ and $\hat{\tau}_i^s := \hat{\tau}' : \hat{\sigma}^*\hat{M}_i^{s-1} = (\hat{\sigma}^*)^s\hat{M}_i' \to \hat{M}_i'$. Clearly *Red* $\underline{\hat{M}} = \underline{\hat{M}}'$. Therefore we can identify

$$\begin{array}{c}\left(\bigoplus_{0\leq j<s}(\hat{\sigma}^*)^j(\hat{M}/\mathfrak{a}_i\hat{M}), (\hat{\tau}^s \bmod \mathfrak{a}_i) \oplus \bigoplus_{0<j<s} \mathrm{id}\right)\\ \Big\downarrow{\scriptstyle \bigoplus_{0\leq j<s}\hat{\tau}^j \bmod \mathfrak{a}_{i+j}}\ \cong\\ \left(\bigoplus_{0\leq j<s}\hat{M}/\mathfrak{a}_{i+j}\hat{M}, \bigoplus_{0\leq j<s}\hat{\tau} \bmod \mathfrak{a}_{i+j}\right) = (\hat{M}, \hat{\tau}).\end{array}$$

Note that the morphism $\hat{\tau}^j$ has its poles and zeros away from $V(\mathfrak{a}_{i+j})$.

The isomorphism between the Tate modules follows from the observation that an element $(x_j)_{j\in\mathbb{Z}/s\mathbb{Z}}$ is $\hat{\tau}$-invariant if and only if $x_{j+1} = \hat{\tau}(\sigma^*x_j)$ for all j and $x_i = \hat{\tau}^s((\hat{\sigma}^*)^s x_i)$. □

When $\nu = \nu_i$ is a characteristic place of $\underline{\mathcal{M}}$, then $\ell = \mathbb{F}_\nu \subseteq L$, $s = \deg\nu$ and $\mathbb{F}_{\tilde{\nu}_j} = \mathbb{F}_\nu$, thus we can set

$$\begin{array}{ccccc}\mathrm{H}^1_{\mathbf{Cris}}(-, A_\nu) : \mathcal{M}ot_C^\nu(L) & \longrightarrow & \hat{\sigma}\text{-}\mathbf{Cryst}_{\mathbb{F}_\nu}(L) & \longrightarrow & \hat{\sigma}^s\text{-}\mathbf{Cryst}_{\mathbb{F}_\nu}(L)\\ \underline{\mathcal{M}} & \mapsto & (\hat{M}, \hat{\tau}_{\hat{M}} := \tau_{\mathcal{M}} \otimes 1) & \mapsto & (\hat{M}/\mathfrak{a}_{\nu_{i0}}, \hat{\tau}^s_{\hat{M}})\end{array}$$

2.1.3 Étale realization of C-Motives and Tate Conjecture

When ν is different from the characteristic places ν_i, assigning the Γ_L-module $\mathrm{H}^i_{\text{ét}}(\underline{\hat{M}})$ to the effective crystal $\underline{\hat{M}} := \Gamma_\nu(\underline{\mathcal{M}})$, defines a functor

$$\mathrm{H}^i_{\text{ét}}(-, A_\nu) : \mathcal{M}ot_C^\nu(L) \to A_\nu[\Gamma_L]\textbf{-modules.}$$

Tensoring up with Q_ν we similarly define

$$\mathrm{H}^i_{\text{ét}}(-, Q_\nu) : \mathcal{M}ot_C^\nu(L) \to Q_\nu[\Gamma_L]\textbf{-modules}.$$

We call the above functor the i'th étale realization functor with coefficients in A_ν (resp. Q_ν). We also use the notation $\omega^\nu(-)$ (resp. $\omega^\nu_{Q_\nu}(-)$) for the first cohomology functor $\mathrm{H}^1_{\text{ét}}(-, A_\nu)$ (resp. $\mathrm{H}^1_{\text{ét}}(-, Q_\nu)$).

Proposition 2.11. *The functor $\omega^{\nu}_{Q_\nu}(-)$ (resp. $\omega^{\nu}(-)$) is exact (resp. left exact).*

Proof. This follows from exactness of $\Gamma_\nu(-)$ and Proposition 2.8. ☐

Proposition 2.12. *Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}'$ be in $\mathcal{M}ot^{\underline{\nu}}_C(L)$, and let ν be a place on C different from characteristic places ν_i. Then the obvious morphism $Q\operatorname{Hom}(\underline{\mathcal{M}},\underline{\mathcal{M}}')\otimes_Q Q_\nu \to \operatorname{Hom}_{Q_\nu[\Gamma_L](\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}),\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}'))}$ is injective. In particular $\dim_Q Q\operatorname{Hom}(\underline{\mathcal{M}},\underline{\mathcal{M}}')\le \operatorname{rk}\underline{\mathcal{M}}.\operatorname{rk}\underline{\mathcal{M}}'$.*

Proof. Clearly we have an embedding of $Q\operatorname{Hom}(\underline{\mathcal{M}},\underline{\mathcal{M}}')\otimes_Q Q_\nu$ in $\operatorname{Hom}_{Q_L}(\mathcal{M}_\eta,\mathcal{M}'_\eta)\otimes_Q Q_\nu$. As the latter sits inside $\operatorname{Hom}_{Q_{\nu,L}}(\Gamma_\nu(\underline{\mathcal{M}}),\Gamma_\nu(\underline{\mathcal{M}}'))$ and is compatible with respect to τ and τ', as well as $\Gamma_\nu(\tau)$ and $\Gamma_\nu(\tau')$. Therefore we have an embedding

$$Q\operatorname{Hom}(\underline{\mathcal{M}},\underline{\mathcal{M}}')\otimes_Q Q_\nu \hookrightarrow \operatorname{Hom}_{\mathbf{Cris}}(\Gamma_\nu(\underline{\mathcal{M}}),\Gamma_\nu(\underline{\mathcal{M}}'))\otimes_{A_\nu} Q_\nu$$

The latter equals $\operatorname{Hom}_{Q_\nu(\Gamma_L)}(\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}),\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}'))$, see Remark 2.7. ☐

Remark 2.13. Let L be a finite field. Let $\underline{\hat{M}} := (L[\![z]\!]^r, T\cdot\hat{\sigma})$ and $\underline{\hat{M}}' = (L[\![z]\!]^{r'}, T'\cdot\hat{\sigma})$ be $\hat{\sigma}$-crystals in $\hat{\sigma}$-$\mathbf{Cryst}_{\mathbb{F}_q}(L)$ and let $F:\underline{\hat{M}}\to\underline{\hat{M}}'$ be a morphism. By Lang's theorem there is $\Phi\in\mathrm{GL}_r(L^{\mathrm{sep}}[\![z]\!])$ (resp. $\Phi'\in\mathrm{GL}_{r'}(L^{\mathrm{sep}}[\![z]\!])$) such that $\Phi = T\cdot\hat{\sigma}(\Phi)$ (resp. $\Phi' = T'\cdot\hat{\sigma}(\Phi')$). In particular their columns form a basis for $\mathrm{H}^1_{\text{ét}}(\underline{\hat{M}})$ (resp. $\mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}')$) and we may write $\mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}) = \Phi\cdot L[\![z]\!]^r$ (resp. $\mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}') = \Phi'\cdot L[\![z]\!]^{r'}$)

$$\begin{array}{ccc}
\mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}') = \Phi\cdot L[\![z]\!]^r & \longrightarrow & \mathrm{H}^1_{\text{ét}}(\underline{\hat{M}}') = \Phi'\cdot L[\![z]\!]^{r'} \\
{\scriptstyle \Phi^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle \Phi'^{-1}} \\
L[\![z]\!]^r & \xrightarrow{\Phi'^{-1}\cdot F\cdot\Phi} & L[\![z]\!]^{r'}
\end{array}$$

Theorem 2.14. *Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}'$ be C-motives over a finite field L. Let ν be a closed point of C away from characraccteristic places ν_i. Then there are isomorphisms*

$$\operatorname{Hom}_L(\underline{\mathcal{M}},\underline{\mathcal{M}}')\otimes_A A_\nu \xrightarrow{\sim} \operatorname{Hom}_{A_\nu[\Gamma_L]}(\omega^\nu(\underline{\mathcal{M}}),\omega^\nu(\underline{\mathcal{M}}'))$$

(resp.

$$Q\operatorname{Hom}_L(\underline{\mathcal{M}},\underline{\mathcal{M}}')\otimes_Q Q_\nu \xrightarrow{\sim} \operatorname{Hom}_{Q_\nu[\Gamma_L]}(\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}),\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}')))$$

of A_ν-modules (resp. Q_ν-vector spaces). Moreover if $\underline{\mathcal{M}} = \underline{\mathcal{M}}'$ then the above are ring isomorphisms.

Proof. Let us first state the following lemma

Lemma 2.15. *Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}'$ and ν be as in the above theorem. Then*

$$\operatorname{Hom}_L(\underline{\mathcal{M}},\underline{\mathcal{M}}')\otimes_A A_\nu \xrightarrow{\sim} \operatorname{Hom}_{A_\nu[\Gamma_L]}(\Gamma_\nu(\underline{\mathcal{M}}),\Gamma_\nu(\underline{\mathcal{M}}'))$$

Proof. Consider the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{M}}') \longrightarrow \mathrm{Hom}_{A_L}(\mathcal{M}, \mathcal{M}') \longrightarrow \mathrm{Hom}_A(\mathcal{M}, \mathcal{M}')$$

$$f \mapsto f \circ \tau_{\mathcal{M}} - \tau_{\mathcal{M}'} \circ f$$

Let us set $\underline{\hat{M}} := \Gamma_\nu(\underline{\mathcal{M}})$ and $\underline{\hat{M}}' := \Gamma_\nu(\underline{\mathcal{M}}')$. As $A \to A_\nu$ is flat, we get the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{M}}') \otimes_A A_\nu \longrightarrow \mathrm{Hom}_{A_{\nu,L}}(\hat{M}, \hat{M}') \longrightarrow \mathrm{Hom}_{A_\nu}(\hat{M}, \hat{M}')$$

$$f \mapsto f \circ \hat{\tau}_{\hat{M}} - \hat{\tau}_{\hat{M}'} \circ f,$$

and thus $\mathrm{Hom}_L(\underline{\mathcal{M}}, \underline{\mathcal{M}}') \otimes_A A_\nu \cong \mathrm{Hom}(\underline{\hat{M}}, \underline{\hat{M}}')$.

□

According to Proposition 2.10 we may assume that $\mathbb{F}_\nu = \mathbb{F}_q$. Let us set $\underline{\hat{M}} := \Gamma_\nu(\underline{\mathcal{M}})$ and $\underline{\hat{M}}' := \Gamma_\nu(\underline{\mathcal{M}}')$. Furthermore according to the above lemma it remains to show that

$$\mathrm{Hom}(\underline{\hat{M}}, \underline{\hat{M}}') = \mathrm{Hom}(\omega^\nu(\underline{\mathcal{M}}), \omega^\nu(\underline{\mathcal{M}}'))$$

Let us fix an isomorphism $\hat{M} \cong L[\![z]\!]^r$ (resp. $\hat{M}' \cong L[\![z]\!]^{r'}$) and write $\tau_{\hat{M}} = T \cdot \hat{\sigma}$ (resp. $\tau_{\hat{M}} = T' \cdot \hat{\sigma}$). Let us further denote by $\rho_{\underline{\mathcal{M}},\nu} : \Gamma_L \to \mathrm{Aut}(\omega^\nu(\underline{\mathcal{M}})) \cong \mathrm{GL}_{\mathrm{rk}\,\underline{\mathcal{M}}}(A_\nu)$ (resp. $\rho_{\underline{\mathcal{M}}',\nu} : \Gamma_L \to \mathrm{Aut}(\omega^\nu(\underline{\mathcal{M}}')) \cong \mathrm{GL}_{\mathrm{rk}\,\underline{\mathcal{M}}'}(A_\nu)$) the corresponding representation of Γ_L. We have

$$\mathrm{Hom}(\underline{\hat{M}}, \underline{\hat{M}}') = \{\hat{F} \in Mat_{r' \times r}(L[\![z]\!]); \hat{F}.T = T'.\hat{\sigma}(\hat{F})\}.$$

Extending the underlying field L to the separable closure L^{sep}, we may write the right hand side in the following way

$$\{\hat{F} \in Mat_{r' \times r}(L^{\mathrm{sep}}[\![z]\!]); \hat{F}.T = T'.\hat{\sigma}(\hat{F}) \text{ and } \varphi(\hat{F}) = \hat{F} \text{ for every } \varphi \in \Gamma_L\}$$

By Lang's theorem there is $\Phi \in \mathrm{GL}_r(L^{\mathrm{sep}}[\![z]\!])$ (resp. $\Phi' \in \mathrm{GL}_r(L^{\mathrm{sep}}[\![z]\!])$) such that $\Phi^{-1}.T = \hat{\sigma}(\Phi)^{-1}$ (resp. $(\Phi')^{-1}.T' = \hat{\sigma}(\Phi')^{-1}$) and therefore we may rewrite the above set in the following way

$$\{G \in Mat_{r' \times r}(L^{\mathrm{sep}}[\![z]\!]); \Phi' G \Phi^{-1}.T = T'.\hat{\sigma}(\Phi' G \Phi^{-1}) \text{ and}$$
$$\varphi(\Phi' G \Phi^{-1}) = \Phi' G \Phi^{-1} \text{ for every } \Gamma_L\}$$

since $\varphi(\Phi)^{-1} = \rho_{\underline{\mathcal{M}},\nu}(\varphi)^{-1}.\Phi^{-1}$ and $\varphi(\Phi') = \Phi'.\rho_{\underline{\mathcal{M}}',\nu}(\varphi)$ we observe that the above set equals

$$\{G \in Mat_{r'\times r}(L^{\mathrm{sep}}[\![z]\!]); \hat{\sigma}(G) = G \text{ and } \rho_{\underline{\mathcal{M}}',\nu}.G.\rho^{-1}_{\underline{\mathcal{M}},\nu} = G \ \forall \varphi \in \Gamma_L\}$$
$$= \{G \in Mat_{r'\times r}(L[\![z]\!]) \ ; \ \rho_{\underline{\mathcal{M}}',\nu}(\varphi).G.\rho_{\underline{\mathcal{M}},\nu}(\varphi)^{-1} = G \ \forall \varphi \in \Gamma_L\}$$

□

3 The Endomorphism Ring Over Finite Fields

Throughout this section we assume that the field L is a finite field extension of $\mathbb{F}_q$.

