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The Hodge Conjecture For Function Fields

Die Hodge-Vermutung für Funktionenkörper

Diplomarbeit

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Münster, April 2010

To my father, Manfred Juschka,
in loving memory.

ABSTRACT

Following Papanikolas in [Pap08], when $Q = \mathbb{F}_q(t)$ we define a Tannakian category of pure dual t-motives. We assign such a pure dual t-motive to a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$. Then $\underline{P}$ generates a strictly full Tannakian subcategory over Q and we call the linear algebraic group obtained by Tannakian duality the *Galois group of* $\underline{P}$.

Secondly, as done by Pink in [Pin97a], again when $Q = \mathbb{F}_q(t)$, we introduce the Tannakian category of pure Q-Hodge-Pink structures and consider the strictly full Tannakian subcategory generated by a pure Q-Hodge-Pink structure $\underline{H}$. We then call the linear algebraic group defined by Tannakian duality the *Hodge-Pink group of* $\underline{H}$.

Further, we may also assign a pure Q-Hodge-Pink structure to a pure rigid analytically trivial dual Anderson A-motive through pure uniformizable Anderson A-modules. Using this functor and the formal inversion of the dual Carlitz t-motive, we associate a pure Q-Hodge-Pink structure $\underline{H}$ with a pure dual t-motive $\underline{P}$ over $\mathbb{C}_\infty$. This induces a map from the Hodge-Pink group of $\underline{H}$ to the Galois group of $\underline{P}$. From the *Hodge conjecture for function fields*, one expects that the Galois group and Hodge-Pink group are isomorphic, which we prove with the help of Tannakian theory [DMOS82, Prop. 2.21]; the classification of σ-bundles and corresponding σ-modules that were respectively introduced in [HP04] and [Har10]; and the rigid analytic GAGA principle.

Combining the isomorphism with Papanikolas's transcendence result [Pap08, Thm. 5.2.2], we obtain *Grothendieck's period conjecture for function fields.* As an application, we consider a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ of rank r over $\overline{Q} \subset \mathbb{C}_\infty$ with sufficiently many complex multiplication through E and determine its associated Hodge-Pink group if E/Q is either separable or purely inseparable. The dimension of the computed Hodge-Pink group is then r and equals the transcendence degree of the periods and quasi-periods of the pure uniformizable Anderson A-module $\underline{E}$ corresponding to $\underline{\mathrm{M}}$. Finally, with [Pin97a, Thm. 10.3] we determine the transcendence degree of the periods and quasi-periods of a pure uniformizable Drinfeld $\mathbb{F}_q[t]$-module $\underline{E}$ of rank 2 over $\overline{Q} \subset \overline{Q_\infty}$. We provide the precise analog for the conjectured transcendence degree of the periods and quasi-periods of an elliptic curve over $\overline{\mathbb{Q}} \subset \mathbb{C}$.

CONTENTS

0. INTRODUCTION

Function field arithmetic enjoys several analogies with classical algebraic number theory. We shall first explain the Hodge conjecture in the classical situation to motivate the function field analog which we prove in this thesis. Together with Papanikolas's main result in [Pap08], we then have the precise analog of Grothendieck's period conjecture on the transcendence degree of periods and quasi-periods of abelian varieties. We describe both conjectures framed in Grothendieck's theory of motives and their analog for function fields. Then we give an outline of the following chapters illustrating both the proof of the Hodge conjecture and transcendence results.

0.1 Motivation: Conjectures and Analogies in Algebraic Number Theory

Number fields are finite extensions of the rational numbers $\mathbb{Q}$. *Fermat's conjecture*, also known as *Fermat's Last Theorem*, is one of the most well-known problems in the theory of number fields.

Theorem 0.1.1 ([Wil95, Thm. 0.5]). *The Fermat equation $a^n + b^n = 1$ has no non-trivial solutions for $n \geq 3$; that is, there is no solution $(a, b) \in \mathbb{Q}^2$ with $ab \neq 0$.*

Around 1637, Fermat remarked in the margin of a book that he had found a marvelous proof and many number theorists tried in vain to show it. Finally in 1994, A. Wiles, along with R. Taylor, G. Frey, J.-P. Serre and K. Ribet, was able to give a proof through the use of elliptic modular functions of elliptic curves and won the Wolfskehl prize. The long story of the proof of Fermat's Last Theorem was taken as material for a catching novel [Sin98], which shows the potent driving force of conjectures as well as the powerful interplay of different modern theories in mathematics. Wiles used recently achieved results in the theory of arithmetic cohomology that is again based on the theory of algebraic curves over finite fields.

We consider a *function field* Q of a smooth projective curve C over a field $k/\mathbb{F}_q$, which is therefore a finite extension of the rational function field $\mathbb{F}_q(t)$. The arithmetic of such function fields shows fascinating parallels with classical algebraic number theory, despite fundamental differences such as finite characteristic. Basic number theory applies to both theories, whence number fields and function fields are often studied together as *global fields.* Global fields are the only fields with the notion of absolute values that satisfy a product formula [AW45]. Completions of global fields with respect to such absolute values are called *local fields.* Examples are the completions $\mathbb{Q}_p$ and Q_P of $\mathbb{Q}$ and Q with respect to $|\cdot|_p$ and $|\cdot|_P$ where p is a prime number and $P \in C$ a closed point, respectively. Further, let $A \subset Q$ denote the ring of regular functions outside a fixed closed point $\infty \in C$, which is by definition a finite extension of the rational polynomial ring $\mathbb{F}_q[t]$. The ring of integers, $\mathbb{Z}$, and A are the bottom rings of the following correspondence, which compares the number field and function field theories:

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] & \mathbb{Q}_p \arrow[rr] & & \mathbb{C}_p \\ \mathbb{Z} \arrow[r] \arrow[u] & \mathbb{Q} \arrow[r] \arrow[u] & \mathbb{R} \arrow[r] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} A_P \arrow[r] & Q_P \arrow[rr] & & \mathbb{C}_P \\ A \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & Q_\infty \arrow[r] & \mathbb{C}_\infty, \end{tikzcd}$$

where $\mathbb{C}_P$ denotes the completion of a fixed algebraic closure $\overline{Q_P}$ of Q_P. The basic example with the usual choice of ∞ is

$$C = \mathbb{P}^1_{\mathbb{F}_q}, \quad A = \mathbb{F}_q[t], \quad Q = \operatorname{Quot}(A) = \mathbb{F}_q(t) \quad \text{and} \quad Q_\infty = \mathbb{F}_q((1/t)).$$

What now is understood as *function field arithmetic* started in 1935, when L. Carlitz attached an entire analytic exponential function $\exp_C : \mathbb{C}_\infty \to \mathbb{C}_\infty$ to the rational polynomial ring $A = \mathbb{F}_q[t]$ that satisfies

$$\exp_C(az) = C_a(\exp(z)) \quad \text{for } a \in A \text{ and additive polynomials } C_a \in k[\tau] \text{ [Car35].}$$

Then $\exp_C$ provides an additive analog of the classical exponential function $\exp : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^z$, with the multiplicative functional equation $\exp(jz) = (e(z))^j$, $j \in \mathbb{Z}$. The *Carlitz module* $\mathrm{C} : A \to k[\tau]$ over k is given by $\mathrm{C}(a) := \mathrm{C}_a$ and one defines its *period* $\tilde{\pi} \in \mathbb{C}_\infty$ by requiring $\exp_C(\tilde{\pi}A) = 0$ (uniquely up to signs in $\mathbb{F}_q^\times$). The latter is an analog of the period $2\pi i \in \mathbb{C}$ of the exponential function satisfying $\exp(2\pi i\mathbb{Z}) = 1$ (also unique up to multiplication by ± 1). Being interested in further analogies, Carlitz's student L. I. Wade showed that $\tilde{\pi}$ and $\tilde{e} := \exp_{\mathrm{C}}(1)$ are transcendental over Q [Wad41].

The interest in Carlitz's approach increased in 1974, with V. G. Drinfeld's invention, *Drinfeld A-modules*. Drinfeld generalized the construction of the Carlitz module of rank 1 to higher ranks. Even more, the exponential function of a Drinfeld A-module is assigned to an arbitrary function field with an arbitrary choice of ∞. With the help of rigid analytic spaces he proved that any Drinfeld A-module φ is *uniformizable*, that is, $\mathbb{C}_\infty/\Lambda_\varphi \cong \mathbb{C}_\infty$ where the *period lattice* Λ_φ is the kernel of $\exp_\varphi$. Drinfeld originally called Drinfeld A-modules *elliptic modules* because the properties of Drinfeld A-modules resemble those of elliptic curves. These analogies are especially strong for a Drinfeld A-module φ of rank 2. Similarly as for elliptic curves, basis vectors λ_1 and λ_2 of the period lattice Λ_φ give rise to *quasi-periods* η_1 and η_2, respectively. These satisfy an analog of the *Legendre relation* for elliptic curves

$$\lambda_1\eta_2 - \lambda_2\eta_1 = \frac{\tilde{\pi}}{\xi} \quad \text{for some } \xi \in k \quad \text{[Tha04, Thm. 6.4.6].}$$

Moreover, mirroring another classical result, J. Yu proved that all periods [Yu86, Thm. 5.1] and quasi-periods [Yu90, Thm. 3.1] of a Drinfeld A-module over $\overline{Q}$ are transcendental over Q, where $\overline{Q}$ denotes an algebraic closure of Q inside $\overline{Q_\infty}$.

Finally in 1986, G. W. Anderson extended the theory of one-dimensional Drinfeld A-modules to higher-dimensional *t-modules* when $A = \mathbb{F}_q[t]$. Anderson also discussed purity, isogenies and uniformization of Anderson A-modules, thereby building up a theory similar to the theory of abelian varieties over number fields. We modify their definition slightly so that Q may be an arbitrary function field with ring of integers A.

Definition 0.1.2. Let $(k, \gamma : A \to k)$ be an A-field[1] and d, r positive integers. An *(abelian) Anderson A-module of rank r, dimension d and characteristic γ over k* is a pair $\underline{E} = (E, \varphi)$,

[1] See Definition 1.1.1.

where $E \cong \mathbb{G}_{a,k}^d$ is the d-dimensional additive group scheme over k and

$$\varphi : A \to \mathrm{End}_{k,\mathbb{F}_q}(E) \cong \mathrm{Mat}_{d\times d}(k[\tau]),\ a \mapsto \varphi_a := \varphi(a)$$

is a ring homomorphism such that

$$(\mathrm{T}_0(\varphi_a) - \gamma(a))^d = 0 \quad \text{on the tangent space } \mathrm{T}_0 E \cong \mathrm{Mat}_{d\times 1}(k) \text{ at the identity}$$

and the group of $\mathbb{F}_q$-linear homomorphisms $\mathcal{M}^*(E) := \mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, E)$ is a locally free A_k-module of rank r under:

$$\begin{aligned} A \ni a:&\quad \mathrm{m} \mapsto \varphi_a \circ \mathrm{m} \\ k \ni b:&\quad \mathrm{m} \mapsto \mathrm{m} \circ b. \end{aligned}$$

Yu established a whole transcendence theory for Drinfeld A-modules and t-modules, and the topic gained interest of several fellow researchers.[2] Papanikolas recently achieved in 2008 a new result on the transcendence degree of periods and quasi-periods. We shall explain the classical conjecture by Grothendieck on the periods and quasi-periods of an abelian variety that motivated Papanikolas's work.

The first Betti homology group $H_1(X(\mathbb{C}), \mathbb{Q})$ of an abelian variety X of dimension d over a number field $K \subset \mathbb{C}$ carries a *Hodge structure over* $\mathbb{Q}$. In [DMOS82], P. Deligne shows that the category of Hodge structures over $\mathbb{Q}$ is a *Tannakian category over* $\mathbb{Q}$, that is, an abelian category with tensor products and duals together with a $\mathbb{Q}$-linear functor ω. By *Tannakian duality* one can define an algebraic group G such that the category $\mathscr{R}ep_{\mathbb{Q}}(G)$ of finite-dimensional representations of G over $\mathbb{Q}$ is equivalent to the Tannakian category over $\mathbb{Q}$. The *Hodge group*, also *Mumford-Tate group*, G_X of the abelian variety X is defined to be the algebraic groups associated with the Tannakian subcategory generated by the rational Hodge structure $H_1(X, \mathbb{Q})$ and the Tate twist $\mathbb{Q}(1)$. Moreover, the natural isomorphism $H^1_{DR}(X) \otimes_K \mathbb{C} \xrightarrow{\sim} H^1_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$ is given by period integrals. Its defining matrix P is called the *period matrix of* X, that is

$$P = \left(\int_{\lambda_i} \delta_j\right)_{1\le i,j\le 2d} = (\lambda_{mn}|\eta_{mn})_{1\le m\le 2d,\ 1\le n\le d} \in \mathrm{Mat}_{2d\times 2d}(\mathbb{C}),$$

where λ_{mn} are the periods and η_{mn} the quasi-periods of X. Deligne then shows the following result on their transcendence degree through its Hodge group:

Corollary 0.1.3 ([DMOS82, Prop. I.6.4]). *Let X be an abelian variety of dimension d over $\overline{\mathbb{Q}}$, P its period matrix and G_X the Hodge group of X. Then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}\, \overline{\mathbb{Q}}\,(P_{ij}|1 \le i,j \le 2d) \le \dim G_X.$$

Grothendieck's period conjecture then says that the above inequality is an equality and Papanikolas wants to obtain an analog in the function field setting.

Corresponding to the classical number field situation in 1997, R. Pink developed a Tannakian category of *Hodge-Pink structures over function fields* [Pin97a] to show the analog of the Mumford-Tate conjecture on the Hodge group of an abelian variety [Pin97c, Pin97b]. The Betti cohomology realization of a pure uniformizable Anderson A-module $\underline{E}$ is given in terms of its period lattice $\Lambda_{\underline{E}}$

$$H_{\mathrm{B}}(\underline{E}, A) := \Lambda_{\underline{E}}, \quad H_{\mathrm{B}}(\underline{E}, B) := \Lambda_{\underline{E}} \otimes_A B \quad \text{and} \quad H^1_{\mathrm{B}}(\underline{E}, B) := \mathrm{Hom}_A(\Lambda_{\underline{E}}, B)$$

[2] For various aspects of transcendence in positive characteristic, see [Tha04, §10].

for any A-algebra B. Similarly as in the classical case, $H_{\mathrm{B}}(\underline{E}, Q)$ determines a pure *Q-Hodge-Pink structure* $\underline{H}$. We define its *Hodge-Pink group* to be the algebraic group corresponding by Tannakian duality to the strictly full Tannakian subcategory over Q generated by $\underline{H}$. We can then translate Grothendieck's period conjecture to the function field case as follows.

Conjecture 0.1.4 (Grothendieck's period conjecture for function fields)**.** Let $\underline{E}$ be a pure uniformizable Anderson A-module of dimension d over $\overline{Q} \subset \overline{Q_\infty}$, P its period matrix and $G_{\underline{E}}$ its associated Hodge-Pink group. Then

$$\operatorname{tr.deg}_{\overline{Q}} \overline{Q}\,(P_{ij} | 1 \leq i, j \leq d) = \dim G_{\underline{E}}.$$

We will show this conjecture through another conjecture on the Hodge group of an abelian variety - the Hodge conjecture, which has its origins in Hodge's book [Hod41]. Both conjectures belong to the theory of motives, which was invented by Grothendieck. A pure motive is assigned to a smooth projective variety in order to obtain a universal cohomology theory. Deligne goes on in his article [DMOS82] to construct a Tannakian category $\underline{\mathcal{M}}$ of pure motives over $\mathbb{Q}$ in several steps. Moreover, Deligne describes a functor h^1 from the category of abelian varieties up to isogeny to a subcategory $\underline{\mathcal{M}}^{+1}$ of $\underline{\mathcal{M}}$ that is an anti-equivalence of categories [DMOS82, Prop. 6.21]. The motive $h^1(X)$ assigned to an abelian variety X over $K \subseteq \mathbb{C}$ generates a strictly full Tannakian subcategory over $\mathbb{Q}$. The corresponding algebraic group Γ_X defined by Tannakian duality is called the *motivic Galois group of X*.

Conjecture 0.1.5 (Hodge conjecture)**.** The Hodge group G_X and motivic Galois group Γ_X assigned to an abelian variety X over $K \subseteq \mathbb{C}$ are isomorphic.

Correspondingly over the rational function field, Anderson introduced *t-motives* and the notion of isogenies, purity and rigid analytically triviality of t-motives in [And86]. Furthermore, Anderson defined a functor from the category of pure uniformizable t-modules up to isogeny to the category of pure rigid analytically trivial t-motives up to isogeny that is in fact an anti-equivalence of categories. In joint work of Anderson, W. D. Brownawell and Papanikolas in 2004, the definition of t-motives was changed slightly due to technical advantages [ABP04]. The resulting objects are called pure dual t-motives, whose definition we generalize to arbitrary function fields as follows:

Definition 0.1.6. Let $(k, \gamma : A \to k)$ be an A-field, $r, d \in \mathbb{N}$ and ς^* the endomorphism of $A_k := A \otimes_{\mathbb{F}_q} k$, which maps an $a \otimes \beta$ to $a \otimes \beta^{q^{-1}}$ for $a \in A$ and $\beta \in k$. A *dual Anderson A-motive of rank r, dimension d and characteristic γ over k* is a pair $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ where M is a locally free A_k-module of rank r and $\sigma_{\mathrm{M}} : \varsigma^*\mathrm{M} := \mathrm{M} \otimes_{A_k, \varsigma^*} A_k \to \mathrm{M}$ is an injective A_k-homomorphism such that M is finitely generated over $k[\sigma]$ where $\sigma : \mathrm{M} \to \mathrm{M}$ is the ς^*-linear map induced by σ_{M},[3] $\dim_k \operatorname{coker} \sigma_{\mathrm{M}} = d$ and $(a \otimes 1 - 1 \otimes \gamma(a))^d = 0$ on $\operatorname{coker} \sigma_{\mathrm{M}}$.

As done by Anderson, we define isogenies, purity and rigid analytic triviality of dual Anderson A-motives. The latter is equivalent to the existence of a matrix Ψ called *rigid analytic trivialization*. Moreover, we prove that the category of pure rigid analytically trivial dual Anderson A-motives of positive rank and dimension over k up to isogeny is equivalent to the category of pure uniformizable Anderson A-modules over k up to isogeny.

Papanikolas used the fact that, over the rational function field, the matrix $\Psi(\theta)$ corresponding to such a rigid analytically trivial dual Anderson A-motive with $\theta := \gamma(t)$ is related to the period matrix of the corresponding Anderson A-module. In order to have a Q-linear

[3] See Lemma 1.1.3.

theory, he introduced *pre-t-motives*. Rigid analytical triviality of pre-t-motives is similarly defined as for dual Anderson A-motives and equivalent to the existence of a rigid analytic trivialization Ψ. Rigid analytically trivial pre-t-motives are called t-motives whose category Papanikolas proved to be a Tannakian category over Q [Pap08, Thm. 3.3.15]. We extend the definition of pre-t-motives to arbitrary function fields as follows:

Definition 0.1.7. Let $r \in \mathbb{N}$ be a non-negative integer and ς_Q^* the endomorphism of $Q_k := \mathrm{Quot}(A_k)$ induced by $\varsigma^* : A_k \to A_k$. A *(dual) Papanikolas Q-motive of rank r and characteristic γ over k* is a pair $\underline{P} = (P, \sigma_P)$ where P is a Q_k-vector space of dimension r and $\sigma_P : \varsigma_Q^* P \to P$ a Q_k-isomorphism.

Roughly speaking, when $Q = \mathbb{F}_q(t)$ we assign by "tensoring with Q" a rigid analytically trivial Papanikolas Q-motive called a *pure dual t-motive* $\underline{P}$ to a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$. The *Galois group* $\Gamma_{\underline{P}}$ is defined to be the algebraic group given by Tannakian duality to the strictly full Tannakian subcategory generated by $\underline{P}$.

Theorem 0.1.8 ([Pap08, Thm. 5.2.2])**.** *Let $\underline{P}$ be a pure dual t-motive of rank r over $\overline{Q} \subset \overline{Q_\infty}$, Ψ a rigid analytic trivialization and $\Gamma_{\underline{P}}$ its associated Galois group. Then*

$$\mathrm{tr.\,deg}_{\overline{Q}}\,\overline{Q}\,(\Psi(\theta)_{ij} | 1 \leq i, j \leq r) = \dim \Gamma_{\underline{P}}.$$

Following Taelman in [Tae09], we show that a pure dual t-motive $\underline{P}$ over $\mathbb{C}_\infty$ consists of a pure rigid analytically trivial dual Anderson A-motive and a tensor power of the function field analog of the Tate twist. This allows us to assign a Hodge-Pink structure together with its Hodge-Pink group to $\underline{P}$. We may then prove the analog of the Hodge conjecture (Theorem 4.2.19).

Theorem 0.1.9 (Hodge conjecture for function fields)**.** *Let $\underline{P}$ be a pure dual t-motive over $\mathbb{C}_\infty$. Then its associated Galois group and Hodge-Pink group are isomorphic over Q.*

By Tannakian duality, the corresponding Tannakian categories must be equivalent. Remarkable about this result is that it relates Hodge-Pink theory and Papanikolas's theory, and therefore objects which are constructed in an entirely different way.

In combination with Papanikolas's transcendence result, we obtain Grothendieck's period conjecture for function fields as desired. Depending on whether an elliptic curve has sufficiently many complex multiplication, the classical conjecture can be stated as follows for elliptic curves.

Conjecture 0.1.10 (Cf. [DMOS82, Rem. 1.8])**.** Let E be an elliptic curve over $\overline{\mathbb{Q}}$, $P = (\int_{\lambda_i} \delta_j)$ its period matrix and G_E the Hodge group of E. Then

$$\mathrm{tr.\,deg}_{\overline{\mathbb{Q}}}\,\overline{\mathbb{Q}}(\int_{\lambda_i} \delta_j) = \dim G_E = \begin{cases} 2 & \text{if } E \text{ is of CM-type,} \\ 4 & \text{otherwise.} \end{cases}$$

Similarly as done in complex multiplication theory of abelian varieties, we then introduce Anderson A-modules of CM-type. As an application, we determine the Hodge-Pink group of a pure uniformizable Anderson A-module of rank r over $k \subseteq \mathbb{C}_\infty$ of CM-type under some conditions. Its dimension is r and together with Pinks main theorem in [Pin97a], we may determine the transcendence of the periods and quasi-periods of a Drinfeld $\mathbb{F}_q[t]$-module of rank 2 over $\overline{Q} \subset \overline{Q_\infty}$. We obtain the precise analog of the previous conjecture for elliptic curves (Theorem 5.2.16).

Summing up, the interest of most researchers in the arithmetic of function fields is twofold: On the one side, the results are beautiful by their own. For this, the references [And04, DMOS82, Gos96] respectively introduce the theory of motives, the Tannakian categories discussed here and function field arithmetic. On the other hand, the analogies between function field arithmetic and classical algebraic number theory are intriguing and results in one theory may reveal and inspire connections in the other one. For this purpose, D. S. Thakur's book [Tha04] and Goss's article [Gos94] may serve as an overview. The latter also discusses L. Denis's proof of Fermat's Last Theorem for function fields - in the sense of A. Weil [Wei79, p. 408]:

> Nothing is more fruitful - all mathematicians know it - than obscure analogies, those disturbing reflections of one theory on another; those furtive caresses, those inexplicable discords; nothing also gives more pleasure to the researcher.

0.2 Outline

In Chapter 1 we first fix notations and recall basics that are needed in definitions that are spread over the thesis. In particular, affine group schemes are introduced through the notion of representable functors. These are also needed in the second section, where neutral Tannakian categories are defined and the principle of Tannakian duality is explained. Of importance is Proposition 1.2.15 that gives equivalent conditions to show that affine group schemes obtained by Tannakian duality are isomorphic and hence the corresponding Tannakian categories are equivalent. The chapter ends with a short introduction to Tate's theory of rigid analytic spaces and the rigid analytic GAGA principle.

The goal of the second chapter is to associate a pure dual t-motive with a pure rigid analytically trivial dual Anderson A-motive that generates a Tannakian category over Q when $Q = \mathbb{F}_q(t)$. We start with the general definition of dual Anderson A-motives and isogenies between dual Anderson A-motives. We show that being isogenious is an equivalence relation and define Papanikolas Q-motives. Next we introduce algebraic σ-sheaves, purity and tensor products of dual Anderson A-motives and Papanikolas Q-motives. Afterwards we discuss rigid analytic σ-sheaves and rigid analytically triviality of dual Anderson A-motives and Papanikolas Q-motives. We denote the category of pure rigid analytically trivial dual Anderson A-motives up to isogeny and the category of pure rigid analytically trivial Papanikolas Q-motives by $\mathscr{PRDA}^I$ and $\mathscr{PR}$ respectively. We give a well-defined fully faithful functor $\mathcal{P} : \mathscr{PRDA}^I \to \mathscr{PR}$, and show that $\mathscr{PR}$ is a neutral Tannakian category over Q. As done by Papanikolas when $Q = \mathbb{F}_q(t)$, we let the category $\mathscr{PT}$ of pure dual t-motives be the Tannakian category generated by the essential image of $\mathcal{P} : \mathscr{PRDA}^I \to \mathscr{PR}$. Following Taelman, we then define a Tannakian category $\mathscr{PT}'$ of a pure dual t-motives that is equivalent to $\mathscr{PT}$ and a fully faithful functor $\mathcal{P}' : \mathscr{PRDA}^I \to \mathscr{PT}'$. The linear algebraic group associated with the Tannakian subcategory generated by a pure dual t-motive $\underline{P}$ by Tannakian duality is called the *Galois group of* $\underline{P}$. We finally mention systems of σ-linear equations invented by Papanikolas, which lead to his main transcendence result on the periods and quasi-periods of a pure dual t-motive.

In Chapter 3, we first recall the definitions of filtrations and pure resp. mixed Q-Hodge-Pink structures. We then define the category $\mathscr{Hodge}_Q$ of pure Q-Hodge-Pink structures, which is a Tannakian category over Q. We may consider the Tannakian subcategory generated by a pure Q-Hodge-Pink structure $\underline{H}$ and call the associated linear algebraic group given by Tannakian duality *Hodge-Pink group of* $\underline{H}$. Some later needed properties of Hodge-Pink groups are mentioned and Hodge-Pink additivity of a Hodge-Pink structure is defined. The latter allows us to define Hodge-Pink cocharacters, which will be useful in the main proof of Chapter 5.