Definition 3.1. Let $L/\mathbb{F}_q$ be a field extension of degree e and let $\underline{\mathcal{M}} := (\mathcal{M}, \tau)$ be a C-motive in $\mathcal{M}ot^{\nu}_{C}(L)$. The *Frobenius isogeny* $\pi := \pi_{\underline{\mathcal{M}}} \in Q\,\mathrm{End}(\underline{\mathcal{M}})$ is given by

$$\pi := \tau \circ (\sigma^*)\tau \cdots \circ (\sigma^*)^{e-1}\tau : \underline{\mathcal{M}} := (\sigma^*)^e \underline{\mathcal{M}} \to \underline{\mathcal{M}}$$

Proposition 3.2. *Let $\underline{\mathcal{M}} := (\mathcal{M}, \tau)$ be a C-motive over a finite field $L = \mathbb{F}_{q^e}$ and let ν be a place on C distinct from characteristic places ν_i. Then*

(a) The action of the generator $\varphi_L \in \Gamma_L$ on $\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}})$ equals the action of π_ν^{-1}. Here π_ν denotes $\omega^{\nu}_{Q_\nu}(\pi)$.

(b) The image of the continues morphism $A_\nu[\Gamma_L] \to \mathrm{End}_{A_\nu}(\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}))$ equals $A_\nu[\pi_\nu]$.

Proof. a)
One can reduce to the case $\mathbb{F}_\nu = \mathbb{F}_q$. Let $(\hat{M}, \hat{\tau})$ be the crystal associated with $\underline{\mathcal{M}}$ under $\Gamma_\nu(-)$. Fix an isomorphism $(\underline{\hat{M}}, \tau) \tilde{=} (L[\![z]\!]^r, T \cdot \hat{\sigma})$. Take $\Phi \in \mathrm{GL}_r(L^{\mathrm{sep}}[\![z]\!])$, invariant under $T \cdot \hat{\sigma}$. We may write

$$\pi_\nu = \Phi^{-1} T \hat{\sigma}(T) \cdots \hat{\sigma}^{e-1}(T) \Phi = \hat{\sigma}^e(\Phi)^{-1} \cdot \Phi = (\Phi \cdot \hat{\sigma}^e(\Phi))^{-1} = \rho_{\underline{\mathcal{M}},\nu}(\varphi_L)^{-1}$$

b) Write $\Phi \equiv \sum_{i=0}^{n-1} \Phi_i z^i (\bmod z^n)$ and let $K = L(\Phi_i; 0 \le i \le n-1)$. For $\varphi \in \Gamma_L$, observe that

$$\rho_{\underline{\mathcal{M}},\nu}(\varphi) \equiv \left(\sum_{i=0}^{n-1} \Phi_i z^i\right)^{-1} \varphi \left(\sum_{i=0}^{n-1} \Phi_i z^i\right) \ (\bmod z^n)$$

only depends on the image of φ under the natural morphism $\Gamma_L \to \mathrm{Gal}(K/L) = \{\varphi_L^j;\ 0 \le j < [K:L]\}$. Therefore $\rho_{\underline{\mathcal{M}},\nu}(\varphi)\ (\bmod z^n)$ lies in $\{\rho_{\underline{\mathcal{M}},\nu}(\varphi_L^j);\ 0 \le j < [K:L]\} = A_\nu[\pi_\nu]/z^n A_\nu[\pi_\nu]$.

As $A_\nu[\rho_{\underline{\mathcal{M}},\nu}(\Gamma_L)]$ and $A_\nu[\pi_\nu]$ coincide modulo z^n and the rings are z-adic complete, the statement follows. □

Fix a C-motive $\underline{\mathcal{M}}$ over L and a place ν different from characteristic places ν_i. Consider $F := Q[\pi] \subseteq E := Q\operatorname{End}(\underline{\mathcal{M}})$. Note that F/Q is an algebraic extension by Proposition 2.12 and write $F := Q[x]/\mu_\pi$, where μ_π is the minimal polynomial of π over Q. Note that according to Theorem 2.14 we have $E \otimes_Q Q_\nu \cong E_\nu := \operatorname{End}_{Q_\nu[\Gamma_L]}(\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}))$ and observe that under this isomorphism $F \otimes_Q Q_\nu$ gets identified with image F_ν of $Q_\nu[\Gamma_L]$ in E_ν. Let χ_ν denote the characteristic function corresponding to $\pi_\nu := \pi \otimes 1$ in E_ν.

Remark 3.3. Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}'$ be in $\mathcal{M}ot^\nu_C(L)$ and let $V_\nu := \omega^\nu_{Q_\nu}(\underline{\mathcal{M}})$ and $V'_\nu := \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}')$ at a place $\nu \in C$, different from characteristic places ν_i. Furthermore, assume that the corresponding Frobenius endomprphisms $\pi_\nu \in \operatorname{End}_{Q_\nu[\Gamma_L]}(V_\nu)$ and $\pi'_\nu \in \operatorname{End}_{Q_\nu[\Gamma_L]}(V'_\nu)$ are semisimple. Consider the decomposition $\chi_\nu = \prod_\mu \mu^{m_\mu}$ (resp. $\chi'_\nu = \prod_\mu \mu^{m'_\mu}$) of the characteristic polynomial χ_ν (resp. χ'_ν) to its irreducible factors and set $K_\mu := Q_\nu[x]/\mu$. There are decompositions $V_\nu \cong \bigoplus_\mu (K_\mu)^{m_\mu}$ and $V'_\nu \cong \bigoplus_\mu (K_\mu)^{m'_\mu}$, and therefore

$$\operatorname{Hom}_{Q_\nu[\Gamma_L]}(V_\nu, V'_\nu) \cong \bigoplus_i Mat_{m'_\mu \times m_\mu}(K_\mu).$$

Theorem 3.4. *Consider the following diagram*

$$\begin{array}{ccc}
\mathcal{M}ot^\nu_C(L)^\circ & \xrightarrow{Q\operatorname{End}(-)} & Q\text{-}Algebras \\
 & \searrow{\scriptstyle \operatorname{End}_{\Gamma_L}(\omega^\nu_{Q_\nu}(-))} & \downarrow{\scriptstyle -\otimes Q_\nu} \\
 & & Q_\nu\text{-}Algebras.
\end{array}$$

The following statements are equivalent

(a) $\underline{\mathcal{M}} \in \mathcal{M}ot^\nu_C(L)$ *is semi-simple.*

(b) $E := Q\operatorname{End}(\underline{\mathcal{M}})$ *is semi-simple.*

(c) $\operatorname{End}_{Q_\nu[\Gamma_L]}(\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}))$ *is semisimple.*

(d) $F_\nu := Q_\nu[\pi_\nu]$ *is semisimple.*

(e) F *is semi-simple.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) is obvious. (c) $\Rightarrow$ (d) follows from [Bou58, Corollaire de Proposition 6.4/9]. Since Q_ν/Q is separable and $F_\nu = F \otimes Q_\nu$, one can argue that (d) and (e) are equivalent by [Bou58, Corollaire 7.6/4]. Also if π_ν is semi-simple then $\operatorname{End}_{Q_\nu[\Gamma_L]}(\omega^\nu_{Q_\nu}(\underline{\mathcal{M}})) = \bigoplus_\mu Mat_{m_\mu \times m_\mu}(K_\mu)$, see Remark 3.3. Thus (d) implies (c). Again we argue that (c) implies (b) by [Bou58, Corollaire 7.6/4].

It remains to justify $(b) \Rightarrow (a)$. First observe that we can reduce to the this statement that if $Q\operatorname{End}(\underline{\mathcal{M}})$ is a division algebra, then $\underline{\mathcal{M}}$ is simple. To see this, suppose $Q\operatorname{End}(\underline{\mathcal{M}}) = \oplus_{j=1}^{m} Mat_{r_j\times r_j}(E_j)$ be the decomposition to the matrix algebras over division algebras E_j over Q. Let $\{e_{j,i_j}\}_{1\leq i_j\leq r_j}$ be the corresponding set of idempotents with $\sum e_{i_j} = \mathrm{id}_{r_j} \in Mat_{r_j\times r_j}(E_j)$ and $e_{i_j} Q\operatorname{End}(\underline{\mathcal{M}}) e_{i_j} = E_j$. Now consider the quasi-isogeny $\sum_{i,j} e_{i_j} : \underline{\mathcal{M}} \to \oplus_{i,j}\underline{\mathcal{M}}_{i_j}$, where $\underline{\mathcal{M}}_{i_j} := \operatorname{im} e_{i_j}$ and observe that $Q\operatorname{End}(\underline{\mathcal{M}}_{i_j}) = e_{i_j} Q\operatorname{End}(\underline{\mathcal{M}}) e_{i_j} = E_j$.
Now assume that $E := Q\operatorname{End}(\underline{\mathcal{M}})$ is a division algebra. We show that $\underline{\mathcal{M}}$ has no non-trivial quotient. Let $\underline{\mathcal{M}}'$ be a quotient of $\underline{\mathcal{M}}$ under $f : \underline{\mathcal{M}} \to \underline{\mathcal{M}}'$. Since E is a division algebra, it is enough to show that there is an element $g \in Q\operatorname{Hom}(\underline{\mathcal{M}}', \underline{\mathcal{M}})$ such that $g \circ f \neq 0$. According to Proposition 2.11, the realization functor $\omega^{\nu}_{Q_\nu}(-)$ is exact, and thus by applying $\omega^{\nu}_{Q_\nu}(-)$ we get a surjection

$$\omega^{\nu}_{Q_\nu}(f) : \omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}) \twoheadrightarrow \omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}').$$

Note that $\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}})$ and $\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}')$ are Q_ν-vector spaces and therefore there exist

$$f'_\nu \in \operatorname{Hom}_{Q_\nu[\Gamma_L]}(\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}'), \omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}))$$

such that $f'_\nu \circ f_\nu = \mathrm{id}_{\omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}})}$. According to Theorem 2.14 this morphism induces a morphism $f' \in \operatorname{Hom}(\underline{\mathcal{M}}', \underline{\mathcal{M}})$. Precisely we have $f' \circ f \neq 0$.

□

Let K be a field and $f, g \in K[x]$. Consider the factorizations $f = \prod \mu^{n_\mu}$ and $g = \prod \mu^{m_\mu}$, where μ runs over irreducible polynomials in $K[x]$. Set $r_K(f,g) := \prod_\mu m_\mu . n_\mu . \deg\mu$.

Proposition 3.5. *Let $\underline{\mathcal{M}}$ be a C-motive over a finite field $L = \mathbb{F}_{q^e}$ with semi-simple Frobenius π. Then $F = Q[\pi]$ is the center $\mathcal{Z}(E)$ of the semi-simple Q-algebra $E = Q\operatorname{End}_\ell(\underline{\mathcal{M}})$.*

Proof. Since F_ν is semisimple, the F_ν-module $\omega^\nu(\underline{\mathcal{M}})$ is semisimple; see Theorem 3.4. As $\omega^\nu(\underline{\mathcal{M}})$ is finitely generated module over E_ν which itself is finite dimensional over Q_ν. Therefore we have $Bicom_{F_\nu}(\omega^\nu(\underline{\mathcal{M}})) = F_\nu$; see Remark 0.2(b). Hence $\mathcal{Z}(E_\nu) = E_\nu \cap F_\nu = F_\nu$ and $F \otimes_Q Q_\nu = F_\nu = \mathcal{Z}(E \otimes_Q Q_\nu) = \mathcal{Z}(E) \otimes_Q Q_\nu$, see Remark 0.2(a). We conclude that $\dim_Q F = \dim_Q \mathcal{Z}(E)$. Note that $F \subseteq \mathcal{Z}(E)$, since for every $f \in E$ we have $f \circ \tau_{\mathcal{M}} = \tau_{\mathcal{M}} \circ f$ and thus $f \circ \pi = \pi \circ f$.

□

Proposition 3.6. *Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}'$ be C-Motives over L and let $V_\nu := \omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}})$ and $V'_\nu := \omega^{\nu}_{Q_\nu}(\underline{\mathcal{M}}')$, at a place $\nu \in C$, different from characteristic places ν_i. Assume further that $\pi_\nu \in \operatorname{End}_{Q_\nu[\Gamma]}(V_\nu)$ (resp. $\pi'_\nu \in \operatorname{End}_{Q_\nu[\Gamma]}(V_{\nu'})$) is semisimple. Then the dimension of $Q\operatorname{Hom}(\underline{\mathcal{M}}, \underline{\mathcal{M}}')$ as a Q-vector space equals $r_{Q_\nu}(\chi_\nu, \chi'_\nu)$.*

Proof. Consider the decomposition $\chi_\nu = \prod_\mu \mu^{m_\mu}$ (resp. $\chi'_\nu = \prod_\mu \mu^{m'_\mu}$) of the characteristic polynomial χ_ν (resp. χ'_ν) to the irreducible factors and set $K_\mu := Q_\nu[x]/\mu$. Then decompose $V_\nu \cong \bigoplus_\mu (K_\mu)^{m_\mu}$ and $V'_\nu \cong \bigoplus_\mu (K_\mu)^{m'_\mu}$. We get

$$\mathrm{Hom}_{Q_\nu[\Gamma_L]}(V_\nu, V'_\nu) \cong \bigoplus_i Mat_{m'_\mu \times m_\mu}(K_\mu).$$

Now the assertion follows from Theorem 2.14. □

Definition 3.7. We say that $\underline{\mathcal{M}}$ has complex multiplication if $Q\,\mathrm{End}(\underline{\mathcal{M}})$ contains a commutative, semi-simple Q-algebra of dimension $\mathrm{rk}\,\underline{\mathcal{M}}$.