Most work is done in the fourth chapter, where we show the Hodge conjecture for function fields. We first construct a map μ from the Galois group of a pure dual t-motive over $\mathbb{C}_\infty$ to the Hodge-Pink group of a pure Q-Hodge-Pink structure. In order to do this, we introduce the category $\mathscr{PUM}^I$ of pure uniformizable Anderson A-modules up to isogeny, which we show to be equivalent to the category $\mathscr{PRDA}_+^I$ of pure rigid analytically trivial dual Anderson A-motives of positive rank and dimension up to isogeny. Then a pure uniformizable Anderson A-module over $\mathbb{C}_\infty$ gives rise to a pure Q-Hodge-Pink structure that we associate with the corresponding pure rigid analytically trivial Anderson A-motive over $\mathbb{C}_\infty$. We give a functor $\mathcal{T} : \mathscr{PT}' \to \mathscr{Hodge}_Q$ that induces a group scheme homomorphism μ from the Hodge-Pink group of $\mathcal{T}(\underline{P})$ to the Galois group of a pure dual t-motive $\underline{P}$ over $\mathbb{C}_\infty$.

In order to prove the Hodge conjecture for function fields, we need to show that μ is an isomorphism. By the equivalent conditions stated in [DMOS82, Prop. 2.21], we want in particular to find a corresponding pure dual sub-t-motive $\underline{R}'$ of a pure dual t-motive $\underline{R}$ in the Tannakian category generated by $\underline{P}$ to any pure sub-Q-Hodge-Pink structure $\underline{H}'$ of the pure Q-Hodge-Pink structure $\mathcal{T}(\underline{R})$ such that $\mathcal{T}(\underline{R}') = \underline{H}'$. In order to do this, we define F-modules following [Har10] which live on rigid analytic disks centered around ∞. By using the additional information that purity gives at ∞ we may associate an F-module with $\underline{R}$. Following unpublished ideas of Pink for non-dual Anderson A-motives over $\mathbb{C}_\infty$, we find a sub-F-module to the pure sub-Q-Hodge-Pink structure $\underline{H}'$ through the classification of F-modules that was studied first in [HP04]. Roughly speaking, by applying the rigid analytic GAGA principle to the underlying rigid sheaves, we obtain the desired algebraic pure dual sub-t-motive $\underline{R}'$ over $\mathbb{C}_\infty$ that satisfies $\mathcal{T}(\underline{R}') = \underline{H}'$.

In the last chapter we combine Papanikolas's main theorem with the just proven isomorphism, yielding Grothendieck's period conjecture for function fields. We define complex multiplication (CM) of dual Anderson A-motives and determine the Hodge-Pink group assigned to a pure rigid analytically trivial dual Anderson A-motive over a complete field $Q_\infty \subset k \subset \mathbb{C}_\infty$ that has sufficiently many complex multiplication through a Q-algebra E if E/Q is either separable or purely inseparable. This result and Pink's main theorem of [Pin97a], allow us to calculate the dimension of the Hodge-Pink group of a pure uniformizable Drinfeld $\mathbb{F}_q[t]$-module over $\overline{Q} \subset \overline{Q_\infty}$. We obtain the precise analog of Grothendieck's period conjecture for an elliptic curve over $\overline{\mathbb{Q}}$.

0.3 Acknowledgements

I want to take this opportunity to thank the people whose help set me on the road which led to this thesis.

On the mathematics side, I am grateful for the support of my advisor Urs Hartl who spent hours discussing proofs and other relations with me. Secondly, I would also like to thank Matt Papanikolas who suggested the topic of this thesis in [Pap08] for the friendly and helpful email contact. I am thankful that Greg Anderson and Richard Pink shared unpublished notes resp. ideas with me. I further thank Anne Schindler and Simon Hüsken for helpful discussions during the preparation of my thesis. Thanks also go to Chris Nowlin, Kevin Brighton and Rick Spjut for correcting my English.

Besides the mathematical support I have gotten, I am grateful for having a wonderful family and friends all over the world whose support I can count on.

1. PRELIMINARIES

This chapter has three components: required basic definitions needed throughout the thesis, a short introduction to Tannakian theory and an overview over rigid analytic geometry. Most importantly, we shall explain in the second section the concept of "Tannakian duality", by which Galois groups and Hodge-Pinks groups are defined. The material of the last section is for instance needed to define rigid analytic trivializations of dual Anderson A-motives and Papanikolas Q-motives in terms of Tate's rigid analytic spaces, and the explanation of the rigid analytic GAGA principle. We use the latter in Section 4.2, where we prove that the Hodge-Pink group and Galois group are isomorphic.

As this thesis addresses people with algebraic geometry background, we assume that the reader is familiar with the geometry of schemes and basics of category theory. We refer the others to [Har77] and [Fre03] that give a detailed introduction to algebraic geometry and abelian categories respectively.

1.1 Basic definitions

For an index of the most important notation occurring in this thesis, we refer to the List of Symbols. In this section, we first fix notation concerning our base curve C and review afterwards basics of the n-fold twist, f^*-linear maps and $\mathbb{F}_q$-linear polynomials, which lead to the definition of the ring of twisted Laurent polynomials. In order to define affine R-group schemes and the Weil restriction, we introduce representable functors. Through the latter we also explain the principle of "Tannakian duality" in the next section. We take a closer look at additive k-group schemes at the end of this section, which we need to study Anderson A-modules.

1.1.1 Notations

Throughout this thesis, we let C be a smooth projective geometrically irreducible curve over $\mathbb{F}_q$, where $\mathbb{F}_q$ is the finite field of characteristic p with $q = p^r$ elements. Denote the function field of C by $Q := \mathbb{F}_q(C)$ and let $\infty \in C(\mathbb{F}_q)$ be a fixed $\mathbb{F}_q$-rational point. Further, we define $A := \mathcal{O}_C(C \setminus \infty)$ to be the ring of functions regular outside ∞ so that $Q = \mathrm{Quot}(A)$ holds (cf. [Har08, Lem. 1.1.2]).

The basic example to keep in mind is

$$C = \mathbb{P}^1_{Fq}, \quad Q = \mathbb{F}_q(t), \quad \infty = (1:0) \quad \text{and} \quad A = \mathcal{O}_C(C \setminus \{\infty\}) = \mathbb{F}_q[t].$$

If Q is not the rational function field, we fix from now on a ring homomorphism

$$i^* : \mathbb{F}_q[t] \hookrightarrow A, \quad t \mapsto a \quad \text{for a non-constant } a \in A \tag{1.1}$$

that induces a finite dominant morphism $i : C \to \mathbb{P}^1_{\mathbb{F}_q}$ so that we may view A as a free $\mathbb{F}_q[t]$-module of rank $\tilde{r} := \deg i = -[\kappa(\infty) : \mathbb{F}_q] \cdot \mathrm{ord}_\infty(a)$ (cf. [Har08, Exmp. 2.1.8]). Note that i^* induces a homomorphism $i^* : \mathbb{F}_q(t) \hookrightarrow Q$ that makes Q into a free $\mathbb{F}_q(t)$-module of rank $\tilde{r}$. We will use this for instance in Section 2.5, allowing us to restrict ourselves to the case $A = \mathbb{F}_q[t]$ and $Q = \mathbb{F}_q(t)$.

Moreover, we let k be a field that contains $\mathbb{F}_q$ and assume throughout the thesis that k is *perfect*, that is, the q^{th} Frobenius map $x \to x^q$ is an automorphism of k [Gos96, Def. 1.6.4]. Let $\overline{k}$ be a fixed algebraic closure of k. We put $A_k := A \otimes_{\mathbb{F}_q} k$, $Q_k := \mathrm{Quot}(A_k)$ and $C_k := C \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,k$, so that we have in the setting of the fundamental example above

$$A_k = k[t], \quad \mathrm{Spec}\,A_k = \mathbb{A}^1_k \quad \text{and} \quad C_k = \mathbb{P}^1_k.$$

In the definition of Anderson A-modules and dual Anderson A-motives, k is required to be an A-field, defined as follows:

Definition 1.1.1. (i) An *A-field* (k, γ) is a perfect field k equipped with a ring homomorphism $\gamma : A \to k$.

(ii) The *A-characteristic of* (k, γ) is the prime ideal $A\text{-char}(k, \gamma) := \ker \gamma$. We say k has *generic characteristic* if $A\text{-char}(k, \gamma) = (0)$ and *finite characteristic* if $A\text{-char}(k, \gamma)$ is a maximal ideal.

In order to define purity of dual Anderson A-motives and Hodge-Pink structures over local function fields, we also need to introduce completions of the rings just defined. Suppose $P \in C \setminus \{\text{generic point}\}$ is a closed point of C. Since C is normal, the local ring $\mathcal{O}_{C,P}$ is a discrete valuation ring with $\mathrm{Quot}(\mathcal{O}_{C,P}) = Q = \mathbb{F}_q(C)$. Denote the valuation associated to P as ν_P, that is, $\nu_P(a) := \mathrm{ord}_P(a)$ for all $a \in A$. Furthermore, ν_P defines a normalized absolute value $|a|_P := q^{-(\deg P)\nu_P(a)}$ where $\deg P$ is the degree of the divisor P over $\mathbb{F}_q$. Write $A_P := \widehat{\mathcal{O}_{C,P}}$ for the completion of $\mathcal{O}_{C,P}$ with respect to $|\cdot|_P$. By [Ser79, Ch. 2 §4] we see that $A_P \cong \mathbb{F}_P[\![z_P]\!]$ if z_P is a uniformizing parameter of A_P. Let $Q_P \cong \mathbb{F}_P(\!(z_P)\!)$ be the function field of A_P and fix an algebraic closure $\overline{Q_P}$ together with the canonical extension of ν_P (cf. [Gos96, §2]), denoted also by ν_P. Define the completion of $\overline{Q_P}$ with respect to $|\cdot|_P$ to be $\mathbb{C}_P$ and equip it with the extension of ν_P. Note that $\mathbb{C}_P$ is an algebraically closed field [Gos96, Prop. 2.1]. Further, we also write $|\cdot|_P$ for the associated absolute values to the extensions of ν_P to $\overline{Q_P}$ and $\mathbb{C}_P$. Finally, set

$$\begin{aligned} A_{P,k} &:= A_P \widehat{\otimes}_{\mathbb{F}_q} k := \left(\widehat{\mathcal{O}_{C,P} \otimes_{\mathbb{F}_q}}\right)^{|\cdot|_P} \cong (\mathbb{F}_P \otimes_{\mathbb{F}_q} k)[\![z_p]\!] \text{ and} \\ Q_{P,k} &:= \mathrm{Quot}(A_{P,k}) \cong (\mathbb{F}_P \otimes_{\mathbb{F}_q} k)(\!(z_P)\!) \end{aligned}$$

where $\widehat{\otimes}$ is the complete tensor product. We assume that $\mathbb{F}_\infty = \mathbb{F}_q$ so that $A_{\infty,k} \cong k[\![z]\!]$ and $Q_{\infty,k} \cong k(\!(z)\!)$ for a uniformizing parameter z of A_∞.

Moreover, we fix an algebraic closure $\overline{Q} \subset \mathbb{C}_\infty$ of Q. We impose further restrictions on k as we progress, finally focusing on the case that $k \subseteq \mathbb{C}_\infty$ is a perfect and complete field that contains Q_∞. Note that this means that k has generic characteristic through the inclusion $\gamma : A \hookrightarrow Q \hookrightarrow Q_\infty \hookrightarrow k$.

1.1.2 The n-fold twisting operation

We call a formal power series $f = \sum_{i \in \mathbb{Q}} \alpha_i t^i$ a *Hahn series f in t with coefficients in k* if $\mathrm{supp}(f) := \{i \in \mathbb{Q} \mid \alpha_i \neq 0\}$ is a well-ordered subset of $\mathbb{Q}$. This implies that the Hahn series in t with coefficients in k form a field under the ordinary addition and multiplication, which we denote by $k[\![t^{\mathbb{Q}}]\!]$. For later purposes, we introduce the n-fold twisting operation on Hahn series and matrices consisting of such.

Definition 1.1.2. Let $n \in \mathbb{Z}$ be an integer.

(i) We define an automorphism $\varsigma^*_{k[\![t^{\mathbb{Q}}]\!]} : k[\![t^{\mathbb{Q}}]\!] \to k[\![t^{\mathbb{Q}}]\!]$ by setting

$$\varsigma^*_{k[\![t^{\mathbb{Q}}]\!]}\left(\sum_{i\in\mathbb{Q}} \alpha_i t^i\right) := \sum_{i\in\mathbb{Q}} \alpha_i^{\frac{1}{q}} t^i.$$

Its inverse is the automorphism $F^*_{k[\![t^{\mathbb{Q}}]\!]} : k[\![t^{\mathbb{Q}}]\!] \to k[\![t^{\mathbb{Q}}]\!]$ given by

$$F^*_{k[\![t^{\mathbb{Q}}]\!]}\left(\sum_{i\in\mathbb{Q}} \alpha_i t^i\right) := \sum_{i\in\mathbb{Q}} \alpha_i^{q} t^i.$$

(ii) The *n-fold twist* of a Hahn series $f = \sum_{i\in\mathbb{Q}} \alpha_i t^i \in k[\![t^{\mathbb{Q}}]\!]$ is

$$f^{(n)} := \sum_{i\in\mathbb{Q}} \alpha_i^{q^n} t^i.$$

(ii) For a matrix X with Hahn series entrywise, we define its *n-fold twist* $X^{(n)}$ by the rule

$$(X^{(n)})_{ij} := (X_{ij})^{(n)}.$$

Note that $\varsigma^*_{k[\![t^{\mathbb{Q}}]\!]}$ defines automorphisms of several subrings of $k[\![t^{\mathbb{Q}}]\!]$; for example, k, $k[t]$ and $k(t)$. For any such subring R of $k[\![t^{\mathbb{Q}}]\!]$, we denote the induced automorphism by ς^*_R, or by abuse of notation ς^*, and put $R^{\varsigma^*} := \{f \in R \mid \varsigma^*(f) = f\}$.

Moreover, consider an R-algebra B and a ring extension $R \subset R' \subset k[\![t^{\mathbb{Q}}]\!]$ such that $\varsigma^*_{k[\![t^{\mathbb{Q}}]\!]}$ induces an automorphism of R' and R and $R^{\varsigma^*} = R$ holds. We then extend the n-fold twisting operation to the R'-algebra $B \otimes_R R'$ by requiring it to act as the identity on B. Furthermore, we define the n-fold twisting operation on matrices with entries in $B \otimes_R R'$ entrywise as above. Observe that this applies in particular to the $k[t]$-algebra $A_k = A \otimes_{\mathbb{F}_q[t]} k[t]$ and the $k(t)$-algebra $Q_k = Q \otimes_{\mathbb{F}_q(t)} k(t)$.

1.1.3 f^*-linear maps

The notion of an f^*-linear map is mostly needed in Chapter 2 when we define dual Anderson A-motives and Papanikolas Q-motives over k. For example, we want to see that we may replace the A_k-homomorphism underlying a dual Anderson A-motive with a ς^*_A-linear map.

In oder to do this, we let R and R' be commutative rings, $f^* : R \to R'$ a ring homomorphism, M an R-module and N an R'-module. A map $\phi : \mathrm{M} \to \mathrm{N}$ is *f^*-linear* if

$$\phi(r\mathrm{m}) = f^*(r)\phi(\mathrm{m}) \quad \text{and} \quad \phi(\mathrm{m} + \mathrm{m}') = \phi(\mathrm{m}) + \phi(\mathrm{m}')$$

for all $r \in R$ and $\mathrm{m}, \mathrm{m}' \in \mathrm{M}$. Put

$$f^*\mathrm{M} := \mathrm{M} \otimes_{R,f^*} R',$$

and make $f^*\mathrm{M}$ into an R'-module by setting $r'(\mathrm{m} \otimes s') := \mathrm{m} \otimes s'r'$ for all $r', s' \in R'$, $\mathrm{m} \in \mathrm{M}$. The next lemma then shows that one may equivalently define an f^*-linear map $\phi : \mathrm{M} \to \mathrm{N}$ instead of an R'-homomorphism $\psi : f^*\mathrm{M} \to \mathrm{N}$.

Lemma 1.1.3. *(i) The map $f^*_{\mathrm{M}} : \mathrm{M} \to f^*\mathrm{M}$, $\mathrm{m} \mapsto \mathrm{m} \otimes 1$, is an f^*-linear map.*

(ii) If $\phi : \mathrm{M} \to \mathrm{N}$ is an f^*-linear map, then ϕ induces an R'-homomorphism

$$\phi^{\mathrm{lin}} : f^*\mathrm{M} \to \mathrm{N}, \quad \phi^{\mathrm{lin}}\left(\sum_{i=1}^{n} \mathrm{m}_i \otimes r'_i\right) := \sum_{i=1}^{n} r'_i \phi(\mathrm{m}_i) \quad \text{for all } r'_i \in R,\ \mathrm{m}_i \in \mathrm{M},$$

which satisfies $\phi = \phi^{\mathrm{lin}} \circ f^*_{\mathrm{M}}$.

(iii) An R'-homomorphism $\psi : f^*\mathrm{M} \to \mathrm{N}$ defines an f^*-linear map

$$\psi^{f^*\text{-lin}} : \mathrm{M} \to \mathrm{N}, \quad \mathrm{m} \mapsto \psi(\mathrm{m} \otimes 1),$$

so that $(\psi^{f^*\text{-lin}})^{\mathrm{lin}} = \psi$.

Proof. Part (i) is clear since we have for all $r \in R$ and $\mathrm{m}, \mathrm{m}' \in \mathrm{M}$

$$f^*_{\mathrm{M}}(r\mathrm{m}) = (r\mathrm{m}) \otimes 1 = \mathrm{m} \otimes f^*(r) = f^*(r) f^*_{\mathrm{M}}(\mathrm{m})$$

and

$$f^*_{\mathrm{M}}(\mathrm{m} + \mathrm{m}') = (\mathrm{m} + \mathrm{m}') \otimes 1 = \mathrm{m} \otimes 1 + \mathrm{m}' \otimes 1 = f^*_{\mathrm{M}}(\mathrm{m}) + f^*_{\mathrm{M}}(\mathrm{m}').$$

As ϕ^{lin} is f^*-linear by definition and $(\phi^{\mathrm{lin}} \circ f^*_{\mathrm{M}})(\mathrm{m}) = \phi^{\mathrm{lin}}(\mathrm{m} \otimes 1) = \phi(\mathrm{m})$ for all $\mathrm{m} \in \mathrm{M}$, we only need to check that ϕ^{lin} is well-defined. This is the fact because

$$\phi^{\mathrm{lin}}(r\mathrm{m} \otimes r') = r'\phi(r\mathrm{m}) = r' f^*(r)\phi(\mathrm{m}) = f^*(r) r' \phi(\mathrm{m}) = \phi^{\mathrm{lin}}\left(\mathrm{m} \otimes f^*(r) r'\right)$$

for all $\mathrm{m} \in \mathrm{M}$, $r \in R$ and $r' \in R'$.

To see (iii), note that for all $r \in R$ and $\mathrm{m}, \mathrm{m}' \in \mathrm{M}$,

$$\psi^{f^*\text{-lin}}(r\mathrm{m}) = \psi(r\mathrm{m} \otimes 1) = \psi(\mathrm{m} \otimes f^*(r)) = f^*(r)\psi(\mathrm{m} \otimes 1) = f^*(r)\psi^{f^*\text{-lin}}(\mathrm{m})$$

and

$$\psi^{f^*\text{-lin}}(\mathrm{m} + \mathrm{m}') = \psi((\mathrm{m} + \mathrm{m}') \otimes 1) = \psi(\mathrm{m} \otimes 1) + \psi(\mathrm{m}' \otimes 1) = \psi^{f^*\text{-lin}}(\mathrm{m}) + \psi^{f^*\text{-lin}}(\mathrm{m}')$$

hold and thus $\psi^{f^*\text{-lin}}$ is f^*-linear as desired. Finally, it does satisfy

$$(\psi^{f^*\text{-lin}})^{\mathrm{lin}}(\sum_{i=1}^{n} \mathrm{m}_i \otimes r'_i) = \sum_{i=1}^{n} r'_i \psi^{f^*\text{-lin}}(\mathrm{m}_i) = \sum_{i=1}^{n} r'_i \psi(\mathrm{m}_i \otimes 1) = \psi(\sum_{i=1}^{n} \mathrm{m}_i \otimes r'_i)$$

for all elements $\sum_{i=1}^{n} \mathrm{m}_i \otimes r'_i \in f^*\mathrm{M}$. □

To ease notation, we occasionally denote the image $f^*_{\mathrm{M}}(\mathrm{m})$ of an $\mathrm{m} \in \mathrm{M}$ under f^*_{M} by $f^*\mathrm{m}$. Let us now consider an R'-homomorphism $\psi : f^*\mathrm{M} \to \mathrm{N}$ and the induced f^*-linear map $\psi^{f^*\text{-lin}} : \mathrm{M} \to \mathrm{N}$. Observe that $(\psi^{f^*\text{-lin}})^n : \mathrm{M} \to \mathrm{N}$ is an $(f^*)^n$-linear map. We write

$$(\psi)^n := \psi \circ f^*\psi \circ \cdots \circ (f^*)^{n-1}\psi : (f^*)^n\mathrm{M} \to \mathrm{N}$$

for the R'-homomorphism $((\psi^{f^*\text{-lin}})^n)^{\mathrm{lin}} : (f^*)^n\mathrm{M} \to \mathrm{N}$ given by the above lemma.

1.1.4 The ring of twisted Laurent polynomials

Let us recall the definition of absolutely additive and $\mathbb{F}_q$-linear polynomials.

Definition 1.1.4. Let $\{x_i, y_i\}_{i=1}^e$ be a family of independent commuting variables and $f = f(x_1, \ldots, x_e) \in k[x_1, \ldots, x_e]$ a polynomial.

(i) We say that f is *absolutely additive* over k if

$$f(x_1 + y_1, \ldots, x_e + y_e) = f(x_1, \ldots, x_e) + f(y_1, \ldots, y_e).$$

(ii) Let f be an absolutely additive polynomial. We say that f is $\mathbb{F}_q$*-linear* if it satisfies

$$f(cx_1, \ldots, cx_e) = cf(x_1, \ldots, x_e)$$

for all $c \in \mathbb{F}_q$.

We will use frequently the following equivalent conditions for a polynomial in $k[x_1, \ldots, x_e]$ to be absolutely additive or $\mathbb{F}_q$-linear.

Lemma 1.1.5. *Let $\{x_i, y_i\}_{i=1}^e$ be a family of independent commuting variables and $f = f(x_1, \ldots, x_e) \in k[x_1, \ldots, x_e]$ a polynomial.*

(i) f is absolutely additive if and only if f is a p-polynomial*, that is, f is of the form*

$$f = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} x_j^{p^i} = \sum_{i=0}^{\infty} (\alpha_{i,1}, \ldots, \alpha_{i,e}) \begin{pmatrix} x_1^{p^i} \\ \vdots \\ x_e^{p^i} \end{pmatrix},$$

with $\alpha_{i,j} \in k$, and $\alpha_{i,j} = 0$ for $i \gg 0$.

(ii) f is $\mathbb{F}_q$-linear if and only if f is a q-polynomial*, that is, f is of the form*

$$f = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} x_j^{q^i} = \sum_{i=0}^{\infty} (\alpha_{i,1}, \ldots, \alpha_{i,e}) \begin{pmatrix} x_1^{q^i} \\ \vdots \\ x_e^{q^i} \end{pmatrix},$$

with $\alpha_{i,j} \in k$ and $\alpha_{i,j} = 0$ for $i \gg 0$.

Proof. For (i), c.f. [Hum75, §20.3 Lem. A].

To see (ii), suppose $f = f(x_1, \ldots, x_e) = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} x_j^{p^i}$, with $\alpha_{i,j} \in k$ and $\alpha_{i,j} = 0$ for $i \gg 0$ is a p-polynomial and let $c \in \mathbb{F}_q$ arbitrarily. Define r such that $q = p^r$ and consider

$$f(cx_1, \ldots, cx_e) = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} (cx_j)^{p^i} = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} c^{p^i} x_j^{p^i}.$$

This is equal to

$$cf(x_1, \ldots, x_e) = c \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} x_j^{p^i} = \sum_{i=0}^{\infty} \sum_{j=0}^{e} c\alpha_{i,j} x_j^{p^i}$$

if and only if $\alpha_{i,j} = 0$ whenever r does not divide j, giving us the desired result. □

Example 1.1.6. (i) The polynomials $\tau_p^i : k \to k,\ x \mapsto x^{p^i},\ i \in \mathbb{N}$, are absolutely additive over k. Viewing τ_p as a non-commuting variable, we may write each absolutely additive polynomial $f(x) = \sum_{i=0}^{\infty} \alpha_i x^{p^i}$ in $k[x]$ as a polynomial in τ_p, that is, $f = \sum_{i=0}^{\infty} \alpha_i \tau_p^i$.

(ii) For all $i \in \mathbb{N}$, consider the polynomial

$$\tau^i : k^e \to k^e, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_e \end{pmatrix} = x \mapsto x^{(i)} = \begin{pmatrix} x_1^{q^i} \\ \vdots \\ x_e^{q^i} \end{pmatrix},$$

which is easily seen to be an $\mathbb{F}_q$-linear polynomials over k^e. Viewing τ as a non-commuting variable, we may write each $\mathbb{F}_q$-linear polynomial

$$f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{e} \alpha_{i,j} x_j^{q^i} \quad (\text{with } \alpha_{i,j} \in k \text{ and } \alpha_{i,j} = 0 \text{ for } i \gg 0)$$

in $k[x] = k[x_1, \ldots, x_e]$ as a polynomial in τ, that is $f = \sum_{i=0}^{\infty} (\alpha_{i,1}, \ldots, \alpha_{i,e}) \tau^i$.

Since k is perfect we may define the inverse $\sigma : k^e \to k^e$ of τ and $\sigma^i := (\tau^{-1})^i : k^e \to k^e,\ x \mapsto x^{(-i)},\ i \in \mathbb{N}$. Viewing τ as an invertible non-commuting variable, we also obtain "polynomials" in $\sigma = \tau^{-1}$. Considering the ring spanned by linear combinations of the monomials $\tau^i,\ i \in \mathbb{Z}$, leads to the following definition:

Definition 1.1.7. The *ring of twisted Laurent polynomials* $k[\tau, \tau^{-1}]$ is obtained by adjoining an invertible non-commuting variable τ to k subject to the relations

$$\tau\alpha = \alpha^q \tau \text{ for all } \alpha \in k.$$

By definition each element $\varphi \in k[\tau, \tau^{-1}]$ can be written as a unique sum $\varphi = \sum_{i \in \mathbb{Z}} \alpha_i \tau^i$ for some $\alpha_i \in k, \alpha_i = 0$ for $|i| \gg 0$. We add such elements termwise and multiply them by the rule

$$\left(\sum_{i \in \mathbb{Z}} \alpha_i \tau^i\right) \left(\sum_{j \in \mathbb{Z}} \beta_j \tau^j\right) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \alpha_i \beta_j^{q^i} \tau^{i+j}.$$

We further define the following rings related to $k[\tau, \tau^{-1}]$:

$$\begin{aligned} k[\tau] &:= \text{The subring generated by } k \text{ and } \tau, \\ k[t;\tau] &:= \text{the ring obtained by adjoining a central variable } t \text{ to } k[\tau], \\ k[\sigma] &:= \text{the subring generated by } k \text{ and } \sigma = \tau^{-1}, \\ k[t;\sigma] &:= \text{the ring obtained by adjoining a central variable } t \text{ to } k[\sigma]. \end{aligned}$$

Note that the following relation holds in $k[\sigma]$:

$$\sigma\alpha = \alpha^{\frac{1}{q}} \sigma \text{ for all } \alpha \in k.$$

Properties of $k[\tau]$ as the existence of right and left division algorithms are discussed in [Gos96, §1.6] and carry over to $k[\sigma]$.