Proposition 3.8. *Let $\underline{\mathcal{M}}$ be a C-motive of rank r over L with Frobenius endomorphism π. Set $E := Q\,\mathrm{End}(\underline{\mathcal{M}})$. Assume that $F = Q[\pi]$ is a field and let $h := [F : Q] = \deg \mu_\pi$. Then*

(a) $h|r$ and $\dim_Q E = r^2/h$ and $\dim_F E = r^2/h^2$.

(b) For any place ν of Q different from characteristic places ν_i we have $E \otimes Q_\nu \cong Mat_{r/h\times r/h}(F \otimes_Q Q_\nu)$ and $\chi_\nu = (\mu_\pi)^{r/h}$, independent of ν.

In particular if $\underline{\mathcal{M}}$ is CM then $F = E = Q\,\mathrm{End}(\underline{\mathcal{M}})$ is commutative

Proof. Since F is a field, π_ν is semi-simple, see Theorem 3.4. Therefore the minimal polynomial μ_{π_ν} equals $\prod_\mu \mu$, with pairwise different monic irreducible polynomials $\mu \in Q_\nu[x]$. The characteristic polynomial χ_ν then equals $\prod_\mu \mu^{m_\mu}$. We have $E_\nu \cong \prod_\mu E_\mu$, where $E_\mu = Mat_{m_\mu\times m_\mu}(K_\mu)$ and $K_\mu = Q_\nu[x]/\mu$, see Remark 3.3. We get $Q_\nu[\pi_\nu] = F_\nu = F \otimes_Q Q_\nu \twoheadrightarrow Q_\nu[x]/(\mu) = K_\mu$ and the surjection

$$E_\nu \otimes_{F_\nu} K_\mu = (E \otimes_Q Q_\nu) \otimes_{F_\nu} K_\mu = E \otimes_F (F \otimes_Q Q_\nu) \otimes_{F_\nu} K_\mu = E \otimes_F K_\mu \twoheadrightarrow E_\mu$$

In particular $m_\mu^2 \le \dim_F E$. Thus

$$\begin{aligned}\dim_F E \cdot [F : Q] = \dim_Q E = \dim_{Q_\nu} E_\nu = \textstyle\sum_\mu m_\mu^2 \cdot \deg_x \mu \le (\dim_F E) \cdot \sum_\mu \deg_x \mu \\ = (\dim_F E) \cdot \deg_x \mu_{\pi_\nu} = [F : Q] \cdot \dim_F E\end{aligned}$$

Therefore $m_\mu^2 = \dim_F E$ for every μ and

$$r = \deg_x \chi_\nu = \sum_\mu m_\mu \deg_x \mu = \sqrt{\dim_F E} \;\cdot\; \sum_\mu \deg_x \mu = \sqrt{\dim_F E} \;\cdot\; [F : Q].$$

Therefore $r = m_\mu \cdot h$ and $\dim_F E = r^2/h^2$. Now to see part (b) write $E_\nu \cong \oplus_\mu Mat_{r/h\times r/h}(K_\mu) \cong Mat_{r/h\times r/h}(\oplus_\mu K_\mu) \cong Mat_{r/h\times r/h}(F_\nu)$. □

-**Endomorphism Q-algebra** Let us state the following proposition, regarding the two extreme cases for $F \subseteq E$.

Proposition 3.9. *Let $\underline{\mathcal{M}}$ be a C-motive over L with semisimple Frobenius endomorphism $\pi := \pi_{\underline{\mathcal{M}}}$, i.e. $F = Q[\pi]$ is a product of fields. Let ν be a place on C apart from characteristic places ν_i. Let χ_ν denotes the characteristic polynomial of $\pi_\nu := \omega^\nu(\pi)$. We have the following statements*

(a) $F = Q(\pi)$ *is the center of the semisimple Q-algebra* $E = Q\operatorname{End}_L(\underline{\mathcal{M}})$.

(b) $\operatorname{rk}\underline{\mathcal{M}} \leq [E : Q] := \dim_Q E \leq (\operatorname{rk}\underline{\mathcal{M}})^2$

(c) The following are equivalent

- *i)* $E = F$
- *ii)* E *is commutative,*
- *iii)* $[F : Q] = \operatorname{rk}\underline{\mathcal{M}}$
- *iv)* $[E : Q] = \operatorname{rk}\underline{\mathcal{M}}$
- *v)* χ_ν *is product of pairwise different irreducible polynomials in* $Q_\nu[x]$

(d) The following are equivalent

- *i)* $F = Q$
- *ii)* $E \cong Mat_{n\times n}(D)$, *for a division algebra D with center $\mathcal{Z}(D) = Q$*
- *iii)* $[F : Q] = 1$
- *iv)* $[E : Q] = (\operatorname{rk}\underline{\mathcal{M}}))^2$
- *v)* $\chi_\nu = \mu^{\operatorname{rk}\underline{\mathcal{M}}}$*for a linear polynomial $\mu \in Q_\nu[x]$. This is the minimal polynomial* μ_π

Proof. (a) was proved in Proposition 3.5. For (b) let $\chi_\nu = \prod_\mu \mu^{m_\mu}$ with $\mu \in Q_\nu[x]$ irreducible pairwise different. We have the decomposition $E \otimes_Q Q_\nu \cong \prod_\mu Mat_{m_\mu \times m_\mu}(Q_\nu[x]/(\mu))$ and thus

$$[E : Q] = \dim_Q E = \sum_\mu m_\mu^2 \cdot \deg_x \mu$$

and

$$\sum_\mu m_\mu \cdot \deg_x \mu = \deg_x \chi_\nu = \dim_{Q_\nu} \omega_Q^\nu(\underline{\mathcal{M}}) = \operatorname{rk}\underline{\mathcal{M}}.$$

Therefore

$$\operatorname{rk}\underline{\mathcal{M}} = \sum_\mu m_\mu \cdot \deg_x \mu \leq \sum_\mu m_\mu^2 \deg_x \mu = [E : Q] \leq (\sum_\mu m_\mu \deg_x \mu)^2 = (\operatorname{rk}\underline{\mathcal{M}})^2. \tag{3.1}$$

We proceed the proof of part c) in the following way

$$i) \Leftrightarrow ii) \Rightarrow iii) \Rightarrow i) \Rightarrow iv) \Leftrightarrow v) \Rightarrow iii).$$

The first implication follows from (a). As we have seen above $[E:Q] = \operatorname{rk} \underline{\mathcal{M}}$ if and only if $m_\mu = 1$ for all μ. This implies that $[F:Q] = \operatorname{rk} \underline{\mathcal{M}}$. We know that μ_π is the minimal polynomial of $\pi_\nu := \omega^\nu(\pi)$ and since divides the characteristic polynomial, we argue that $\mu_\pi = \chi_\nu$. Hence

$$[F:Q] = \deg_x \mu_\pi = \deg_x \chi_\nu = \dim_{Q_\nu} \omega^\nu(\underline{\mathcal{M}}) = \operatorname{rk} \underline{\mathcal{M}}.$$

Let's show that $E = F$ if and only if $[E:Q] = \operatorname{rk} \underline{\mathcal{M}}$. We have

$$\operatorname{rk} \underline{\mathcal{M}} = \deg_x \chi_\nu \geq \deg_x \mu_\pi = [F:Q] = [E:Q] \geq \operatorname{rk} \underline{\mathcal{M}},$$

conversely,

$$\sum_\mu \deg_x \mu = \deg_x \mu_\pi = \operatorname{rk} \underline{\mathcal{M}} = \deg_x \chi_\nu = \sum_{i=1}^n m_\mu \cdot \deg_x \mu$$

Therefore $m_\mu = 1$, for all μ and thus $[E:Q] = \operatorname{rk} \underline{\mathcal{M}} = [F:Q]$, which implies $E = F$. It remains to prove (d). If $F = Q$, then by a) the center of E is a field, and therefor ii) follows from Artin-Wedderburn theorem. Conversely if $E \cong Mat_{n\times n}(D)$ then $F = \mathcal{Z}(E) = \mathcal{Z}(Mat_{n\times n}(D)) = \mathcal{Z}(D) = Q$. The equivalence $[E:Q] = \operatorname{rk} \underline{\mathcal{M}} \Leftrightarrow \chi_\nu = \mu_\pi^{\operatorname{rk} \underline{\mathcal{M}}}$ follows from 3.1. Finally since π is semi-simple $\mu_\pi = \mu$, therefore $[F:Q] = \deg_x \mu_\pi = \deg_x \mu = 1$. On the other hand if $\mu = \mu_\pi$ is linear then $\chi_\nu = (\mu_\pi)^{\operatorname{rk} \underline{\mathcal{M}}}$ by Cayley-Hamilton.

□

4 Relation to G-Shtukas

Let $\mathfrak{G}$ be a flat affine group scheme of finite type over the curve C with generic fiber G. Let $\mathscr{H}^1(C, \mathfrak{G})$ denote the stack whose S points parameterize $\mathfrak{G}$-bundles on $C_S := C \times_{\mathbb{F}_q} S$. We have the following statement [AH14b, Theorem 2.4].

Theorem 4.1. *The stack $\mathscr{H}^1(C, \mathfrak{G})$ is a smooth Artin-stack locally of finite type over $\mathbb{F}_q$. It admits a covering by connected open substacks of finite type over $\mathbb{F}_q$.*

4.1 G-motives and functoriality

One may endow the category of C-motives $\mathcal{M}ot_C^\nu(S)$ with a G-structure. This leads to the following definition.

Definition 4.2. Let $\operatorname{Rep}\mathfrak{G}$ denote the category of representations of $\mathfrak{G}$ in finite free $\mathcal{O}_C$-modules $\mathcal{V}$. By a $\mathfrak{G}$-motive (resp. G-motive) over S we mean a tensor functor $\underline{\mathcal{M}}_{\mathfrak{G}} : \operatorname{Rep}\mathfrak{G} \to \mathcal{M}ot_C^{\underline{\nu}}(S)$ (resp. $\underline{\mathcal{M}}_G : \operatorname{Rep}_Q\, G \to \mathcal{M}ot_C^{\underline{\nu}}(S)$). We say that two $\mathfrak{G}$-motives (resp. G-motives) are isomorphic if they are isomorphic as tensor functors. We denote the resulting category of $\mathfrak{G}$-motives (resp. G-motives) over S by $\mathfrak{G}\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$ (resp. $G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$).

Note that the construction of the category $G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$ is functorial both in G and C. Namely

- morphism $\rho : G \to G'$ (resp. $\mathfrak{G} \to \mathfrak{G}'$) induces a functor $\operatorname{Rep} G' \to \operatorname{Rep} G$ (resp. $\operatorname{Rep}\mathfrak{G}' \to \operatorname{Rep}\mathfrak{G}$) and this further induces a functor $\rho_* : G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S) \to G'\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$ (resp. $\rho_* : \mathfrak{G}\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S) \to \mathfrak{G}'\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$).
- Suppose we have a morphism $C \to C'$ of smooth projective geometrically irreducible curves which is of degree d. The characteristic places $\underline{\nu}$ on C induce an n-tuple of characteristic places on C', which we denote by $\underline{\nu}'$. In addition the push forward functor f_* induces a functor from the category of locally free sheaves of rank r over C_S to the category of locally free sheaves of rank $r \cdot d$ over C'_S. This further induces the push forward functor $f_* : G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S) \to G\text{-}\mathcal{M}ot_{C'}^{\underline{\nu}'}(S)$.

4.2 G-Shtukas and functoriality

Let's now discuss the geometrization of the category $G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$. To this goal let us first recall the following definition of the moduli (stacks) of global $\mathfrak{G}$-shtukas.

Definition 4.3. A global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ over an $\mathbb{F}_q$-scheme S is a tuple $(\mathcal{G}, \underline{s}, \tau)$ consisting of

- a $\mathfrak{G}$-bundle $\mathcal{G}$ over C_S,
- an n-tuple $\underline{s}$ of (characteristic) sections and
- an isomorphism $\tau\colon \sigma^*\mathcal{G}|_{C_S \smallsetminus \Gamma_{\underline{s}}} \xrightarrow{\sim} \mathcal{G}|_{C_S \smallsetminus \Gamma_{\underline{s}}}$.

We let $\nabla_n\mathscr{H}^1(C, \mathfrak{G})$ denote the stack whose S-points parameterizes global $\mathfrak{G}$-shtukas over S. Sometimes we will fix the sections $\underline{s} := (s_i)_i \in C^n(S)$ and simply call $\underline{\mathcal{G}} = (\mathcal{G}, \tau)$ a global $\mathfrak{G}$-shtuka over S. We let $\nabla\mathscr{H}^1(C, \mathfrak{G})(S)_Q$ denote the category which has the same objects as $\nabla\mathscr{H}^1(C, \mathfrak{G})(S)$, but the set of morphisms is enlarged to *quasi-isogenies* of $\mathfrak{G}$-shtukas. A quasi-isogeny $f : \underline{\mathcal{G}} \to \underline{\mathcal{G}}'$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{G}' \\ {\scriptstyle\tau}\uparrow & & \uparrow{\scriptstyle\tau'} \\ \sigma^*\mathcal{G} & \xrightarrow{\sigma^* f} & \sigma^*\mathcal{G}', \end{array}$$

defined outside $\Gamma_{\underline{s}} \cup D \times_{\mathbb{F}_q} S$ for a closed subscheme $D \subseteq C$. We denote by $QIsog_S(\underline{\mathcal{G}}, \underline{\mathcal{G}}')$ the set of quasi-isogenies between $\underline{\mathcal{G}}$ and $\underline{\mathcal{G}}'$.