An element in the matrix space $\mathrm{Mat}_{d \times e}(k[\tau, \tau^{-1}])$ has a unique presentation of the form $\sum_{i \in \mathbb{Z}} \alpha_{(i)} \tau^i$ for some $\alpha_{(i)} \in \mathrm{Mat}_{d \times e}(k), \alpha_{(i)} = 0$ for $|i| \gg 0$.

Addition in $\mathrm{Mat}_{d\times e}(k[\tau,\tau^{-1}])$ is then termwise, and multiplication is given by the rule

$$\left(\sum_{i\in\mathbb{Z}}\alpha_{(i)}\tau^i\right)\left(\sum_{j\in\mathbb{Z}}\beta_{(j)}\tau^j\right)=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\alpha_{(i)}\beta_{(j)}^{(i)}\tau^{i+j}.$$

Definition 1.1.8. (i) We define the *asterisk operation* to be the unique involutive antiautomorphism $k[\tau,\tau^{-1}]\to k[\tau,\tau^{-1}]$, given by $\phi\mapsto\phi^*$ such that $\tau^*=\tau^{-1}$ and $x^*=x$ for all $x\in K$.

Thus we can "asterisk" an elements in $k[\tau,\tau^{-1}]$ as follows:

$$\left(\sum_{i\in\mathbb{Z}}\alpha_i\tau^i\right)^*=\sum_{i\in\mathbb{Z}}\tau^{-i}\alpha_i=\sum_{i\in\mathbb{Z}}\alpha_i^{q^{-i}}\tau^{-i}=\sum_{i\in\mathbb{Z}}\alpha_i^{(-i)}\sigma^i.$$

(ii) We define the *dagger operation*$(\phi\mapsto\phi^\dagger):\mathrm{Mat}_{d\times e}(k[\tau,\tau^{-1}])\to\mathrm{Mat}_{e\times d}(k[\tau,\tau^{-1}])$ entrywise by setting

$$(\phi^\dagger)_{ij}=(\phi_{ji})^*.$$

Thus we can "dagger" an element $\sum_{i\in\mathbb{Z}}\alpha_{(i)}\tau^i\in\mathrm{Mat}_{e\times d}(k[\tau,\tau^{-1}])$ as follows:

$$\left(\sum_{i\in\mathbb{Z}}\alpha_{(i)}\tau^i\right)^\dagger=\sum_{i\in\mathbb{Z}}\tau^{-i}\alpha_{(i)}^{\mathrm{tr}}=\sum_{i\in\mathbb{Z}}\left(\alpha_{(i)}^{(-i)}\right)^{\mathrm{tr}}\tau^{-i}=\sum_{i\in\mathbb{Z}}\left(\alpha_{(i)}^{(-i)}\right)^{\mathrm{tr}}\sigma^i.$$

Remark 1.1.9. (i) The dagger operation maps a matrix in $\mathrm{Mat}_{d\times e}(k)$ to its usual transpose in $\mathrm{Mat}_{e\times d}(k)$.

(ii) We have for all $f\in\mathrm{Mat}_{e\times d}(k[\tau,\tau^{-1}])$, $g\in\mathrm{Mat}_{d\times e}(k[\tau,\tau^{-1}])$ and $c\in k[\tau,\tau^{-1}]$:

$$(fg)^\dagger=g^\dagger f^\dagger,\ (cf)^\dagger=f^\dagger c^*\quad\text{and}\quad(fc)^\dagger=c^*f^\dagger.$$

1.1.5 Representable functors and affine R-group schemes

Throughout this section, we let R and R' denote arbitrary rings. Affine group schemes over $k/\mathbb{F}_q$ are the major objects occurring in the definition of Anderson A-modules and (dual) Anderson A-motives, both defined over k. Recall that an *R-scheme* X is a scheme X equipped with a morphism $\pi:X\to\operatorname{Spec}R$; X is alternatively called a *scheme over R*.

At the end of this section we define additive algebraic group schemes over k whose corresponding k-algebras are isomorphic to $k[x_1,\dots,x_n]$ and investigate their composition rings of $\mathbb{F}_q$-linear homomorphisms (Definition 1.1.20) that will be isomorphic to matrix spaces with entries in $k[\tau]$.

Let us now define the functor of points h_X of an object X in a category $\mathscr{C}$. We later see that h_X fully determines X by Yoneda's Lemma (Lemma 1.1.12).

Definition 1.1.10. Let $\mathscr{C}$ be a category and X an object in $\mathscr{C}$.

(i) The *functor of points* of X is the covariant functor

$$h_X:\mathscr{C}^\circ\to\mathscr{S}ets$$

from the opposite category of $\mathscr{C}$ to the category of sets given by

$$\begin{array}{lll}\mathrm{Ob}(\mathscr{C}) & \ni Y\mapsto\mathrm{Hom}(Y,X) & \text{and}\\ \mathrm{Hom}(Y,Z) & \ni f\mapsto\mathrm{Hom}(h_X(Z),h_X(Y)) & \text{with } Y,Z\in\mathrm{Ob}(\mathscr{C}),\end{array}$$

the latter by taking $g\in h_X(Z)=\mathrm{Hom}(Z,X)$ to $g\circ f\in h_X(Y)=\mathrm{Hom}(Y,X)$.

(ii) We call the set $h_X(Y) = \mathrm{Hom}(Y, X)$ the *set of Y-valued points.*

We then consider the covariant functor

$$h : \mathscr{C} \to \mathscr{Fun}(\mathscr{C}^\circ, \mathscr{Sets}), \quad X \mapsto h_X,$$

from the category of schemes to the *category of covariant functors from $\mathscr{C}^\circ$ to $\mathscr{Sets}$.* The morphisms in the category of functors are functorial morphisms, defined as follows:

Definition 1.1.11. (i) Let F and G be functors from a category $\mathscr{C}$ to a category $\mathscr{D}$. We define a *functorial morphism* or *natural transformation η from F to G* by associating a morphism $\eta_X : F(X) \to G(X)$ in $\mathscr{D}$ with each object X in $\mathscr{C}$, such that for all morphisms $f : X \to Y$ in $\mathscr{C}$ the following diagram is commutative

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_X} & G(Y). \end{array}$$

(ii) A functorial morphism η is called a *functorial isomorphism* if η_X is an isomorphism in $\mathscr{D}$ for all $X \in \mathrm{Ob}(\mathscr{C})$.

(iii) Two functors F and G are said to be *isomorphic* if there exists a functorial isomorphism η between them.

Suppose X and X' are objects in a category $\mathscr{C}$ and $F : \mathscr{C}^\circ \to \mathscr{Sets}$ is a covariant functor. Then there is a canonical mapping

$$\varphi : F(X) \to \mathrm{Hom}(h_X, F), \quad \eta \mapsto \nu \mapsto (F(\nu))(\eta).$$

Lemma 1.1.12 (Yoneda's Lemma, cf. [BLR90, §4.1 Prop. 1])**.** *(i) φ is bijective; that is, the natural transformations from h_X to F are in a natural correspondence with the elements of $F(X)$.*

(ii) The functor $h : \mathscr{C} \to \mathscr{Fun}(\mathscr{C})$ is an equivalence of $\mathscr{C}$ with a full subcategory of the category of functors.

Now we can finally define what one means by saying X represents a functor F.

Definition 1.1.13 ([Die82, CI §2])**.** (i) Let $\mathscr{C}$ be a category. A covariant functor $F : \mathscr{C}^\circ \to \mathscr{Sets}$ is called *representable* if there exists an $X \in \mathrm{Ob}(\mathscr{C})$ and an element $\eta \in F(X)$ such that $\varphi(\eta)$ is an isomorphism of functors. The pair (X, η), or by abuse of notation X, *represents* F.

(ii) Let F be a representable functor. We call the object of $F(X)$ corresponding to $\mathrm{id}_X \in h_X(X)$ the *universal family.*

It follows in particular from Yoneda's Lemma that if (X', η') is another pair representing F, there is a unique isomorphism $w : X \xrightarrow{\sim} X'$ such that $\eta = F(w)(\eta')$, that is, (X, η) is unique up to unique isomorphisms.

Let $S = \mathrm{Spec}\, R$ and define the induced covariant functor $h^*_X(A) := h_X(\mathrm{Spec}\, A)$ for an R-algebra A from the category R$-\mathscr{algebras}$ of R-algebras to the category of sets. As mentioned earlier, we make the following definition:

Definition 1.1.14 ([Wat79, §1.4]). (i) A *Hopf algebra A over R* is an R-algebra A together with R-algebra maps

$$\mu^* : A \to A \otimes_k A \quad e^* : A \to R \quad i^* : A \to A$$

called comultiplication, counit and coinverse, respectively such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes_R A \otimes_R A & A \otimes_R A \arrow[l, "\mathrm{id} \otimes \mu^*"'] \\
A \otimes_R A \arrow[u, "\mu^* \otimes \mathrm{id}"'] & A \arrow[l, "\mu^*"'] \arrow[u, "\mu^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \otimes_R A & A \otimes_R A \arrow[l, "e^* \otimes \mathrm{id}"'] \\
R \otimes_R A \arrow[u, "\sim"'] & A \arrow[l, "\sim"'] \arrow[u, "\mu^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A & A \otimes_R A \arrow[l, "(i^*{,}\mathrm{id})"'] \\
A \arrow[u, "\sim"'] & A. \arrow[l, "\sim"'] \arrow[u, "\mu^*"']
\end{tikzcd}$$

(ii) We call an R-scheme G an *affine group scheme over R* if one of the following equivalent conditions (Lemma 1.1.12) is satisfied:

(a) Its functor of points h_G^* :R$-\mathscr{algebras} \to \mathscr{Sets}$ factors through the forgetful functor $\mathscr{Groups} \to \mathscr{Sets}$ from the category of groups to the category of sets.

(b) There is a Hopf Algebra A over R such that $G = \operatorname{Spec} A$ so that the R-morphisms $\mu : G \times G \to G$ (*group multiplication*), $e : \operatorname{Spec} R \to G$ (*unit*) and $i : G \to G$ (*inverse*) induced by μ^*, e^* and i^*, respectively, then satisfy the usual group laws associativity, left unit and left inverse.

Furthermore we call an affine group scheme $G = \operatorname{Spec} A$ over R *algebraic*, or an *algebraic group*, if the corresponding R-algebra A is finitely generated.

(iii) An R-morphism $\varphi : G \to G'$ (that is, $\pi = \pi' \circ \varphi$) is called a *homomorphism of R-group schemes* if

$$\varphi \circ i = i' \circ \varphi : G \to G', \;\; e' = \varphi \circ e : R \to G' \text{ and } \mu' \circ (\varphi, \varphi) = \varphi \circ \mu : G \times_R G \to G'.$$

We set

$$\operatorname{Hom}_R(G, G') := \{\varphi : G \to G' \text{ homomorphism of } R\text{-group schemes}\}$$

and $\operatorname{End}_R(G) := \operatorname{Hom}_R(G, G)$.

We now give the examples of closed algebraic groups that occur later in this thesis.

Example 1.1.15. (i) We call $\mathbb{G}_{a,R} := \operatorname{Spec} R[t]$ the *additive group over R*. Its maps are given by their induced maps on the R-algebras as follows:

$$e^*(t) = 0, \quad i^*(t) = -t \quad \text{and} \quad \mu^*(t) = \underbrace{t \otimes 1 + 1 \otimes t}_{=:t_1+t_2} \in R[t] \otimes_R R[t] \cong R[t_1, t_2].$$

If $R = K$ is a field, we define the *affine line over K* to be $\mathbb{A}^1_K := \mathbb{G}_{a,K}$. This way we may regard the *affine n-space $\mathbb{A}^n_K$ over K* as

$$\mathbb{A}^n_K := \mathbb{G}^n_{a,K} = \mathbb{G}_{a,K} \times_{\operatorname{Spec} K} \dots \times_{\operatorname{Spec} K} \mathbb{G}_{a,K} = \operatorname{Spec} K[t_1, \dots, t_n].$$

(ii) The *general linear group over* R is $\mathrm{GL}_{n,R}$, the group of all invertible $n \times n$ matrices with entries in R with the ordinary matrix multiplication as group operation. Thus $\mathrm{GL}_{n,R}$ is represented by $\operatorname{Spec} R[a_{ij}, \det(a_{ij})^{-1}]$ and its maps are given by

$$\begin{aligned} &e^*(a_{ij}) = \delta_{ij}, \quad \mu^*(a_{ij}) = \textstyle\sum_{k=1}^n a_{ik} \otimes a_{kj} \quad \text{and} \\ &i^*(a_{ij}) = (-1)^{i+j} \det(a_{ij})^{-1} \det(a_{km})_{k\neq j, m\neq i}. \end{aligned}$$

The matrices with determinant 1 and entries in R form the *special linear group* $\mathrm{SL}_{n,R}$.

The group of upper triangular $(n\times n)$-matrices with entries in R and all diagonal entries 1 is the *unipotent group* $\mathrm{U}_{n,R}$ *over* R. Its name comes from the fact that all elements M in $\mathrm{U}_{n,R}$ are *unipotent* since $1 - M$ is *nilpotent*; that is, $(1 - M)^n = 0$.