Let us recall the following feature of the quasi-isogenies between global $\mathfrak{G}$-shtukas. Namely, they have rigidity property, in the sense that a quasi-isogeny lifts over infinitesimal thickenings. The proof was given in [AH14a, Proposition 5.9]. Also see [Dri76] for the corresponding fact for p-divisible groups (and abelian varieties).

Theorem 4.4 (Rigidity of quasi-isogenies for global $\mathfrak{G}$-shtukas). *Let S be a scheme in $\mathcal{N}ilp_{\widehat{\mathcal{O}}_{C,\nu}}$ and let $j : \overline{S} \to S$ be a closed immersion defined by a sheaf of ideals $\mathcal{I}$ which is locally nilpotent. Let $\underline{\mathcal{G}} = (\mathcal{G}, \tau)$ and $\mathcal{G}' = (\mathcal{G}', \tau')$ be two global $\mathfrak{G}$-shtukas over S. Then*

$$QIsog_S(\underline{\mathcal{G}}, \underline{\mathcal{G}}') \to QIsog_S(j^*\underline{\mathcal{G}}, j^*\underline{\mathcal{G}}'),\ \ f \mapsto j^* f$$

is a bijection of sets. f is an isomorphism at a place $\nu \in \underline{\nu}$ if and only if $j^* f$ is an isomorphism at ν.

Remark 4.5. The category of vector bundles of rank n on C_S can be identified with the category of GL_n-bundles on C_S. Consequently, one can identify the category $\nabla\mathscr{H}^1(C, \mathrm{GL}_n)(S)_Q$ with the subcategory C-motives $\mathcal{M}ot_C^{\underline{\nu}}(L)^\circ$ of rank n. In particular the above rigidity property of quasi-isogenies also holds for C-motives.

The assignment of the moduli stack $\mathscr{H}^1(C, \mathfrak{G})$ to a group $\mathfrak{G}$, is functorial. In other words a morphism $\rho : \mathfrak{G} \to \mathfrak{G}'$ of algebraic groups gives rise to a morphism

$$\mathscr{H}^1(C, \mathfrak{G}) \to \mathscr{H}^1(C, \mathfrak{G}'),$$
$$\mathcal{G} \mapsto \rho_* \mathcal{G} := \mathcal{G} \times_{\mathfrak{G}, \rho} \mathfrak{G}'$$

In particular any representation $\rho : \mathfrak{G} \to \mathrm{GL}(\mathcal{V})$ induces a natural morphism of $\mathbb{F}_q$-stacks

$$\mathscr{H}^1(C, \mathfrak{G}) \to \mathscr{H}^1(C, \mathrm{GL}(\mathcal{V})), \mathcal{G} \mapsto \rho_* \mathcal{G}. \tag{4.2}$$

Note that by tannakian theory having a $\mathfrak{G}$-bundle over C_L is equivalent with giving a tensor functor f from $\mathrm{Rep}\,\mathfrak{G}$ to the category $Vect_C(L)$ of vector bundles over C_L.

Regarding the functoriality of $\mathscr{H}^1(C, -)$, the assignment of the moduli stack $\nabla\mathscr{H}^1(C, \mathfrak{G})$ of global $\mathfrak{G}$-shtukas to a group $\mathfrak{G}$ is functorial in $\mathfrak{G}$, i.e. a morphism $\mathfrak{G} \to \mathfrak{G}'$ gives rise to the morphism

$$\rho_*(-) : \nabla\mathscr{H}^1(C, \mathfrak{G}) \to \nabla\mathscr{H}^1(C, \mathfrak{G}').$$

Given a $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}$ and a representation $\overline{\rho} : G \to \mathrm{GL}(V)$ of the group G on a Q-vector space, we may consider a lift $\rho : \mathfrak{G} \to \mathrm{GL}(\mathcal{V})$ and use it to push forward $\underline{\mathcal{G}}$ to produce a $\mathrm{GL}(\mathcal{V})$-shtuka $\rho_*(\underline{\mathcal{G}})$ in $\nabla\mathscr{H}^1(C, \mathrm{GL}(\mathcal{V}))(L)$. According to Remark 4.5, $\rho_*(\underline{\mathcal{G}})$ can be viewed as a C-motive in $\mathcal{M}ot_C^{\underline{\nu}}(L)$. Note that taking a different lift ρ' of $\overline{\rho}$, the representations $\rho'_* \mathcal{G}$ and $\rho_* \underline{\mathcal{G}}$ are canonically isomorphic in $\mathcal{M}ot_C^{\underline{\nu}}(L)$. This is because ρ and ρ' agree over an open $U \subset C$. In particular we obtain the following pairing

$$\nabla\mathscr{H}^1(C,\mathfrak{G})(L) \times \operatorname{Rep}_Q G \to \mathcal{M}ot_C^{\underline{\nu}}(L).$$

This pairing is a perfect pairing in the following sense. Namely, it induces an equivalence of categories

$$\nabla\mathscr{H}^1(C,\mathfrak{G})(L)_Q \tilde{\to}\ G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(L). \tag{4.3}$$

4.3 Ind-algebraic structure on $\nabla\mathscr{H}^1(C,\mathfrak{G})$

Recall that, for a field F, an object $M \in DM_{gm}^{eff}(F)$ is *geometrically mixed Tate* if it becomes mixed Tate over an algebraic closure F^{alg}. It can be shown that a motive which becomes mixed Tate over an algebraic closure, is already mixed Tate over a finite separable extension; see Proposition 4.6 bellow. As a consequence of the geometrization of the category $G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(S)$, one can immediately see that a similar fact also holds for $(G\text{-})C$-motives.

Proposition 4.6. *Let F be a field and let $M \in DM_{gm}^{eff}(F^{\mathrm{sep}})$. Then there is a finite separable extension $F \subseteq E \subseteq F^{\mathrm{sep}}$ and $M' \in DM_{gm}^{eff}(E)$ such that $M = i^*M'$. Similarly a geometrically mixed Tate motive M comes from a mixed Tate motive over a finite extension E/F.*

Proof. For the mixed Tate motives it follows from the fact that they admit a finite filtration. For the geometric motives observe that any object of this category can be realized as a finite (homotopy) colimit of a diagram whose terms are of the form $M(X)(n)[k]$, where X is some smooth scheme over F^{sep}. The maps are also provided by algebraic cycles, and therefore come from some finite extension E. See also continuity property for DM_{gm} [CD, Proposition 4.3.4]. □

Analogously the following holds for the category of $(G\text{-})C$-motives.

Proposition 4.7. *Let F be a field over $\mathbb{F}_q$. Any G-C-motive $\underline{\mathcal{M}}_G$ in $G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(L^{\mathrm{sep}})$ comes by base change from a G-C-motive in $G\text{-}\mathcal{M}ot_C^{\underline{\nu}}(E)$ for a finite extension E/F.*

Proof. The proof goes in a different manner than 4.6. Namely, this observation can be made by looking at $\nabla\mathscr{H}^1(C,\mathfrak{G})$ as a moduli space for motives, regarding equivalence of categories (4.3), and then using the existence of ind-algebraic structure on $\nabla\mathscr{H}^1(C,\mathfrak{G})$, which we explain in this section, see Theorem 4.11 below. The proposition immediately follows. □

In the rest of this section we discuss the ind-algebraic structure on $\nabla\mathscr{H}^1(C,\mathfrak{G})$. Note that this was established in [AH14b]. Here we briefly recall it for the sake of completeness.

First of all let us recall the definition of ind-algebraic stack

Definition 4.8. Let T be a scheme.

(a) By an inductive system of algebraic stacks over T we mean an inductive system (C_a, i_{ab}) indexed by a countable directed set I, such that each C_a is an algebraic Artin-stack over T and $i_{ab} : C_a \to C_b$ is a closed immersion of stacks for all $a \leq b$ in I.

(b) A stack C over T is an ind-algebraic T-stack if there is an inductive system of algebraic stacks (C_a, i_{ab}) over T together with morphisms $j_a : C_a \to C$ satisfying $j_b \circ i_{ab} = j_a$ for all $a \leq b$, such that for all quasi-compact T-schemes S and all objects $c \in C(S)$ there is an $a \in I$, an object $c_a \in C_a(S)$ and an isomorphism $j_a(c_a) \cong c$ in C. In this case we say that C is the inductive limit of (C_a, i_{ab}) and we write $C = \varinjlim C_a$.

(c) If in (b) all C_a are locally of finite type (resp. separated) over T we say that C is locally of ind-finite type (resp. ind-separated) over T.

Now notice that the morphism (4.2) is representable by a morphism of schemes which is quasi-affine and of finite presentation. To observe this, let $\mathcal{V}$ be a vector bundle on C of rank r as before. For a $GL(\mathcal{V})$-bundle $\mathcal{G}$ in $\mathscr{H}^1(C, \mathrm{GL}(\mathcal{V}))(S)$ we have the following 2-Cartesian diagram of stacks

$$\begin{array}{ccc} p_{S*}(\mathcal{G}/\mathfrak{G}_S) & \longrightarrow & S \\ \downarrow & & \downarrow \mathcal{G} \\ \mathscr{H}^1(C, \mathfrak{G}) & \longrightarrow & \mathscr{H}^1(C, \mathrm{GL}(\mathcal{V})). \end{array}$$

Here $p_S : C_S \to S$ is the projection map viewed as a morphism of big étale sites $\mathbf{\acute{E}t}(C_S) \to \mathbf{\acute{E}t}(S)$. For any scheme Y over C_S let $p_{S*}(Y)$ denote the sheaf which sends an S-scheme T to $\mathrm{Hom}_{C_S}(C_T, Y)$. It can be shown that $p_{S*}(\mathcal{G}/\mathfrak{G}_S)$ is a quasi-affine S-scheme of finite presentation; see [AH14b, Lemma 2.6]. As we mentioned earlier, the stack $\mathscr{H}^1(C, \mathrm{GL}(\mathcal{V}))$ is isomorphic to the stack $Vect_C^r$ whose S-valued points parameterize vector bundles of rank r on C_S. The stack $Vect_C^r$ is an Artin-stack locally of finite type over $\mathbb{F}_q$ and it admits a covering by connected open substacks of finite type over $\mathbb{F}_q$; see also [Wan11, Theorem 1.0.1]. For a $\mathrm{GL}(\mathcal{V})$-torsor $\mathcal{G}$ over C_S we let $\mathcal{V}(\mathcal{G})$ denote the associated vector bundle over C_S.

Let us recall the definition of the (unbounded ind-algebraic) Hecke stacks.

Definition 4.9. For a natural number n, let $Hecke_n(C, \mathfrak{G})$ be the stack fibered in groupoids over the category of $\mathbb{F}_q$-schemes, whose S valued points are tuples $\big(\mathcal{G}, \mathcal{G}', \underline{s}, \tau\big)$ where

- $\mathcal{G}$ and $\mathcal{G}'$ are in $\mathscr{H}^1(C, \mathfrak{G})(S)$,
- $\underline{s} := (s_i)_i \in C^n(S)$ are sections, and
- $\tau \colon \mathcal{G}'_{|_{C_S \smallsetminus \Gamma_{\underline{s}}}} \xrightarrow{\sim} \mathcal{G}_{|_{C_S \smallsetminus \Gamma_{\underline{s}}}}$ is an isomorphism.

Note that forgetting the isomorphism τ defines a morphism

$$Hecke_n(C,\mathfrak{G}) \to \mathscr{H}^1(C,\mathfrak{G}) \times \mathscr{H}^1(C,\mathfrak{G}) \times C^n. \tag{4.4}$$

Remark 4.10. Note that the moduli stack $\nabla_n\mathscr{H}^1(C,\mathfrak{G})$ of global $\mathfrak{G}$-shtukas is the preimage in $Hecke_n(C,\mathfrak{G})$ of the graph of the Frobenius morphism on $\mathscr{H}^1(C,\mathfrak{G})$. In other words, we have the following Cartesian diagram

$$\begin{array}{ccc} \nabla_n\mathscr{H}^1(C,\mathfrak{G}) & \longrightarrow & Hecke_n(C,\mathfrak{G}) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle pr_1\times pr_2} \\ \mathscr{H}^1(C,\mathfrak{G}) & \xrightarrow{\Delta^\sigma:\mathcal{G}\mapsto\mathcal{G}\times\sigma^*\mathcal{G}} & \mathscr{H}^1(C,\mathfrak{G}) \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G}), \end{array}$$

i.e.

$$\nabla_n\mathscr{H}^1(C,\mathfrak{G}) := \operatorname{equi}(\sigma_{\mathscr{H}^1(C,\mathfrak{G})} \circ pr_1, pr_2\colon Hecke_n(C,\mathfrak{G}) \rightrightarrows \mathscr{H}^1(C,\mathfrak{G})),$$

where pr_i are the projections to the first, resp. second factor in (4.4). Therefore it is enough to construct an ind-algebraic structure $\varinjlim Hecke_n^{\omega}(C,\mathfrak{G})$ on the stack $Hecke_n(C,\mathfrak{G})$ on $Hecke_n(C,\mathfrak{G})$. The ind-algebraic structure $Hecke_n(C,\mathfrak{G}) = \varinjlim Hecke_n^{\omega}(C,\mathfrak{G})$ then induces an ind-algebraic structure $\varinjlim_{\omega} \nabla_n^{\omega}\mathscr{H}_D^1(C,\mathfrak{G})$ on $\nabla_n\mathscr{H}_D^1(C,\mathfrak{G})$. To do this, in [AH14b], the authors proposed the following method.

According to [AH14b], there is a faithful representation $\rho : \mathfrak{G} \to \mathrm{GL}(\mathcal{V})$ for a vector bundle $\mathcal{V}$ on C together with an isomorphism $\alpha : \wedge^{top}\mathcal{V} \tilde{\to} \mathcal{O}_C$ such that ρ factors through

$$\mathrm{SL}(\mathcal{V}) := \ker\big(\det : GL(V) \to \mathrm{GL}(\wedge^{top}\mathcal{V})\big)$$

and the quotients $\mathrm{SL}(\mathcal{V})/\mathfrak{G}$ and $\mathrm{GL}(\mathcal{V})/\mathfrak{G}$ are quasi-affine schemes over C. We fix such a representation $\rho : \mathfrak{G} \to \mathrm{SL}(\mathcal{V}_0) \subseteq \mathrm{GL}(\mathcal{V}_0)$.
The deep reason for taking a representation in special linear group $\mathrm{SL}(\mathcal{V})$ rather than the general linear group $\mathrm{GL}(\mathcal{V})$, lies in the lines of [PR08, Proposition 9.9]. In fact if we start with a representation of $\mathfrak{G}$ in $\mathrm{GL}(\mathcal{V})$, at the end, the resulting ind-algebraic structure may not cover the whole stack $Hecke_n(C,\mathfrak{G})$.

Define the *relative affine Grassmannian* $\mathcal{G}r_{\mathfrak{G},n,r}$ as the stack over $C^n \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G})$ which parametrizes tuples $\big((s_i)_{i=1,\dots,n}, \mathcal{G}, \mathcal{V}', \alpha', \varphi\big)$, where

$$\big((s_i)_i, \mathcal{G}, \mathcal{V}'\big) \in C^n \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G}) \times_{\mathbb{F}_q} Vect_C^r,$$

$\alpha'\colon \wedge^r \mathcal{V}' \xrightarrow{\sim} \mathcal{O}_{C_S}$ is a trivialization and $\varphi\colon \mathcal{V}(\rho_*\mathcal{G})|_{C_S\smallsetminus\bigcup_{i\in I}\Gamma_{s_i}} \xrightarrow{\sim} \mathcal{V}'|_{C_S\smallsetminus\bigcup_{i\in I}\Gamma_{s_i}}$ is an isomorphism between the vector bundle $\mathcal{V}(\rho_*\mathcal{G})$ associated with $\rho_*\mathcal{G}$ and $\mathcal{V}'$ outside the graphs $\bigcup_i \Gamma_{s_i}$ such that $\alpha_{\mathcal{G}} = \alpha' \circ \wedge^r\varphi$ on $C_S \smallsetminus \cup\Gamma_{s_i}$. Here $\alpha_{\mathcal{G}}\colon \wedge^r \mathcal{V}(\rho_*\mathcal{G}) \xrightarrow{\sim} \mathcal{O}_{C_S}$ is the canonical isomorphism induced from the fact that ρ factors through $\mathrm{SL}(\mathcal{V}_0)$. In particular $\wedge^r\varphi = (\alpha')^{-1} \circ \alpha_{\mathcal{G}}$

extends to an isomorphism $\wedge^r\varphi\colon \wedge^r \mathcal{V}(\rho_*\mathcal{G}) \xrightarrow{\sim} \wedge^r\mathcal{V}'$ on all of C_S. If $\mathcal{S}l'$ is the $\mathrm{SL}(\mathcal{V}_0)$-torsor associated with $(\mathcal{V}',\alpha')$ then φ induces an isomorphism $\varphi\colon \rho_*\mathcal{G}|_{C_S\smallsetminus\bigcup_{i\in I}\Gamma_{s_i}} \xrightarrow{\sim} \mathcal{S}l'|_{C_S\smallsetminus\bigcup_{i\in I}\Gamma_{s_i}}$. Note that the morphism ρ_* yields a morphism

$$Hecke_n(C,\mathfrak{G}) \to \mathcal{G}r_{\mathfrak{G},n,r},$$

sending the tuple $\big(\mathcal{G},\mathcal{G}',(s_i)_i\,\tau\big)$ to the tuple $\big(\mathcal{G},\mathcal{V}(\rho_*\mathcal{G}'),\alpha_{\mathcal{G}'},(s_i)_i,\mathcal{V}(\rho_*\tau)\big)$. We now establish an ind-algebraic structure on $\mathcal{G}r^{\omega}_{\mathfrak{G},n,r} = \varinjlim_{\underline{\omega}} \mathcal{G}r^{\underline{\omega}}_{\mathfrak{G},n,r}$, and use this morphism to produce an ind-algebraic structure on $Hecke_n(C,\mathfrak{G})$. Let $\underline{\omega} := (\omega_i)_{i=1,\dots,n}$ be an n-tuple of coweights of SL_r given as $\omega_i\colon x\mapsto \mathrm{diag}(x^{\omega_{i,1}},\dots,x^{\omega_{i,r}})$ for integers $\omega_{i,1}\geq\ldots\geq\omega_{i,r}$ with $\omega_{i,1}+\ldots+\omega_{i,r}=0$ for all i. The inequality means that all ω_i are dominant with respect to the Borel subgroup of upper triangular matrices. Let $\mathcal{G}r^{\underline{\omega}}_{\mathfrak{G},n,r}$ denote the substack of $\mathcal{G}r_{\mathfrak{G},n,r}$ defined by the condition that the universal isomorphism φ_{univ} is *bounded* by $\underline{\omega}$, i.e. it satisfies

$$\begin{gathered}\textstyle\bigwedge^{\ell}_{C_S}\varphi_{univ}\big(\mathcal{V}(\rho_*\mathcal{G}_{univ})\big) \;\subset\; \Big(\bigwedge^{\ell}_{C_S}\mathcal{V}'\Big)\big(\sum_i(-\omega_{i,r-\ell+1}-\ldots-\omega_{i,r})\cdot\Gamma_{s_i}\big)\\ \text{for all } 1\leq\ell\leq r \text{ with equality for } \ell=r\end{gathered} \tag{4.5}$$

where the notation $\big(\wedge^{\ell}_{C_S}\mathcal{V}'\big)\big(\sum_i(-\omega_{i,r-\ell+1}-\ldots-\omega_{i,r})\cdot\Gamma_{s_i}\big)$ means that we allow poles of order $-\omega_{i,r-\ell+1}-\ldots-\omega_{i,r}$ along the Cartier divisor Γ_{s_i} on C_S; compare [HV11, Lemma 4.3]. Note that the condition for $\ell=r$ is equivalent to the requirement that $\wedge^r\varphi$ is an isomorphism on all of C_S, which in turn is equivalent to the condition that $\alpha_{\mathcal{G}}=\alpha'\circ\wedge^r\varphi$ for an isomorphism $\alpha'\colon \wedge^r\mathcal{V}'\xrightarrow{\sim}\mathcal{O}_{C_S}$. By Cramer's rule (e.g. [Bou58, III.8.6, Formulas (21) and (22)]) condition (4.5) is equivalent to

$$\begin{gathered}\textstyle\bigwedge^{\ell}_{C_S}\varphi^{-1}_{univ}(\mathcal{V}') \;\subset\; \Big(\bigwedge^{\ell}_{C_S}\mathcal{V}(\rho_*\mathcal{G}_{univ})\Big)\big(\sum_i(\omega_{i,1}+\ldots+\omega_{i,\ell})\cdot\Gamma_{s_i}\big)\\ \text{for all } 1\leq\ell\leq r \text{ with equality for } \ell=r\end{gathered} \tag{4.6}$$

Again the condition for $\ell=r$ is equivalent to the condition that $\alpha_{\mathcal{G}}=\alpha'\circ\wedge^r\varphi$ for an isomorphism $\alpha'\colon \wedge^r\mathcal{V}'\xrightarrow{\sim}\mathcal{O}_{C_S}$.

The relative affine Grassmannian $\mathcal{G}r^{\underline{\omega}}_{\mathfrak{G},n,r}$ is relatively representable over $C^n\times_{\mathbb{F}_q}\mathscr{H}^1(C,\mathfrak{G})$ by a projective morphism of schemes. To see this we look at the fiber of $\mathcal{G}r^{\underline{\omega}}_{\mathfrak{G},n,r}\to C^n\times_{\mathbb{F}_q}\mathscr{H}^1(C,\mathfrak{G})$ over an S-valued point $((s_i)_{i\in I},\mathcal{G})$ in $(C^n\times_{\mathbb{F}_q}\mathscr{H}^1(C,\mathfrak{G}))(S)$. Then (4.5) and (4.6) imply that $\mathcal{V}(\rho_*\mathcal{G})(\sum_{i\in I}\omega_{i,1}\cdot\Gamma_{s_i})/\varphi^{-1}(\mathcal{V}')$ must be a quotient of the sheaf

$$\mathcal{F} \;:=\; \mathcal{V}(\rho_*\mathcal{G})(\textstyle\sum_i\omega_{i,1}\cdot\Gamma_{s_i})\big/\mathcal{V}(\rho_*\mathcal{G})(\sum_i\omega_{i,r}\cdot\Gamma_{s_i})$$

on the effective relative Cartier divisor $X:=\sum_i(\omega_{i,1}-\omega_{i,r})\cdot\Gamma_{s_i}$. Note that X is a finite flat S-scheme. From the case $\ell=r$ in (4.5) and (4.6) we also obtain the isomorphism

$$\alpha' := \alpha_{\mathcal{G}}\circ(\wedge^r\varphi)^{-1}\colon\; \wedge^r\mathcal{V}'\xrightarrow{\sim}\mathcal{O}_{C_S}.$$

Therefore $\mathcal{G}r^{\underline{\omega}}_{\mathfrak{G},n,r} \times_{(C^n \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G}))} S$ is represented by a closed subscheme of Grothendieck's Quot-scheme $\mathrm{Quot}^{\Phi}_{\mathcal{F}/X/S}$, see [FGA, n°221, Théorème 3.1], for constant Hilbert polynomial $\Phi = r \cdot \sum_i \omega_{i,1}$.
Now define the stack $Hecke^{\underline{\omega}}_n(C,\mathfrak{G})$ by the Cartesian diagram

$$\begin{array}{ccc} Hecke^{\underline{\omega}}_n(C,\mathfrak{G}) & \longrightarrow & \mathcal{G}r^{\underline{\omega}}_{\mathfrak{G},n,r} \\ \downarrow & & \downarrow \\ Hecke_n(C,\mathfrak{G}) & \longrightarrow & \mathcal{G}r_{\mathfrak{G},n,r}\ . \end{array}$$

Note further that one can show that $Hecke_{\mathfrak{G},n} \to \mathcal{G}r_{\mathfrak{G},n,r}$ is represented by a locally closed and quasi-compact (resp. a closed) immersion. This was proved in [AH14b, Proposition 3.10]. This implies that the morphism of stacks $Hecke^{\underline{\omega}}_{\mathfrak{G},n} \to C^n \times_{\mathbb{F}_q} \mathscr{H}^1(C,\mathfrak{G})$ sending $\big(\mathcal{G}, \mathcal{G}', (s_i)_i, \tau\big)$ to $\big((s_i)_i, \mathcal{G}\big)$ is relatively representable by a morphism of schemes which is quasi-compact and quasi-projective, and even projective if there is a representation ρ with affine quotient $\mathrm{SL}(\mathcal{V}_0)/\mathfrak{G}$. We can finally state the following theorem.

Theorem 4.11. *The stack $\nabla_n\mathscr{H}^1(C,\mathfrak{G}) = \varinjlim_{\underline{\omega}} \nabla^{\underline{\omega}}_n\mathscr{H}^1(C,\mathfrak{G})$ is an ind-algebraic stack over C^n which is ind-separated and locally of ind-finite type. Here $\underline{\omega}$ runs over n-tuple of coweights of* SL_r.

□

4.4 Local $\mathbb{P}$-Shtukas

Let $\mathbb{F}$ be a finite field and $\mathbb{F}[\![z]\!]$ be the power series ring over $\mathbb{F}$ in the variable z. We let $\mathbb{P}$ be a smooth affine group scheme over $\mathbb{D} := \mathrm{Spec}\,\mathbb{F}[\![z]\!]$ with connected generic fiber P. Set $\dot{\mathbb{D}} := \mathrm{Spec}\,\mathbb{F}(\!(z)\!)$.

In contrast with global situation one defines the category $\mathbb{P}$-$\hat{\sigma}$-$\mathbf{Cryst}_{\mathbb{F}}(L)$ of $\mathbb{P}$-$\hat{\sigma}$-crystals (or σ-crystals with $\mathbb{P}$-structure) over L as the category whose objects are the tensor functors

$$\mathrm{Rep}\,\mathbb{P} \to \hat{\sigma} - \mathbf{Cryst}_{\mathbb{F}}(L).$$

The morphisms are natural transformations of functors.
This category is related to the category of *local* $\mathbb{P}$*-shtukas* in a similar way that $\mathfrak{G}$-C-motives are related to global $\mathfrak{G}$-shtukas.
Before recalling the definition of local $\mathbb{P}$-shtukas let us recall some background materials.

Definition 4.12. The *group of positive loops associated with* $\mathbb{P}$ is the affine group scheme $L^+\mathbb{P}$ over $\mathbb{F}$ whose R-valued points for an $\mathbb{F}$-algebra R are

$$L^+\mathbb{P}(R) := \mathbb{P}(R[\![z]\!]) := \mathbb{P}(\mathbb{D}_R) := \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}_R, \mathbb{P})\,.$$

The *group of loops associated with* $\mathbb{P}$ is the *fpqc*-sheaf of groups LP over $\mathbb{F}$ whose R-valued points for an $\mathbb{F}$-algebra R are

$$LP := P(R(\!(z)\!)) := P(\dot{\mathbb{D}}_R) := \mathrm{Hom}_{\dot{\mathbb{D}}}(\dot{\mathbb{D}}_R, P)\,,$$

where we write $R(\!(z)\!) := R[\![z]\!][\frac{1}{z}]$ and $\dot{\mathbb{D}}_R := \operatorname{Spec} R(\!(z)\!)$. It is representable by an ind-scheme of ind-finite type over $\mathbb{F}$; see [PR08, § 1.a], or [BD, §4.5]. Let $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P}) := [\operatorname{Spec}\mathbb{F}/L^+\mathbb{P}]$ (respectively $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, P) := [\operatorname{Spec}\mathbb{F}/LP]$) denote the classifying space of $L^+\mathbb{P}$-torsors (respectively LP-torsors). It is a stack fibered in groupoids over the category of $\mathbb{F}$-schemes S whose category $\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P})(S)$ consists of all $L^+\mathbb{P}$-torsors (resp. LP-torsors) on S. The inclusion of sheaves $L^+\mathbb{P} \subset LP$ gives rise to the natural 1-morphism

$$\mathscr{H}^1(\operatorname{Spec}\mathbb{F}, L^+\mathbb{P}) \longrightarrow \mathscr{H}^1(\operatorname{Spec}\mathbb{F}, LP),\ \mathcal{L}_+ \mapsto \mathcal{L}\,. \tag{4.7}$$

Now we recall the definition of the category of local $\mathbb{P}$-shtukas.