Furthermore, the *multiplicative group over* R is $\mathbb{G}_{m,R} := \mathrm{GL}_{1,R} = R^\times$ with $\mu^*(x, y) = xy$, $i^*(x) = x^{-1}$ and $e^* = 1$. If $R = K$ is a field, then $\mathbb{G}_{m,K}$ is the affine open subset $K^\times$ of the affine line $\mathbb{A}^1_K$.

Following [Wat79, §4.2], we call $\mathrm{SL}_{n,K}$ and any closed reduced subgroup of $\mathrm{SL}_{n,K}$ an *algebraic matrix group over* K if K is an infinite field. Moreover, the term *linear algebraic group over* K found in the literature corresponds to the definition of a smooth affine K-group scheme, that is, an affine K-group scheme G such that $G_{\overline{K}} := G \times_K \operatorname{Spec} \overline{K}$ is an algebraic matrix group over $\overline{K}$, where $\overline{K}$ denotes an algebraic closure of K (cf. [Wat79, §4.5, §11.6]).

In fact, all affine K-group schemes in this thesis are algebraic and by [Wat79, Thm. 3.4] isomorphic to closed subgroups of $\mathrm{GL}_{n,K}$ for some $n \in \mathbb{N}$ and therefore algebraic matrix groups over K if K is infinite. Obviously, any algebraic matrix group is a linear algebraic group and of special interest will be the reductive linear algebraic groups.

Definition 1.1.16 (Cf. [Wat79, §4])**.** (i) An algebraic matrix group over K is called *reductive* if its unipotent radical is trivial.

(ii) A linear algebraic group G over K is said to be *reductive* if the algebraic matrix group $G_{\overline{K}}$ is reductive.

Before taking a closer look at algebraic groups isomorphic to $\mathbb{G}^d_{a,K}$, we want do discuss the existence of the Weil restriction which we need in Section 5.1.

Definition 1.1.17. Let $\pi^* : R \to R'$ be a ring homomorphism. For any R'-scheme X', consider the covariant functor

$$\mathfrak{R}_{R'/R}(X') : R\text{-}\mathscr{schemes}^\circ \to \mathscr{Sets}, \quad Y \mapsto \operatorname{Hom}_{R'}(Y \times_R \operatorname{Spec} R', X')$$

from the category of R-schemes to the category of sets. If $\mathfrak{R}_{R'/R}(X')$ is representable, we denote the corresponding R-scheme by $\mathfrak{R}_{R'/R}X'$ and call it the *Weil restriction of* X'.

If the Weil restriction $\mathfrak{R}_{R'/R}X'$ of an R'-scheme X' exists, it is hence characterized by its *universal property* in form of a functorial isomorphism

$$\operatorname{Hom}_R(Y, \mathfrak{R}_{R'/R}X') \xrightarrow{\sim} \operatorname{Hom}_{R'}(Y \times_R \operatorname{Spec} R', X')$$

of functors for all R-schemes Y.

The following criterion for the existence of the Weil restriction is due to Grothendieck, here given in a simple version:

Theorem 1.1.18 (Cf. [BLR90, Thm. 7.6/4]). *Let R' be a free R-module of rank d and X' an affine R'-scheme. Then there is an R-scheme X which represents $\mathfrak{R}_{R'/R}(X')$. In particular, $\mathfrak{R}_{R'/R}\mathbb{G}_{a,R'} \cong \mathbb{G}_{a,R}^d$.*

Lemma 1.1.19. *Let R be a ring and R' a finite R-algebra. Then*

$$\dim \mathfrak{R}_{R'/R}\mathbb{G}_{m,R'} = [R' : R].$$

Proof. By definition, we find that $\mathbb{G}_{m,R'} = (R')^\times \to \mathbb{G}_{a,R'}$ is an open immersion. It follows from [BLR90, §7.6 Prop. 2] that the Weil restriction respects open immersions; that is, $\mathfrak{R}_{R'/R}\mathbb{G}_{m,R'} \to \mathfrak{R}_{R'/R}\mathbb{G}_{a,R'} \overset{\text{Thm. 1.1.18}}{\cong} \mathbb{G}_{a,R}^{[R':R]}$ is an open immersion and moreover

$$\dim \mathfrak{R}_{R'/R}\mathbb{G}_{m,R'} = [R' : R].$$

□

Additive algebraic group schemes

We want to take a closer look at algebraic group schemes isomorphic to $\mathbb{G}_{a,k}^d$ that we call *additive algebraic group schemes* over k and their composition rings of $\mathbb{F}_q$-linear homomorphisms. An Anderson A-module will then, roughly speaking, be a pair $\underline{E} = (E, \varphi)$ where E is such an additive algebraic group scheme and φ is a map into its endomorphism ring of $\mathbb{F}_q$-linear polynomials. Furthermore, endowing the ring of $\mathbb{F}_q$-linear homomorphisms over k from $\mathbb{G}_{a,k}$ to E with the structure of a certain type of a module gives us the associated dual Anderson A-motive to (E, φ). So let us first define the notion of R-linear homomorphisms and R-module schemes over k.

Definition 1.1.20 ([Har08, Def. 1.1.4]).

(i) Let R be a ring with 1. We call a pair (G, φ) consisting of a commutative group scheme G over k and a ring homomorphism $\varphi : R \to \mathrm{End}_k(G)$ sending $r \in R$ to $\varphi_r := \varphi(r)$ an *R-module scheme over k.*

(ii) Let (G, φ) and (G, φ') be two R-modules. We denote the group of R-linear homomorphisms by

$$\begin{aligned} \mathrm{Hom}_{k,R}(G, G') &:= \mathrm{Hom}_{k,R}((G, \varphi), (G', \varphi')) \\ &:= \{f \in \mathrm{Hom}_k(G, G') : \ f \circ \varphi_r = \varphi'_r \circ f \forall\ r \in R\} \end{aligned}$$

and put $\mathrm{End}_{k,R}(G) := \mathrm{End}_{k,R}((G, \varphi)) := \mathrm{Hom}_{k,R}((G, \varphi), (G, \varphi))$.

Then $\mathbb{G}_{a,k}$ is an $\mathbb{F}_q$-module scheme over k since $\mathbb{F}_q \subset k$. In Example 1.1.6 we have seen that we may write an $\mathbb{F}_q$-linear k-homomorphism $f \in \mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}^e, \mathbb{G}_{a,k})$ as an element of $\mathrm{Mat}_{1\times e}(k[\tau])$. Note that we have then $\mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}^e, \mathbb{G}_{a,k}^d) \cong \mathrm{Mat}_{d\times e}(k[\tau])$ and daggering provides an identification

$$\begin{aligned} \mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}^e, \mathbb{G}_{a,k}^d) \cong \mathrm{Mat}_{d\times e}(k[\tau]) &\xrightarrow{\dagger} \mathrm{Mat}_{e\times d}(k[\sigma]) \\ \mathrm{m} &\mapsto \mathrm{m}^\dagger, \end{aligned}$$

preserving addition since $(m_1^\dagger + m_2^\dagger)^\dagger = m_1 + m_2$, but reversing multiplication since $(m_1^\dagger \cdot m_2^\dagger)^\dagger = m_2 \cdot m_1$.

Recall now that if k'/k is a field extension, we get by evaluating

$$\mathbb{G}_{a,k}(k') = \mathrm{Hom}_k(\mathrm{Spec}\, k', \mathbb{G}_{a,k}) = \mathrm{Hom}_k(k[x], k') \cong k'.$$

Hence $\mathbb{G}_{a,k}(k)$ is isomorphic to k with the addition as group action and we similarly get $\mathbb{G}_{a,k}^d(k) \cong \mathrm{Mat}_{d\times 1}(k)$. Moreover, the k-valued points or *k-rational points* of $\mathbb{G}_{a,k}$ are

$$\mathbb{G}_{a,k}(k) = \mathrm{Hom}_k(\mathrm{Spec}\, k, \mathbb{G}_{a,k}) \xrightarrow{\sim} \{x \in \mathbb{G}_{a,k} : \kappa(x) = k\}$$

(cf. [EH00, §VI.1.2]).

An $f = \sum_{i=-\infty}^{\infty} \alpha_{(i)}\tau^i \in \mathrm{Mat}_{e\times d}(k[\tau,\tau^{-1}])$ with $\alpha_{(i)} \in \mathrm{Mat}_{e\times d}(k)$ and $\alpha_{(i)} = 0$ for $|i| \gg 0$ induces a morphism

$$x \mapsto \sum_{i=-\infty}^{\infty} \alpha_{(i)} x^{(i)} : \mathrm{Mat}_{d\times 1}(k) \to \mathrm{Mat}_{e\times 1}(k),$$

which is also denoted by f. We obtain the following result that will help us later to show that the category of Anderson A-modules and the category of dual Anderson A-motives of positive rank and dimension are equivalent.

Lemma 1.1.21 ([ABP, §1.2.8]). *Regard* $\mathrm{Mat}_{d\times 1}(k)$ *and* $\mathrm{Mat}_{e\times 1}(k)$ *as* $\mathbb{G}_{a,k}^d(k)$ *and* $\mathbb{G}_{a,k}^e(k)$*, respectively, and let* $\delta : \mathrm{Mat}_{1\times d}(k[\sigma]) \to \mathrm{Mat}_{d\times 1}(k)$ *be given by*

$$\sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i \mapsto \sum_{i=0}^{\infty} \left(\alpha_{(i)}^{(i)}\right)^{\mathrm{tr}}$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k)$, $\alpha_{(i)} = 0$ *for* $i \gg 0$*. Then the following diagram commutes and has exact rows*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Mat}_{1\times d}(k[\sigma]) \arrow[r, "(\sigma-1)\cdot"] \arrow[d, "\cdot f^\dagger"'] & \mathrm{Mat}_{1\times d}(k[\sigma]) \arrow[r, "\delta"] \arrow[d, "\cdot f^\dagger"'] & \mathrm{Mat}_{d\times 1}(k) \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & \mathrm{Mat}_{1\times e}(k[\sigma]) \arrow[r, "(\sigma-1)\cdot"] & \mathrm{Mat}_{1\times e}(k[\sigma]) \arrow[r, "\delta"] & \mathrm{Mat}_{e\times 1}(k) \arrow[r] & 0
\end{tikzcd}$$

for every $f \in \mathrm{Mat}_{e\times d}(k[\tau])$*, where daggering identifies* $\mathrm{Mat}_{1\times d}(k[\sigma])$ *and* $\mathrm{Mat}_{1\times e}(k[\sigma])$ *with* $\mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k}^d)$ *and* $\mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k}^e)$*, respectively.*

Proof. We first show exactness. Since $(\sigma-1)$ is clearly injective, let us prove $\ker\delta = \mathrm{im}(\sigma-1)$. Suppose

$$\mathrm{m} = \sum_{i=0}^{n} \alpha_{(i)}\sigma^i \in \ker\delta \subseteq \mathrm{Mat}_{1\times d}(k[\sigma])$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k)$. Then $\delta(\mathrm{m}) = \sum_{j=0}^{n} (\alpha_{(j)}^{(j)})^{\mathrm{tr}} = 0$; hence, $\alpha_{(0)} = -\sum_{j=1}^{n} \alpha_{(j)}^{(j)}$, and by considering the following element in the image of $(\sigma-1)$

$$\begin{aligned}
(\sigma - \mathrm{id})\left(\sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \alpha_{(j)}^{(j-i)}\sigma^i\right) &= \sigma\left(\sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \alpha_{(j)}^{(j-i)}\sigma^i\right) - \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \alpha_{(j)}^{(j-i)}\sigma^i \\
&= \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \alpha_{(j)}^{(j-(i+1))}\sigma^{i+1} - \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \alpha_{(j)}^{(j-i)}\sigma^i \\
&= \sum_{i=1}^{n}\sum_{j=i}^{n} \alpha_{(j)}^{(j-i)}\sigma^i - \sum_{i=0}^{n-1}\sum_{j=i+1}^{n} \alpha_{(j)}^{(j-i)}\sigma^i \\
&= \alpha_{(n)}\sigma^n + \sum_{i=1}^{n-1} \alpha_{(i)}\sigma^i - \sum_{j=1}^{n} \alpha_{(j)}^{(j)} = \mathrm{m},
\end{aligned}$$

we find that $\ker\delta \subseteq \mathrm{im}(\sigma-1)$. To see equality, let

$$\mathrm{m} = \sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i \in \mathrm{Mat}_{1\times d}(k[\sigma])$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k),\ \alpha_{(i)} = 0$ for $i \gg 0$. Then

$$\begin{aligned}
\delta\left((\sigma-1)(\mathrm{m})\right) &= \delta\left(\sigma\left(\sum_{i=0}^{\infty}\alpha_{(i)}\sigma^i\right)\right) - \delta\left(\sum_{i=0}^{\infty}\alpha_{(i)}\sigma^i\right) \\
&= \delta\left(\sum_{i=1}^{\infty}\alpha_{(i-1)}^{(-1)}\sigma^i\right) - \delta\left(\sum_{i=0}^{\infty}\alpha_{(i)}\sigma^i\right) \\
&= \left(\sum_{i=1}^{\infty}\alpha_{(i-1)}^{(i-1)}\right)^{\mathrm{tr}} - \left(\sum_{i=0}^{\infty}\alpha_{(i)}^{(i)}\right)^{\mathrm{tr}} \\
&= 0,
\end{aligned}$$

and thus $\ker\delta = \mathrm{im}(\sigma-1)$ as desired.

We see δ is surjective because for all $\alpha \in \mathrm{Mat}_{d\times 1}(k) : \delta(\alpha^{\mathrm{tr}}\sigma^0) = \alpha$.