Definition 4.13. (a) A local $\mathbb{P}$-shtuka over $S \in \mathcal{N}ilp_{\mathbb{F}[\![\zeta]\!]}$ is a pair $\underline{\mathcal{L}} = (\mathcal{L}_+, \tau)$ consisting of an $L^+\mathbb{P}$-torsor $\mathcal{L}_+$ on S and an isomorphism of the associated loop group torsors $\hat{\tau}\colon \hat{\sigma}^*\mathcal{L} \to \mathcal{L}$.

(b) A *quasi-isogeny* $f\colon \underline{\mathcal{L}} \to \underline{\mathcal{L}}'$ between two local $\mathbb{P}$-shtukas $\underline{\mathcal{L}} := (\mathcal{L}_+, \tau)$ and $\underline{\mathcal{L}}' := (\mathcal{L}'_+, \tau')$ over S is an isomorphism of the associated LP-torsors $f\colon \mathcal{L} \to \mathcal{L}'$ such that the following diagram

$$\begin{array}{ccc} \sigma^*\mathcal{L} & \xrightarrow{\tau} & \mathcal{L} \\ \downarrow{\scriptstyle \sigma^* f} & & \downarrow{\scriptstyle f} \\ \sigma^*\mathcal{L}' & \xrightarrow{\tau'} & \mathcal{L}'\,. \end{array}$$

becomes commutative.

(c) We denote by $\mathrm{QIsog}_S(\underline{\mathcal{L}}, \underline{\mathcal{L}}')$ the set of quasi-isogenies between $\underline{\mathcal{L}}$ and $\underline{\mathcal{L}}'$ over S. We denote by $Sht_{\mathbb{P}}^{\mathbb{D}}(S)$ the category of local $\mathbb{P}$-shtukas over S with quasi-isogenies.

Remark 4.14. Like global $\mathfrak{G}$-shtukas, quasi-isogenies of local $\mathbb{P}$-shtukas have the rigidity property. This means that they lift over infinitesimal thickenings. See Theorem 4.4.

Remark 4.15. Let $B \subseteq GL_r$ be the Borel subgroup of upper triangular matrices and let T be the torus of diagonal matrices. Then $X_*(T) = \mathbb{Z}^r$ with simple coroots $e_i - e_{i+1}$ for $i = 1, ..., r-1$. Also $X^*(T) = \mathbb{Z}^r$. Let $\lambda_i = (1, ..., 1, 0, ..., 0)$ with multiplicities i and $r - i$. The Weyl module $V(\lambda_1) = Ind_B^{GL_r}(-\lambda_1)_{dom}$ of highest weight λ_1 is simply the standard representation of GL_r on the space of column vectors with r rows, and $V(\lambda_i) = \wedge_i V(\lambda_1)$. For an $\mathbb{F}_q$-scheme S we have $L^+\,\mathrm{GL}_r(S) = \mathrm{GL}_r\,\Gamma(S, \mathcal{O}_S)[\![z]\!]$. There is an equivalence between the category of $L^+\,\mathrm{GL}_r$-torsors on S and the category of sheaves of $\mathcal{O}_S[\![z]\!]$-modules which Zariski-locally on S are free of rank r with isomorphisms as the only morphisms. According to this equivalence, we send $\mathcal{L}$ to the sheaf $\mathcal{L}_{\lambda_1}$ corresponding to the following presheaf

$$Y \mapsto \big(\mathcal{L}(Y) \times \big(V(\lambda_1) \otimes_{\mathbb{F}_q} \mathcal{O}_S[\![z]\!](Y)\big)\big) \,/\, \mathrm{GL}_r(Y[\![z]\!]);$$

Accordingly, the category of local GL_r-shtukas over $\operatorname{Spec} L$ with quasi-isogenies as morphisms is equivalent to the subcategory of rank r $\hat{\sigma}$-crystals in $\hat{\sigma}$-**Cryst**(L). In particular the quasi-isogenies in $\hat{\sigma}$-**Cryst**(L) are rigid. See Remark 4.14.

From the above explanation we can define

$$Sht_{\mathbb{P}}^{\mathbb{D}}(L) \times \mathrm{Rep}_{\mathbb{F}[\![z]\!]}\,\mathbb{P} \to \hat{\sigma}\text{-}\mathbf{Cryst}(L),\ \underline{\mathcal{L}} \times \rho \mapsto \rho_*\underline{\mathcal{L}}.$$

In other words a local $\mathbb{P}$-shtuka $\underline{\mathcal{L}}$ gives rise to a $\mathbb{P}$-$\hat{\sigma}$-crystal over L.

4.5 Global-Local Functor And Deformation Of Global $\mathfrak{G}$-Shtukas

Recall that to an abelian variety A over k one associates a p-divisible group $A[p^\infty]$, or equivalently a Dieudonné module $M(A[p^\infty])$ over the ring of Witt vectors $W(k)$. The deformation theory of an abelian variety is ruled by the associated Dieudonné module. In this section we briefly discuss the analogous picture over function fields. For a detailed account on this subject we refer the reader to [AH14a, Chapter 5].

Let $\nabla\mathscr{H}^1(C,G)^{\underline{\nu}}$ denote the formal algebraic stack $\nabla\mathscr{H}^1(C,G)^{\underline{\nu}}(S) := \nabla\mathscr{H}^1(C,G) \times_{C^n} \operatorname{Spf}\widehat{\mathcal{O}}_{C^n,\underline{\nu}}$ that is, $s_i : S \to C$ factors through $\operatorname{Spf}\widehat{\mathcal{O}}_{C,\nu_i}$. One may decompose

$$\mathcal{G}\widehat{\times}_{C_S}(\operatorname{Spf}\widehat{\mathcal{O}}_{C,\nu_i}\widehat{\times}_{\mathbb{F}_q}S) = \coprod_{\ell\in\mathbb{Z}/\deg\nu_i} \mathcal{G}\widehat{\times}_{C_S}V(\mathfrak{a}_{\nu,\ell})$$

which is induced from $\operatorname{Spf}\widehat{\mathcal{O}}_\nu\widehat{\times}_{\mathbb{F}_q}S = \coprod_{\ell\in\mathbb{Z}/\deg\nu_i} V(\mathfrak{a}_{\nu,\ell})$, where $V(\mathfrak{a}_{\nu,\ell})$ denotes the component identified by the ideal $\mathfrak{a}_{\nu,\ell} = \langle a\otimes 1 - 1\otimes s^*(a)^{q^\ell} : a \in \mathbb{F}_{\nu_i}\rangle$. Note that $\mathcal{G}\widehat{\times}_{C_S}V(\mathfrak{a}_{\nu,\ell})$ can be viewed as a positive loop torsor. Since σ cyclically permutes these connected components, see also the lines before Definition 2.9, the tuple $(\mathcal{G}\widehat{\times}_{C_S}V(\mathfrak{a}_{\nu,0}), \tau^{\deg\nu_i})$ defines a local $\mathbb{P}_{\nu_i}$-shtuka. This defines the global-local functor

$$\Gamma_{\nu_i} : \nabla_n\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S) \to Sht_{\mathbb{P}_{\nu_i}}^{\mathbb{D}_{\mathbb{F}_{\nu_i}}}(S).$$

We set $\underline{\Gamma} := \prod_{\nu_i}\Gamma_{\nu_i}$.

The Serre-Tate theorem for $\mathbb{P}$-shtukas states that the above functor induces an isomorphism of deformation spaces

$$\mathcal{D}_{S/\overline{S}}(\underline{\mathcal{G}}) = \prod_i \mathcal{D}_{S/\overline{S}}(\Gamma_{\nu_i}(\underline{\mathcal{G}})),$$

for a global $\mathfrak{G}$-shtuka $\overline{\underline{\mathcal{G}}}$ in $\nabla\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}(\overline{S})$ and a nilpotent thickening $j:\overline{S}\to S$. The category $\mathcal{D}_{S/\overline{S}}(\overline{\underline{\mathcal{G}}})$ is the category of lifts of $\overline{\underline{\mathcal{G}}}$ to S which consists of all pairs $(\mathcal{G},\alpha : j^*\underline{\mathcal{G}}\to\overline{\underline{\mathcal{G}}})$ where $\underline{\mathcal{G}}$ belongs to $\nabla\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S)$, where α is an isomorphism of global $\mathfrak{G}$-shtukas over $\overline{S}$, and where morphisms are isomorphisms between them that are compatible with the α's. Similarly for a local $\mathbb{P}$-shtuka $\underline{\mathcal{L}}$ in $Sht_{\mathbb{P}}^{\mathbb{D}}$ we define the category of lifts $\mathcal{D}_{S/\overline{S}}(\overline{\underline{\mathcal{L}}})$ of $\overline{\underline{\mathcal{L}}}$ to S.
The proof essentially relies on rigidity of quasi isogenies, and the following result which says that like abelian varieties, we can pull back a global $\mathfrak{G}$-shtuka along a quasi-isogeny to the corresponding local $\mathbb{P}$-shtukas. More precisely we have

Theorem 4.16. *Let $\underline{\mathcal{G}}\in\nabla_n\mathscr{H}^1(C,\mathfrak{G})^{\underline{\nu}}(S)$ be a global $\mathfrak{G}$-shtuka over S and let ν be a place on C. Let $\underline{\mathcal{L}}_\nu := \Gamma_\nu(\underline{\mathcal{G}})$ be the local $\mathbb{P}_\nu$-shtuka associated with $\underline{\mathcal{G}}$ at ν in the sense of Remark 5.5 (if $\nu\in\underline{\nu}$), respectively Remark 5.6 (if $\nu\notin\underline{\nu}$). Let $f:\underline{\mathcal{L}}'\to\underline{\mathcal{L}}_\nu$ be a quasi-isogeny of local $\mathbb{P}_\nu$-shtukas over S. If $\nu\in\underline{\nu}$ we assume that the Frobenius of $\underline{\mathcal{L}}'$ is an isomorphism outside $V(\mathfrak{a}_{\nu,0})$. If $\nu\notin\underline{\nu}$ we assume that $\underline{\mathcal{L}}$ is étale. Then there exists a unique global $\mathfrak{G}$-shtuka $\underline{\mathcal{G}}'$ in $\mathscr{H}^1(C,G)^{\underline{\nu}}(S)$ and a unique quasi-isogeny $g:\underline{\mathcal{G}}'\to\underline{\mathcal{G}}$ which is an isomorphism outside ν, such that the local $\mathbb{P}_\nu$-shtuka associated with $\underline{\mathcal{G}}'$ is $\underline{\mathcal{L}}'$ and the quasi-isogeny of local $\mathbb{P}_\nu$-shtukas induced by g is f. We denote $\underline{\mathcal{G}}'$ by $f^*\underline{\mathcal{G}}$.*

Note that we have the following diagram of functors

$$\begin{array}{ccc} \nabla\mathscr{H}(C,\mathfrak{G})(L) & \longrightarrow & \mathfrak{G}-\mathcal{M}ot_C^{\underline{\nu}}(L) \\ \Big\downarrow{\scriptstyle \Gamma_\nu(-)} & & \Big\downarrow \\ Sht_{\mathbb{P}_\nu}^{\mathbb{D}_{\mathbb{F}_\nu}} & \longrightarrow & \mathbb{P}_\nu-\sigma-\mathbf{Cryst}_{\mathbb{F}_\nu}(L) \end{array}$$

Corollary 4.17. *Let ν be a place on C. Let $\underline{\mathcal{M}}$ be in $\mathcal{M}ot_C^{\underline{\nu}}(L)$ and let $f:\underline{\hat{M}}'\to\underline{\hat{M}}$ be a quasi-isogeny from $\underline{\hat{M}}'$ to $\underline{\hat{M}} := \Gamma_\nu(\underline{\mathcal{M}})$. Then one can form the pull back $\underline{\mathcal{M}}' := f^*\underline{\mathcal{M}}\in\mathcal{M}ot_C^{\underline{\nu}}(L)$ along $\hat{f}:\underline{\hat{M}}'\to\underline{\hat{M}}$ together with a quasi-isogeny $f:\underline{\mathcal{M}}'\to\underline{\mathcal{M}}$, such that $\hat{f}=\Gamma_\nu(f)$.*

Proof. This follows from Theorem 4.16. See Remark 4.15 and Remark 4.5. □

Corollary 4.18. *Let $\varphi:\underline{\mathcal{M}}'\to\underline{\mathcal{M}}$ be a quasi-isogeny of C-motives in $\mathcal{M}ot_C^{\underline{\nu}}(L)$. Then $\omega_Q^\nu(\varphi)$ identifies $\omega^\nu(\underline{\mathcal{M}}')$ with a Γ_L-stable sublattice of $\omega_Q^\nu(\underline{\mathcal{M}})$. This gives a one to one correspondence between the following sets*

$$\{\textit{quasi-isogenies } \underline{\mathcal{M}}'\to\underline{\mathcal{M}} \textit{ in } \mathcal{M}ot_C^{\underline{\nu}}(L)\}$$

and

$$\{\Gamma_L\textit{-stable sublattice } \Lambda_\nu\subseteq\omega_Q^\nu(\underline{\mathcal{M}}) \textit{ which are contained in } \omega^\nu(\underline{\mathcal{M}})\}.$$

Proof. We may view $\omega^\nu(\underline{\mathcal{M}}')$ as a Γ_L-stable sublattice of $\omega^\nu_Q(\underline{\mathcal{M}})$ contained in $\omega^\nu(\underline{\mathcal{M}})$ by applying $\omega^\nu(-)$ to a given quasi-isogeny $f : \underline{\mathcal{M}}' \to \underline{\mathcal{M}}$

$$\omega^\nu(\underline{\mathcal{M}}') \hookrightarrow \omega^\nu(\underline{\mathcal{M}}) \subseteq \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}).$$

Vice versa consider the inclusion

$$\Lambda_\nu \otimes_{A_\nu} A_{L^{\mathrm{sep}}} \subseteq \omega^\nu(\underline{\mathcal{M}}) \otimes_{A_\nu} A_{\nu,L^{\mathrm{sep}}} = \Gamma_\nu(\underline{\mathcal{M}}) \otimes_{A_{\nu,L}} A_{\nu,L^{\mathrm{sep}}}.$$

Therefore we get a quasi-isogeny $\hat{f}$ from $\underline{\hat{M}}' := (\Lambda_\nu \otimes_{A_\nu} A_{L^{\mathrm{sep}}})^{\Gamma_L}$ to $\underline{\hat{M}} := (\Gamma_\nu(\underline{\mathcal{M}}) \otimes_{A_{\nu,L}} A_{\nu,L^{\mathrm{sep}}})^{\Gamma_L}$. According to Theorem 4.16 we may construct the pull back $\underline{\mathcal{M}}' := \hat{f}^*\underline{\mathcal{M}}$ of $\underline{\mathcal{M}}$ along $\hat{f}$, which comes with a canonical quasi-isogeny $f : \underline{\mathcal{M}}' \to \underline{\mathcal{M}}$. By construction the above assignments are inverse to each other.

□

5 Quasi-isogeny classes and Honda-Tate theory

Recall that a Weil p^n-number is an algebraic number π for which there exists an integer m such that $\pi\overline{\pi} = p^n$ for all $Q[\pi] \to \mathbb{C}$. Here $\overline{\pi}$ denotes the complex conjugate of π. The Honda-Tate theory, [Hon] and [Tat66], states that sending an abelian variety $\mathcal{A}$ over a finite field with q-elements to the eigenvalue of Frobenius endomorphism $\pi_{\mathcal{A}}$ on the first étale cohomolgy group, gives a bijection between isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and the set of Weil p^n-numbers $W(p^n)$ (up to conjugation).

In this chapter we discuss the analogues picture for $\mathcal{M}ot^{\underline{\nu}}_C(\overline{\mathbb{F}}_q)$. Note that, unlike the above case of abelian varieties, as it is mentioned earlier, C-motives are not pure. This means that the eigenvalues of Frobenius endomorphism may have different valuations. So in particular one must modify the group of Weil (q-)numbers.

Proposition 5.1. *Let $\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}'$ be in $\mathcal{M}ot^{\underline{\nu}}_C(L)$. Let $\pi := \pi_{\underline{\mathcal{M}}}$ and $\pi' := \pi_{\underline{\mathcal{M}}'}$ be the corresponding Frobenius endomorphism with minimal polynomials $\mu := \mu_{\pi_{\underline{\mathcal{M}}}}$ and $\mu' := \mu_{\pi_{\underline{\mathcal{M}}}}$. Let χ_ν and χ'_ν denote the characteristic polynomials of π_ν and π'_ν. Consider the following statements*

(a) $\underline{\mathcal{M}}'$ is quasi-isogenous to a quotient of $\underline{\mathcal{M}}$.

(b) $\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}')$ is Γ_L-isomorphic to a Γ_L-quotient space of $\omega^\nu_{Q_\nu}(\underline{\mathcal{M}})$.

(c) χ'_ν divides χ_ν in $Q_\nu[x]$.

(d) μ'_ν divides μ_ν in $Q[x]$ and $\operatorname{rk}\underline{\mathcal{M}} \leq \operatorname{rk}\underline{\mathcal{M}}'$.

then (a) and (b) are always equivalent and imply (c) and (d). Furthermore we have the following statements

i) If π_ν and π'_ν are semi-simple then (c) also implies (b).

ii) If μ is irreducible, then all the above statements are equivalent.

Proof. $(a) \Rightarrow (b)$ is obvious. So let us first show that $(b) \Rightarrow (a)$. The main ingredient to prove this is the analog of Tate conjecture; See Theorem 2.14. Consider the quotient morphism $f_\nu : \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}) \to \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}')$. Multiplying with a suitable power of the uniformizer $z_\nu \in A_\nu$, we may assume that it is defined with integral coefficients $f_\nu : \omega^\nu(\underline{\mathcal{M}}) \to \omega^\nu(\underline{\mathcal{M}}')$ with $z_\nu^N \omega^\nu(\underline{\mathcal{M}}') \subseteq f_\nu(\omega^\nu(\underline{\mathcal{M}}))$, for some integer $N \gg 0$. By Theorem 2.14 f_ν can be viewed as an element of $\mathrm{Hom}_k(\underline{\mathcal{M}}, \underline{\mathcal{M}}') \otimes_A A_\nu$ and thus induces a morphism $f : \underline{\mathcal{M}} \to \underline{\mathcal{M}}'$ such that $\omega^\nu(f) = f_\nu \pmod{z_\nu^N}$. We claim that $\dim_{\mathbb{F}_q((z_\nu))} \omega^\nu_{Q_\nu}(f)(\omega^\nu(\underline{\mathcal{M}})) = r'$. To see this first notice that there exists $x_1, \ldots, x_{r'} \in \omega^\nu_{Q_\nu}(f)(\omega^\nu(\underline{\mathcal{M}}))$ which generate the $\mathbb{F}_q$-vector space

$$z_\nu^N \omega^\nu(\underline{\mathcal{M}}')/z_\nu^{N+1}\omega^\nu(\underline{\mathcal{M}}') \cong \omega^\nu(\underline{\mathcal{M}}')/z_\nu \cdot \omega^\nu(\underline{\mathcal{M}}') \cong \mathbb{F}_q^{r'}.$$

For this we have $H := \sum_{i=1}^{r'} A_\nu \cdot x_i \subseteq \omega^\nu(f)(\omega^\nu(\underline{\mathcal{M}})) \subseteq \omega^\nu(\underline{\mathcal{M}}')$. As $\omega^\nu(\underline{\mathcal{M}}')$ is a free module of rank r', computing modulo z_ν we argue that H is also a free module of rank r'. Therefore

$$H \otimes_{A_\nu} Q_\nu = \sum_{i=1}^{r'} Q_\nu \cdot x_i \subseteq \omega^\nu_{Q_\nu}(f)(\omega^\nu_{Q_\nu}(\underline{\mathcal{M}})) \subseteq \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}')$$

and hence $\dim_{Q_\nu} \omega^\nu_{Q_\nu}(f)(\omega^\nu(\underline{\mathcal{M}})) = r'$. Now observe that $\mathrm{rk}(\underline{\mathrm{im}}f) = r'$. To see this, apply $\omega^\nu(-)$ to the morphism $\underline{\mathcal{M}} \twoheadrightarrow \underline{\mathrm{im}}f \subseteq \underline{\mathcal{M}}'$, to get a surjection $\omega^\nu(f) : \omega^\nu(\underline{\mathcal{M}}) \to \omega^\nu(\underline{\mathrm{im}}f) \subseteq \omega^\nu(\underline{\mathcal{M}}')$. Consequently we have

$$r' = \dim_{Q_\nu} \omega^\nu(\underline{\mathrm{im}}f) = \mathrm{rk}(\underline{\mathrm{im}}f),$$

and therefore $\underline{\mathrm{im}}f \to \underline{\mathcal{M}}'$ is a quasi-isogeny.

$(b) \Rightarrow (c)$ and (d), precisely because of the following commutative diagram

$$\begin{array}{ccccc}
\omega^\nu_Q(\underline{\mathcal{M}}) & \xrightarrow{f_\nu} & \omega^\nu_Q(\underline{\mathcal{M}}') & \longrightarrow & 0 \\
\downarrow{\scriptstyle \pi_\nu} & & \downarrow{\scriptstyle \pi'_\nu} & & \\
\omega^\nu_Q(\underline{\mathcal{M}}) & \xrightarrow{f_\nu} & \omega^\nu_Q(\underline{\mathcal{M}}') & \longrightarrow & 0.
\end{array}$$

Now assume that π_ν and π'_ν are semi-simple with characteristic polynomials χ_ν and χ'_ν. Write $\chi'_\nu = \prod_{i=1}^{n'} P'_i$ for irreducible polynomials $P'_i \in Q_\nu[x]$. By semi-simplicity we may write $\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}') = \oplus_{i=1}^{n'} V'_i$ as $Q_\nu[\pi_\nu]$-module, where $V'_i \cong Q_\nu[x]/P'_i$, see Remark 3.3. Thus $\chi_\nu = \chi'_\nu \cdot u(x)$ for some $u(x) \in Q_\nu[x]$, and hence $\oplus_{i=1}^{n'} V'_i$ appears as a summand of $\omega^\nu_{Q_\nu}(\underline{\mathcal{M}})$.

Assume that μ_π is irreducible. Then $\mu_\pi = \mu'_\pi$ and $F = Q[x]/\mu_\pi$ is a field. Therefore by Proposition 3.8 we have $\chi'_\nu = (\mu'_\pi)^{\operatorname{rk}\underline{\mathcal{M}}'/[F:Q]} \,|\, (\mu_\pi)^{\operatorname{rk}\underline{\mathcal{M}}/[F:Q]} = \chi_\nu$. Furthermore π_ν and π'_ν are semi-simple and (b) follows from (c) as in i).

$\square$

This proposition has the following consequence.

Proposition 5.2. *Keep the notation from the above proposition. Consider the following statements*

(a) $\underline{\mathcal{M}}$ *is quasi-isogenus to* $\underline{\mathcal{M}}'$.

(b) There exists an isomorphism $\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}) \tilde{\to} \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}')$ *in* $\operatorname{Hom}_{Q_\nu[\Gamma_L]}(\omega^\nu_{Q_\nu}(\underline{\mathcal{M}}), \omega^\nu_{Q_\nu}(\underline{\mathcal{M}}'))$.

(c) $\chi_\nu = \chi'_\nu$.

(d) $\mu_\pi = \mu_{\pi'}$ *and* $\operatorname{rk}\underline{\mathcal{M}} = \operatorname{rk}\underline{\mathcal{M}}'$.

(e) There exist an isomorphism of Q-algebras

$$\alpha : Q\operatorname{End}_L(\underline{\mathcal{M}}) \tilde{\to} Q\operatorname{End}_L(\underline{\mathcal{M}}'),$$

with $\alpha(\pi) = \pi'$.

(f) There exist an isomorphism of Q_ν*-algebras*

$$\alpha_\nu : Q\operatorname{End}_L(\omega^\nu(\underline{\mathcal{M}})) \tilde{\to} Q\operatorname{End}_L(\omega^\nu(\underline{\mathcal{M}}')),$$

with $\alpha(\pi_\nu) = \pi'_\nu$.

Then we have the following statements

i) (a) and (b) are equivalent and imply (c), (d) and (e). (e) precisely implies (f)

ii) if π_ν *and* π'_ν *are semisimple then we have*

$$(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (f) \Leftrightarrow (e)$$

and $(c) \Rightarrow (d)$.

iii) if μ_π *and* $\mu_{\pi'}$ *are irreducible in* $Q[x]$*, then all the above statements are equivalent.*

Proof. The statements about (a), (b), (c) and (d) follow from the above Proposition 5.1. Precisely (a) implies (e). Namely, a quasi-isogeny $f : \underline{\mathcal{M}} \to \underline{\mathcal{M}}'$ gives the isomorphism

$$Q\operatorname{End}_k(\underline{\mathcal{M}}) \tilde{\to} Q\operatorname{End}_k(\underline{\mathcal{M}}')$$

by sending $g \mapsto f \circ g \circ f^{-1}$. Furthermore we have $\alpha(\pi) = f \circ \pi \circ f^{-1} = f \circ \tau_{\mathcal{M}} \circ (\sigma^*)\tau_{\mathcal{M}} \cdots (\sigma^*)^{e-1}\tau \circ f^{-1}$, where using $f \circ \tau_{\mathcal{M}} = \sigma^* f \circ \tau_{\mathcal{M}'}$, the later equals $\pi'_{\mathcal{M}}$.
Suppose π and π' are semisimple. Then the assertion (f) implies (c) follows from decomposition $Q\operatorname{End}_k(\underline{\mathcal{M}}) = \oplus_{i=1}^n Mat_{m_\mu \times m_\mu}(K_\mu)$ with $\chi_\nu = \prod_\mu \mu_\mu^{m_\mu}$ and $K_\mu = Q_\nu[x]/\mu$; see remark 3.3.
Suppose μ_π and $\mu_{\pi'}$ are irreducible, then $E = Q[x]/\mu_\pi$ and $E = Q[x]/\mu_{\pi'}$ are fields and therefore π_ν and $\pi_{\nu'}$ are semi-simple. As we have seen above, this implies that $\chi_\nu = \chi'_\nu$. We conclude that $\mu_\pi = \mu_{\pi'}$ by Proposition 3.8(b).