The left square of the diagram obviously commutes so we need to check it for the square at the right. Write

$$\mathrm{m} = \sum_{i=0}^{\infty}\alpha_{(i)}\sigma^i \in \mathrm{Mat}_{1\times d}(k[\sigma]),\ f = \sum_{j=0}^{\infty} f_{(j)}\tau^j \in \mathrm{Mat}_{e\times d}(k[\tau])$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k),\ \alpha_{(i)} = 0$ for $i \gg 0$ and $f_{(j)} \in \mathrm{Mat}_{e\times d}(k),\ f_{(j)} = 0$ for $j \gg 0$. Then $f^\dagger = \sum_{j=0}^{\infty}(f_{(j)}^{(-j)})^{\mathrm{tr}}\sigma^j$ and

$$\begin{aligned}
\delta\left(f^\dagger(m)\right) &= \delta\left(\left(\sum_{i=0}^{\infty}\alpha_{(i)}\sigma^i\right)\left(\sum_{j=0}^{\infty}(f_{(j)}^{(-j)})^{\mathrm{tr}}\sigma^j\right)\right) = \delta\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{(i)}(f_{(j)}^{(-i-j)})^{\mathrm{tr}}\sigma^{i+j}\right) \\
&= \left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f_{(j)}(\alpha_{(i)}^{(i+j)})^{\mathrm{tr}}\right) = \sum_{j=0}^{\infty} f_{(j)}\left(\sum_{i=0}^{\infty}(\alpha_{(i)}^{(i)})^{\mathrm{tr}}\right)^{(j)} \\
&= f(\delta(\mathrm{m})).
\end{aligned}$$

□

We want to get a similar isomorphism to the Zariski tangent space of $\mathbb{G}_{a,k}$. In order to do this, denote the image of $e \in \mathrm{Hom}_k(\mathrm{Spec}\, k, \mathbb{G}_{a,k})$ under the isomorphism

$$\mathbb{G}_{a,k}(k) = \mathrm{Hom}_k(\mathrm{Spec}\, k, \mathbb{G}_{a,k}) \xrightarrow{\sim} \{x \in \mathbb{G}_{a,k} : \kappa(x) = k.\}$$

by $0 \in \mathbb{G}_{a,k}$. Then the local ring $\mathbb{G}_{a,k,0} = k[x]_{(x)}$ of $\mathbb{G}_{a,k}$ at 0 has the maximal ideal $\mathfrak{m}_0 = (x)$. The *Zariski cotangent space to* $\mathbb{G}_{a,k}$ is $\mathfrak{m}_0/\mathfrak{m}_0^2 = kx$ and the *Zariski tangent space to* $\mathbb{G}_{a,k}$ *at the identity* is

$$\mathrm{T}_0\mathbb{G}_{a,k} := \mathrm{Hom}_k(\mathfrak{m}_0/\mathfrak{m}_0^2, k) = k\check{x},$$

so that $\mathrm{T}_0\mathbb{G}_{a,k}$ is a vector space of dimension 1 over k. An $f \in \mathrm{Hom}_k(\mathbb{G}_{a,k}, \mathbb{G}_{a,k})$, then induces a map f^* on the local rings and hence on the Zariski cotangent space to $\mathbb{G}_{a,k}$ at the identity. The dual of that map is

$$\mathrm{T}_0 f := (f^*)^\vee : k\check{x} \to k\check{x}.$$

We can carry [Har08, Prop. 1.1.6] over to $\mathbb{G}_{a,k}^d$, so that $\mathrm{T}_0\mathbb{G}_{a,k}^d = (k\check{x})^{\oplus d} = \mathrm{Mat}_{d\times 1}(k)$ and an $f \in \mathrm{Hom}_k(\mathbb{G}_{a,k}^d, \mathbb{G}_{a,k}^e)$ then induces a map f^* on the cotangent spaces at the identity and moreover $\mathrm{T}_0 f = (f^*)^\vee : (k\check{x})^{\oplus d} \to (k\check{x})^{\oplus e}$ so that

$$\mathrm{T}_0 f := \alpha_{(0)} \quad \text{with } \alpha_{(i)} \in \mathrm{Mat}_{e\times d}(k).$$

The following lemma will also be needed to prove the equivalence of the two categories.

Lemma 1.1.22 ([ABP, §1.2.8])**.** *Regard* $\mathrm{Mat}_{d\times 1}(k)$ *and* $\mathrm{Mat}_{e\times 1}(k)$ *as Zariski tangent spaces to* $\mathbb{G}_{a,k}^d$ *and* $\mathbb{G}_{a,k}^e$ *respectively, at the identity and let* $\delta_0 : \mathrm{Mat}_{1\times d}(k[\sigma]) \to \mathrm{Mat}_{d\times 1}(k)$ *be given by*

$$\delta_0\left(\sum_{i=0}^{n} \alpha_{(i)}\sigma^i\right) := \alpha_{(0)}^{\mathrm{tr}}.$$

Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Mat}_{1\times d}(k[\sigma]) & \xrightarrow{\sigma\cdot} & \mathrm{Mat}_{1\times d}(k[\sigma]) & \xrightarrow{\delta_0} & \mathrm{Mat}_{d\times 1}(k) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \cdot f^\dagger} & & \Big\downarrow{\scriptstyle \cdot f^\dagger} & & \Big\downarrow{\scriptstyle \mathrm{T}_0 f} & & \\
0 & \longrightarrow & \mathrm{Mat}_{1\times e}(k[\sigma]) & \xrightarrow{\sigma\cdot} & \mathrm{Mat}_{1\times e}(k[\sigma]) & \xrightarrow{\delta_0} & \mathrm{Mat}_{e\times 1}(k) & \longrightarrow & 0
\end{array}$$

for every $f \in \mathrm{Mat}_{e\times d}(k[\tau])$*, where daggering identifies* $\mathrm{Mat}_{1\times d}(k[\sigma])$ *and* $\mathrm{Mat}_{1\times e}(k[\sigma])$ *with* $\mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k}^d)$ *and* $\mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, \mathbb{G}_{a,k}^e)$*, respectively.*

Proof. Since σ is clearly injective, let us show that $\ker\delta_0 = \mathrm{im}\,\sigma$ holds. Suppose

$$\mathrm{m} = \sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i \in \ker\delta_0$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k),\ \alpha_{(i)} = 0$ for $i \gg 0$. Then $\delta_0(\mathrm{m}) = \alpha_{(0)}^{\mathrm{tr}} = 0$, hence $\alpha_{(0)} = 0$ and

$$\mathrm{m} = \sum_{i=1}^{\infty} \alpha_{(i)}\sigma^i = \sigma\left(\sum_{i=0}^{\infty} \alpha_{(i+1)}\sigma^i\right) \in \mathrm{im}\,\sigma.$$

Thus $\ker\delta_0 \subseteq \mathrm{im}\,\sigma$. Let

$$\mathrm{m} = \sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i \in \ker\delta_0$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k),\ \alpha_{(i)} = 0$ for $i \gg 0$. Then

$$\begin{aligned}
\delta_0(\sigma(\mathrm{m})) &= \delta_0\left(\sigma\left(\sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i\right)\right) = \delta_0\left(\sum_{i=0}^{\infty} \alpha_{(i)}^{(-1)}\sigma^{i+1}\right) \\
&= 0,
\end{aligned}$$

and thus $\mathrm{im}\,\sigma = \ker\delta_0$ as desired.

Notice δ_0 is surjective because for all $\alpha \in \mathrm{Mat}_{d\times 1}(k) : \delta_0(\alpha^{\mathrm{tr}}\sigma^0) = \alpha$.

The left square of the diagram commutes clearly, so it remains to check commutativity of the square at the right side. Write

$$\mathrm{m} = \sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i \in \ker\delta_0,\ \ f = \sum_{j=0}^{\infty} f_{(j)}\tau^j \in \mathrm{Mat}_{e\times d}(k[\tau])$$

with $\alpha_{(i)} \in \mathrm{Mat}_{1\times d}(k)$, $\alpha_{(i)} = 0$ for $i \gg 0$ and $f_{(j)} \in \mathrm{Mat}_{e\times d}(k)$, $f_{(j)} = 0$ for $j \gg 0$. Then $f^\dagger = \sum_{j=0}^{\infty} (f_{(j)}^{(-j)})^{\mathrm{tr}} \sigma^j \in \mathrm{Mat}_{d\times e}(k[\sigma])$ and

$$\begin{aligned}\delta_0(f^\dagger(\mathrm{m})) &= \delta_0\left(\left(\sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i\right)\left(\sum_{j=0}^{\infty} (f_{(j)}^{(-j)})^{\mathrm{tr}}\sigma^j\right)\right) = \delta_0\left(\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{(i)} (f_{(j)}^{(-(i+j))})^{\mathrm{tr}} \sigma^{i+j}\right) \\ &= f_{(0)}\alpha_{(0)}^{\mathrm{tr}} = \mathrm{T}_0 f(\delta_0(\mathrm{m})). \qquad \square\end{aligned}$$

We want to carry the previous results over to arbitrary additive algebraic group schemes. We therefore make the following definition:

Definition 1.1.23 (Cf. [Gos96, 5.9.4]). Let $E \cong \mathbb{G}_{a,k}^d$ be an additive algebraic group scheme over k. A *coordinate system for* E with $E \cong \mathbb{G}_{a,k}^d$ is an isomorphism of algebraic groups

$$\rho = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_d \end{pmatrix} : E \to \mathbb{G}_{a,k}^d.$$

Then ρ induces an isomorphism

$$\mathrm{T}_0\rho : \mathrm{T}_0 E \xrightarrow{\sim} \mathrm{T}_0\mathbb{G}_{a,k}^d \cong \mathrm{Mat}_{e\times 1}(k).$$

By fixing such coordinate systems ρ_1 and ρ_2 for $E \cong \mathbb{G}_{a,k}^d$ and $E' \cong \mathbb{G}_{a,k}^d$ respectively, we get the following identification:

$$\mathrm{Hom}_{k,\mathbb{F}_q}(E, E') \cong \mathrm{Mat}_{e\times d}(k[\tau]) \xrightarrow{\dagger} \mathrm{Mat}_{d\times e}(k[\sigma]), \tag{1.2}$$

so we may in particular also apply Lemma 1.1.22 and Lemma 1.1.21 to additive algebraic group schemes.

1.2 Tannakian theory

As mentioned in the introduction, the goal of this thesis is to study relations between the Tannakian category over Q generated by a pure dual t-motive and the Tannakian category over Q generated by a pure Q-Hodge-Pink structure defined by Papanikolas in [Pap08] and Pink in [Pin97a]. This kind of category was studied by Deligne in [DMOS82] and [Del90] and we recall Deligne's definition of a neutral Tannakian category over a field K with a fiber functor ω and the properties of Tannakian categories that are of importance to us. In particular, the priciple "Tannakian duality" is explained in form of Theorem 1.2.10. This implies the existence of an affine group scheme G corresponding to such a neutral Tannakian category over K so that this category is equivalent to the category of finite-dimensional representations of G over K. In Chapter 4 we make use of Proposition 1.2.15 to show that the affine group schemes associated with the category generated by a pure dual t-motive and the category generated by a pure Q-Hodge-Pink structure are isomorphic and the two categories are equivalent.

1.2.1 Rigid abelian tensor categories

A Tannakian category is in particular a rigid abelian tensor category. Let us first recall the definition of additive and abelian categories.

Definition 1.2.1. (i) An *additive category* is a category $\mathscr{C}$ such that

(a) $\mathrm{Hom}(X,Y)$ is endowed with an abelian group structure for all objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, such that composition of morphisms is bilinear,

(b) $\mathscr{C}$ has a final object A which is also an initial object, so that $\mathrm{Hom}(A,A) = 0$,

(c) the product of objects X, Y in $\mathscr{C}$ is again an object in $\mathscr{C}$; then it also has a sum which is isomorphic to the product of the two.

(ii) An additive category $\mathscr{C}$ is said to be *abelian* if

(a) For all morphisms $f : X \to Y$, the morphisms f and 0 have a kernel and a cokernel, so that there is an exact sequence

$$\begin{tikzcd} N \arrow[r, "j"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y \arrow[r, "p"] & K. \end{tikzcd}$$

One defines $\ker f := j$ and $\operatorname{coker} f := p$ and, by abuse of notation, we denote N as $\ker f$ and K as $\operatorname{coker} f$. N is a called a *subobject* of X and K a *quotient object* of Y.

(b) Every monomorphism is of the form $\ker f$ and every epimorphism of the form $\operatorname{coker} f$.