□

-The Grothendieck Ring $K_0(\mathcal{M}ot_C^\nu(\mathbb{F}))$

Recall that the Grothendieck Ring $K_0(Var_k)$ is the quotient of the free abelian group generated by isomorphism classes of k-varieties by the relation $[X \smallsetminus Y] = [X] - [Y]$, where Y is a closed subscheme of X; the fibre product over k induces a ring structure defined by $[X].[X'] = [(X \times_k X')_{\text{red}}]$. By construction, there is a morphism

$$K_0(Var_k) \to K_0(DM_{gm}(k)).$$

Note that $K_0(DM_{gm}(k) \cong K_0(Ch_{rat}(k, \mathbb{Q}))$. Indeed, $DM_{gm}(k)$ carries a weight structure, whose heart is the category of Chow motives. The existence of *weight truncation triangles* gives the isomorphism; see [Bon]. Thus, according to Remark 1.3, we obtain $K_0(DM_{gm}(k)) \to \mathbb{Z}\left(\Gamma_{\mathbb{Q}} \backslash W(p^\infty)\right)$.

Let us now discuss the analogues picture over function fields. Set $W_{\underline{\nu}} = \{\alpha \in Q^{\text{alg}}; \nu(\alpha) = 0 \ \forall\ \nu \notin \underline{\nu}\}$. Consider the free $\mathbb{Z}$-module $\mathbb{Z}[\Gamma_Q \backslash W_{\underline{\nu}} \times \mathbb{N}_{\geqslant 1}/\sim]$ generated by the equivalence classes in $\Gamma_Q \backslash W_{\underline{\nu}} \times \mathbb{N}_{\geqslant 1}/\sim$. Here (α, n) and (β, m) are equivalent if $\alpha^{m.l} = \beta^{n.l}$ for some integer $l \in \mathbb{N}_{\geqslant 1}$. The operation $(\alpha, n) \cdot (\beta, n) = (\alpha\beta, n)$ induces a ring structure on $\mathbb{Z}[\Gamma_Q \backslash W \times \mathbb{N}_{\geqslant 1}/\sim]$.

Proposition 5.3. *There is a bijection*

$$\text{set } \Sigma \text{ of simple objects in } \mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}}_q) \leftrightarrow \text{elements of } \Gamma_Q \backslash W_{\underline{\nu}} \times \mathbb{N}_{\geqslant 1}/\sim.$$

Proof. Let $\underline{\mathcal{M}} := (\mathcal{M}, \tau_{\mathcal{M}})$ be a simple object in $\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}}_q)$. Suppose that it comes by base change from a C-motive in $\mathcal{M}ot_C^{\underline{\nu}}(L)$ for a finite extension $L/\mathbb{F}_q$ of degree n, see Proposition 4.7 . Let $\pi := \pi_{\underline{\mathcal{M}}}$ denote the corresponding Frobenius isogeny and let μ_π denote the corresponding minimal polynomial. Let α_π be a zero of the minimal polynomial μ_π. Then sending $\underline{\mathcal{M}}$ to the pair (α, n) gives an assignment $\Sigma \to \Gamma_Q \backslash W_{\underline{\nu}} \times \mathbb{N}_{\geqslant 1}/\sim$. This is one to one by Proposition 5.1. The fact that it is onto was proved in [Röt].

□

Corollary 5.4. *There is a morphism*

$$K_0(\mathcal{M}ot_C^{\underline{\nu}}(\overline{\mathbb{F}}_q)) \to \mathbb{Z}[\Gamma_Q \backslash W_{\underline{\nu}} \times \mathbb{N}_{\geqslant 1}/\sim]$$

of rings.

6 The Zeta-Function

Recall that assigning a zeta function to a variety over a finite field L, factors through the Grothendieck ring of varieties

$$K_0(Var_L) \to 1 + t\mathbb{Z}[\![t]\!].$$

It can be seen that this morphism further factors through $K_0(DM_-^{eff}(L))$ and gives

$$Z(-,t) : K_0(DM_-^{eff}(L)) \to 1 + \mathbb{Z}[\![t]\!].$$

The zeta function satisfies sort of properties manifested in Weil conjectures. A crucial observation to prove these conjectures was to establish a cohomology theory for schemes and expressing the zeta function of X in terms of the action of Frobenius on the corresponding cohomology groups.

Let us briefly explain the analogous picture over function fields. Let us fix a place ν away from characteristic places ν_i. In contrast with the above assignment, we define the zeta function associated to a C-motive $\underline{\mathcal{M}}$ in $\mathcal{M}ot_C^{\underline{\nu}}(L)$ by the following formula

$$Z(\underline{\mathcal{M}}, t) := \prod_i \det(1 - t\pi_\nu |\, \mathrm{H}^i_{\text{ét}}(\underline{\mathcal{M}}, Q_\nu))^{(-1)^{i+1}}.$$

According to Proposition 2.11, this assignment defines a morphism $K_0(\mathcal{M}ot_C^{\underline{\nu}}(L)) \to 1 + Q_\nu[t]$, which can be shown that in fact factors through $Q[\![t]\!]$ and gives

$$Z(-,t) : K_0(\mathcal{M}ot_C^{\underline{\nu}}(L)) \to 1 + tQ[\![t]\!],$$

i.e. the definition is independent of the choice of the place ν. We further define $Z(\underline{\mathcal{M}}, t) := \prod_p Z(i_p^* \underline{\mathcal{M}}, t)$ for a motive $\underline{\mathcal{M}} \in \mathcal{M}ot_C^{\underline{\nu}}(S)$ over a general base scheme S, which is of finite type over $\mathbb{F}_q$.Here the product is over all closed points $i_p : \operatorname{Spec} \kappa(p) \to C$. Let us explain the reason

behind the fact that these definitions are independent of the chosen place ν.
First assume that $\underline{\mathcal{M}}$ is simple. Then $E := Q\operatorname{End}(\underline{\mathcal{M}})$ is a central simple algebra over the field $F := F(\pi)$. For a semi-simple element $f \in E$ we let $\mathfrak{J}$ denote the commutative subalgebra of rank $r := \operatorname{rk}\underline{\mathcal{M}}$ containing f. Consider the norm function $N : E \to Q$ which sends g to $N_{K/Q}(\det(\alpha(g)))$, here K is a splitting field for E and $\alpha : E \otimes_F K \tilde{\to} Mat_{n\times n}(K)$ is an isomorphism. One can see the norm $N(f)$, as the determinant of the Q-endomorphism of $\mathfrak{J} \otimes_F K$ given by multiplication by f. Note that one can identify $\mathfrak{J}_\nu := \mathfrak{J} \otimes_Q Q_\nu$ with $\omega^\nu(\underline{\mathcal{M}})$. Therefore $N(f) = \det(\omega^\nu(f))$.

The above defined norm induces $N(-) : Q\operatorname{End}(\underline{\mathcal{M}}) \to Q$ for semi-simple $\underline{\mathcal{M}}$, for which the equality $N(f) = \det(\omega^\nu(f))$ holds. Now, to see that this equality holds for general element $f \in Q\operatorname{End}(\underline{\mathcal{M}})$, we write $\omega^\nu(f)$ in Jordan normal form $S + N$ over Q_ν^{alg}, and take a power q^N such that $(N)^{K^q} = 0$. So f^{q^N} is semi-simple and from the above arguments we see that $N(f^{q^N}) = \det(\omega^\nu(f))^{q^N}$ and thus $N(f) = \det(\omega^\nu(f))$. Now for every $a \in A$ we have

$$\chi_\nu(a) = \det(a \cdot Id - \pi_\nu) = N(a - \pi),$$

and thus the characteristic polynomial χ_ν is independent of the chosen place ν. This implies that the zeta function $Z(\underline{\mathcal{M}}, t)$ lies in $Q(t)$.

Remark 6.1 (zeta function of a G-shtuka)**.** Let $\underline{\mathcal{G}}$ be in $\nabla\mathscr{H}^1(C, \mathfrak{G})(L)$. Then to any representation ρ we can assign the zeta function of $\rho_*\underline{\mathcal{G}} \in \mathcal{M}ot_C^{\underline{\nu}}(L)$, this gives

$$[Z(\underline{\mathcal{G}}, t)] : R(G) \to Q(t),$$

which assigns a rational function to a given class of representation in the Grothendieck ring of representation $R(G)$.

Remark 6.2. We can similarly define the zeta function of a local $\mathbb{P}$-shtuka. This can be represented by a morphism $R(\mathbb{P}) \to Q_\nu(t)$, where $R(\mathbb{P})$ is the grothendieck ring of representation of $\operatorname{Rep}_{Q_\nu}\mathbb{P}$. According to Satake theory the Grothendieck ring $R(\mathbb{P})$ (with coefficients in Q_ν) is isomorphic to the unramified Hecke algebra H_ν . This in particular gives a set of generators for $R(\mathbb{P})$ and the morphism $R(\mathbb{P}) \to Q_\nu(t)$ is given by the image of these classes.

Remark 6.3. Assume that $\underline{\nu} := (0, \infty)$ for two specified places 0 and $\infty \in C$. Let ζ denote the image of the uniformizar of $\mathcal{O}_{C,0}$ in L. One say's that $\underline{\mathcal{M}}^\flat \in \mathcal{M}ot_C^{\underline{\nu}}(\mathbb{F}_q)$ is analog to $\mathcal{M} \in DM_-^{eff}(\mathbb{F})$ if $Z(\underline{\mathcal{M}}, t)$ is the reduction of a lift of the zeta function $Z(\mathcal{M}, t)$ to $\mathbb{Z}[\![\mathsf{y}, T]\!]$, according to the the following diagram

$$\begin{array}{ccccc}
\mathcal{M}ot_C^{\nu}(\mathbb{F}) & \longrightarrow & K_0(\mathcal{M}ot_C^{\nu}(\mathbb{F})) & \xrightarrow{Z(-,t)} & A[\![t]\!] \\
 & & & & \uparrow {\scriptstyle \mathsf{y}=z-\zeta} \\
 & & & & \mathbb{Z}[\![\mathsf{y},t]\!] \\
 & & & \nearrow & \downarrow {\scriptstyle \mathsf{y}=q} \\
DM_-^{eff}(\mathbb{F}) & \longrightarrow & K_0(DM_-^{eff}(\mathbb{F})) & \xrightarrow{Z(-,t)} & \mathbb{Z}[\![t]\!]
\end{array}$$

For example Carlitz module and $M(\mathbb{G}_m)$ are analog. Supersingular Drinfeld module of rank 2 and "some" elliptic curve E.

7 Semi-simplicity Of The Category Of C-Motives over finite fields

Consider the category $\mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q)$. This is a tannakian category with a fiber functor

$$\omega : \mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q) \to Q.\overline{\mathbb{F}}_q\text{-vector spaces.}$$

This category admits étale and crystalline realizations. Note that according to Proposition 4.7 we may regard the tannakian category $V_\nu(Q_\nu)$ of germs of Q_ν-adic representation of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ as the étale realization category

$$\omega_{Q_\nu}^{\nu}(-) : \mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q) \to V_\nu(Q_\nu)$$

Recall that this category consists of equivalence classes of continuous semisimple representations of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ on the same finite dimensional Q_ν-vector spaces V. Where we say that ρ_1 and ρ_2 are related if they agree on an open subgroup of $U_1 \cap U_2$.

Theorem 7.1. *The category $\mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q)$ with the fiber functor ω, is a semi-simple tannakian category. In particular the kernel $P := \mathcal{M}ot_C^{\nu}(\omega)^{\Delta}$ of the corresponding motivic groupoid $\mathfrak{P} := \mathcal{M}ot_C^{\nu}(\omega)$ is a pro-reductive group.*

Proof. According to Proposition 4.7, we may suppose that a given motive $\underline{\mathcal{M}} \in \mathcal{M}ot_C^{\nu}(\overline{\mathbb{F}}_q)$ comes from a motive over a finite extension $L/\mathbb{F}_q$, which we again denote by $\underline{\mathcal{M}}$.
It is enough to show that after a finite extension $L \subset L' \subset \mathbb{F}$, the image $\underline{\mathcal{M}}'$ of $\underline{\mathcal{M}}$ under the obvious functor $\mathcal{M}ot_C^{\nu}(L) \to \mathcal{M}ot_C^{\nu}(L')$ is semi-simple, or equivalently the endomorphism algebra $E := Q\,\mathrm{End}(\underline{\mathcal{M}}')$ is a semi-simple algebra over Q; see Theorem 3.4.
Let $\underline{\hat{M}}_\nu = (\hat{M}_\nu, \tau_\nu)$ denote the $\hat{\sigma}$-crystal associated to $\underline{\mathcal{M}}$, for a place ν distinct from the characteristic places ν_i. It is enough to show that the endomorphism algebra $E_\nu := E \otimes \widehat{Q}_\nu = Q\,\mathrm{End}(\omega^{\nu}(\underline{\mathcal{M}}'))$ is semi-simple. Note that the last equality follows from Theorem 2.14.

We can equivalently show that $\widehat{Q}_\nu(\pi'_\nu)$ is semi-simple, where π'_ν is the Frobenius endomorphism $\pi'_\nu := \omega^\nu(\pi_{\underline{\mathcal{M}}'})$. To show this take a representative matrix B_{π_ν} for $\pi_\nu \otimes 1 \in End(\underline{\hat{M}}_\nu \otimes_{A_\nu} \widehat{Q}_\nu^{alg})$ and write B_{π_ν} in the Jordan normal form, i.e. $B = S + N$ where S is semi-simple and N is nilpotent. We take L'/L to be a field extension such that $[L' : L]$ is a power of the characteristic of $\mathbb{F}_q$ and $[L' : L] \geq \operatorname{rank} M$. Clearly $B_{\pi_\nu}^{[L':L]} = (S+N)^{[L':L]} = S^{[L':L]} + N^{[L':L]} = S^{[L':L]}$ is semi-simple. This represents the Frobenius endomorphism $\pi' \otimes 1$ in $E_\nu \otimes \widehat{Q}_\nu^{alg}$. Since $\widehat{Q}_\nu^{alg}/Q$ is perfect we may argue by [Bou58, Proposition 9.2/4] that π'_ν is semi-simple, and as we mentioned above, this suffices.
The second Part of the proposition follows from [DM82, Proposition 2.23].

□

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