The basic examples of an abelian category are the category of abelian groups and the category $\mathscr{M}od_R$ of finitely generated R-modules where R is a commutative ring with 1 as usual. We follow Deligne and require the functor $\otimes$ underlying a tensor category to satisfy the compatibility condition *with ACU*:

Definition 1.2.2. Let $\mathscr{C}$ be a category and $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ a functor that maps (X,Y) to $X \otimes Y$. We say that $(\mathscr{C}, \otimes)$, or by abuse of notation $\mathscr{C}$, is a *tensor category* if

(a) $(\mathscr{C}, \otimes)$ has an *identity object*; that is, a pair $(\mathbb{1}, e)$ consisting of an object $\mathbb{1} \in \mathrm{Ob}(\mathscr{C})$ and an isomorphism $e : \mathbb{1} \xrightarrow{\sim} \mathbb{1} \otimes \mathbb{1}$ such that the functor $X \mapsto \mathbb{1} \otimes X : \mathscr{C} \to \mathscr{C}$ is an equivalence of categories,

(b) there is an *associativity constraint* φ *for* $(\mathscr{C}, \otimes)$; that is, a functorial isomorphism

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$$

such that for all $X, Y, Z, T \in \mathrm{Ob}(\mathscr{C})$ the following diagram commutes:

$$\begin{tikzcd} X \otimes (Y \otimes (Z \otimes T)) \arrow[r, "\varphi"] \arrow[d, "1\otimes\varphi"'] & (X \otimes Y) \otimes (Z \otimes T) \arrow[r, "\varphi"] & ((X \otimes Y) \otimes Z) \otimes T \arrow[d, "\varphi\otimes 1"] \\ X \otimes ((Y \otimes Z) \otimes T) \arrow[rr, "\varphi"] & & (X \otimes (Y \otimes Z)) \otimes T, \end{tikzcd}$$

(c) there is a *commutativity constraint ψ for* $(\mathscr{C},\otimes)$; that is, a functorial isomorphism

$$\psi_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

such that

$$\psi_{Y,X} \circ \psi_{X,Y} = \mathrm{id}_{X\otimes Y} : X \otimes Y \to X \otimes Y \quad \text{for all } X, Y \in \mathrm{Ob}(\mathscr{C}),$$

and

(d) the associativity constraint φ and commutativity constraint ψ are *compatible*, which means that for all $X, Y, Z \in \mathrm{Ob}(\mathscr{C})$ the following diagram commutes:

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\varphi"] \arrow[d, "1\otimes\psi"'] & (X \otimes Y) \otimes Z \arrow[r, "\psi"] & Z \otimes (X \otimes Y) \arrow[d, "\varphi"] \\
X \otimes (Z \otimes Y) \arrow[r, "\varphi"] & (X \otimes Z) \otimes Y \arrow[r, "\psi\otimes 1"] & (Z \otimes X) \otimes Y.
\end{tikzcd}$$

It is easily shown that an identity element of a tensor category is unique up to unique isomorphisms [DMOS82, Prop. II.1.3]. Moreover, a *tensor subcategory* $(\mathscr{C}',\otimes)$ is a full subcategory $\mathscr{C}'$ of a tensor category $(\mathscr{C},\otimes)$ that contains an identity object of $\mathscr{C}$, which is closed under the formation of tensor product; that is, $X_1 \otimes X_2 \in \mathrm{Ob}(\mathscr{C}')$ for all $X_1, X_2 \in \mathrm{Ob}(\mathscr{C}')$.

Definition 1.2.3. A tensor category $(\mathscr{C},\otimes)$ is called *additive (resp. abelian)* if

(a) $\mathscr{C}$ is an additive (resp. abelian) category and

(b) $\otimes$ is a bi-additive functor.

One criterion for a tensor category to be rigid is that there exists for all objects X and Y in $\mathscr{C}$ a special object $\mathcal{H}om(X,Y)$ called *inner hom* that leads to the definition of duals of objects as follows:

Definition 1.2.4. Let $(\mathscr{C},\otimes)$ be a tensor category and consider the contravariant functor $F : \mathscr{C}^\circ \to \mathscr{Sets}$, $T \mapsto \mathrm{Hom}(T \otimes X, Y)$ for $X,\ Y \in \mathrm{Ob}(\mathscr{C})$.

(i) If F is representable by an object of $\mathscr{C}$, then we denote it by $\mathcal{H}om(X,Y)$ and the universal family corresponding to $\mathrm{id}_{\mathcal{H}om(X,Y)}$ by

$$\mathrm{ev}_{X,Y} \in F(\mathcal{H}om(X,Y)) = \mathrm{Hom}(\mathcal{H}om(X,Y) \otimes X, Y).$$

(ii) If $\mathcal{H}om(X,Y)$ represents F, one defines the *dual* $X^\vee$ *of* X to be $\mathcal{H}om(X,\mathbb{1})$.

To see why one writes $\mathrm{ev}_{X,Y}$ for the universal family, let us take a look at inner hom's in the category $\mathscr{Mod}_R$. The inner hom of two R-modules M and N is the R-module $\mathrm{Hom}_R(M,N)$ and the universal family is then the evaluation homomorphism $\mathrm{ev}_{M,N}(f \otimes m) = f(m)$. Moreover, the dual of an R-module M is $M^\vee = \mathrm{Hom}_R(M,R)$.

Let X and Y be objects in an arbitrary tensor category $(\mathscr{C},\otimes)$. $\mathcal{H}om(X,Y)$ represents F means there is a functorial isomorphism $h_{\mathcal{H}om(X,Y)} \xrightarrow{\sim} F$, providing the *adjunction formula*

$$\mathrm{Hom}(Z, \mathcal{H}om(X,Y)) = h_{\mathcal{H}om(X,Y)}(Z) \cong F(Z) = \mathrm{Hom}(Z \otimes X, Y) \tag{1.3}$$

for all $Z \in \mathrm{Ob}(\mathscr{C})$. More precisely, under this isomorphism an $f \in \mathrm{Hom}(Z, \mathcal{H}om(X,Y))$ gets mapped to $\mathrm{ev}_{X,Y} \circ (f \otimes \mathrm{id}) \in \mathrm{Hom}(Z \otimes X, Y)$. If we take $Y = \mathbb{1}$, then $\mathrm{ev}_{X,\mathbb{1}} : X^\vee \otimes X \to \mathbb{1}$ induces an isomorphism

$$\mathrm{Hom}(Z, X^\vee) = \mathrm{Hom}(Z, \mathcal{H}om(X, \mathbb{1})) \cong \mathrm{Hom}(Z \otimes X, \mathbb{1})$$

for all $Z \in \mathrm{Ob}(\mathscr{C})$. Similarly, we have

$$\mathrm{Hom}(X, (X^\vee)^\vee) = \mathrm{Hom}(X, \mathcal{H}om(X^\vee, \mathbb{1})) \cong \mathrm{Hom}(X \otimes X^\vee, \mathbb{1}),$$

and we define $i_X : X \to (X^\vee)^\vee$ to be the map corresponding to $\mathrm{ev}_{X,\mathbb{1}} \circ \psi : X \otimes X^\vee \to X^\vee \otimes X \to \mathbb{1}$.

Consider now finite families $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ of objects in $\mathscr{C}$ and the morphism

$$(\otimes_{i\in I} \mathcal{H}om(X_i, Y_i)) \otimes (\otimes_{i\in I} X_i) \longrightarrow \otimes_{i\in I}(\mathcal{H}om(X_i, Y_i) \otimes X_i) \xrightarrow{\mathrm{ev}_{X_i,Y_i}} \otimes_{i\in I} Y_i.$$

By the adjunction formula, there is a corresponding morphism

$$t_{\otimes_{i\in I}X_i, \otimes_{i\in I}Y_i} : \otimes_{i\in I} \mathcal{H}om(X_i, Y_i) \longrightarrow \mathcal{H}om(\otimes_{i\in I}X_i, \otimes_{i\in I}Y_i).$$

Definition 1.2.5. Let $(\mathscr{C}, \otimes)$ be a tensor category. We say $(\mathscr{C}, \otimes)$ is *rigid* if

(a) $\mathcal{H}om(X,Y)$ exists for all objects X, Y in $\mathscr{C}$,

(b) $t_{\otimes_{i\in I}X_i, \otimes_{i\in I}Y_i} : \otimes_{i\in I} \mathcal{H}om(X_i, Y_i) \to \mathcal{H}om(\otimes_{i\in I}X_i, \otimes_{i\in I}Y_i)$ is an isomorphism, and

(c) each object X in $\mathscr{C}$ is *reflexive*; that is, $i_X : X \to (X^\vee)^\vee$ is an isomorphism.

To ease notation, we denote the n-fold tensor power of an object X in a tensor category by $X^{\otimes n}$ or X^n.

Definition 1.2.6. An object X in a tensor category $(\mathscr{C}, \otimes)$ is *invertible* if the functor $(Y \mapsto Y \otimes X) : \mathscr{C} \to \mathscr{C}$ is an equivalence of categories.

Observe that $X \in \mathrm{Ob}(\mathscr{C})$ is invertible if and only if there is an object $X^{-1} \in \mathrm{Ob}(\mathscr{C})$ such that $X^{-1} \otimes X \cong \mathbb{1}$. From the adjunction formula follows that if X is invertible, then X is reflexive and $X^{-1} \cong X^\vee$. Motivated by this, we write $X^{-n} := (X^\vee)^n$ for the n-fold tensor power of its dual $X^\vee$.

1.2.2 Tensor functors and morphisms between them

Having defined tensor categories, we want to look at functors between such categories, called tensor functors.

Definition 1.2.7. Let $(\mathscr{C}, \otimes_1)$ and $(\mathscr{D}, \otimes_2)$ be tensor categories. A *tensor functor* $(\mathscr{C}, \otimes_1) \to (\mathscr{D}, \otimes_2)$ is a pair (F, c), such that $F : \mathscr{C} \to \mathscr{D}$ is a functor and

$$c_{X,Y} : F(X) \otimes_2 F(Y) \xrightarrow{\sim} F(X \otimes_1 Y)$$

is a functorial isomorphism satisfying the compatibility condition with ACU.

We now extend the definition of functorial morphisms between functors to morphisms of tensor functors.

Definition 1.2.8. Let $(F,c),(G,d):\mathscr{C}\to\mathscr{D}$ be tensor functors.

(i) A *morphism* $(F,c)\to(G,d)$ *of tensor functors* is a functorial morphism $\eta: F\to G$ such that there is a commutative diagram

$$\begin{tikzcd} \otimes_{i\in I}F(X_i) \arrow[r, "c"] \arrow[d, "\otimes\eta_{X_i}"'] & F(\otimes_{i\in I}X_i) \arrow[d, "\eta_{\otimes X_i}"] \\ \otimes_{i\in I}G(X_i) \arrow[r, "c"] & G(\otimes_{i\in I}X_i) \end{tikzcd}$$

for all finite families $(X_i)_{i\in I}$ with $X_i\in\mathrm{Ob}(\mathscr{C})$. We denote the set of all morphisms $(F,c)\to(G,d)$ of tensor functors by $\mathrm{Hom}^{\otimes}(F,G)$.

(ii) An *automorphism* $(F,c)\to(F,c)$ *of tensor functors* is a morphism of tensor functors whose underlying functorial morphism is a functorial isomorphism. We write $\mathrm{Aut}^{\otimes}(F)$ for the set of all automorphism of tensor functors of (F,c).

Morphisms between tensor functors give rise to a functor of K-algebras that is crucial to the definition of the affine group scheme associated with a Tannakian category and its fiber functor.

Definition 1.2.9. Let R be a K-algebra and $\mathscr{V}ec_K$ the category of finite-dimensional K-vector spaces. We define a canonical tensor functor $\psi_R:\mathscr{V}ec_K\to\mathscr{M}od_R$ by sending a K-vector space V to $V\otimes_K R$. If $(F,c),(G,d)$ are tensor functors from a tensor category $\mathscr{C}$ to $\mathscr{V}ec_K$, then the functor $\underline{\mathrm{Hom}}^{\otimes}(F,G)$ of K-algebras is given by

$$\underline{\mathrm{Hom}}^{\otimes}(F,G)(R)=\mathrm{Hom}^{\otimes}(\psi_R\circ F,\psi_R\circ G)\quad\text{for all } K\text{-algebras } R.$$

Similarly, we define a functor $\underline{\mathrm{Aut}}^{\otimes}(F)$ of K-algebras by the rule

$$\underline{\mathrm{Aut}}^{\otimes}(F)(R)=\mathrm{Aut}^{\otimes}(\psi_R\circ F)\quad\text{for all } K\text{-algebras } R.$$

Deligne shows that the category $\mathscr{R}ep_K(G)$ of finite-dimensional representations of an affine group scheme G over K is a rigid abelian tensor category with $\mathrm{End}(\mathbb{1})=K$ and one has clearly a forgetful functor $\omega^G:\mathscr{R}ep_K(G)\to\mathscr{V}ec_K$ that is exact, faithful and K-linear [DMOS82, Exmp. II.1.24]. The following theorem, which applies to neutral Tannakian categories, says that there is an affine group scheme G to any rigid abelian tensor category $\mathscr{C}$ with such a functor so that $\mathscr{C}$ and $\mathscr{R}ep_K(G)$ are equivalent.

Theorem 1.2.10 ([DMOS82, Thm. II.2.11])**.** *Let $\mathscr{C}$ be a rigid abelian tensor category such that $K=\mathrm{End}(\mathbb{1})$ is a field and $\omega:\mathscr{C}\to\mathscr{V}ec_K$ an exact faithful K-linear tensor functor. Then*

(i) the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ of K-algebras is representable by an affine group scheme G over K;

(ii) ω defines an equivalence of tensor categories $\mathscr{C}\to\mathscr{R}ep_K(G)$.

1.2.3 Neutral Tannakian categories

The general definition of a Tannakian category over K is given in [DMOS82, Def. II.3.7], but we renounce it here since all Tannakian categories in this thesis are neutral.

Definition 1.2.11. A *(neutral) Tannakian category $\mathcal{T}$ over K* is a rigid abelian tensor category $\mathcal{T}$ with $\mathrm{End}(\mathbb{1}) = K$ and a K-linear exact faithful tensor functor $\omega : \mathcal{T} \to \mathscr{V}\!ec_K$ which one calls the *fiber functor of $\mathcal{T}$*.

Thus by *Tannakian duality* (Theorem 1.2.10) any neutral Tannakian category with fiber functor ω is equivalent to the category of finite-dimensional representations of an affine group scheme G that represents $\underline{\mathrm{Aut}}^{\otimes}(\omega)$. In particular, $\mathscr{R}\!ep_K G$ is a neutral Tannakian category with fiber functor ω^G and we are interested in how far properties of G correspond to representations in $\mathscr{R}\!ep_K(G)$.

Lemma 1.2.12 ([DMOS82, Cor. II.2.4])**.** *An affine group scheme G over K is algebraic if and only if it has a faithful finite-dimensional representation over K.*

There is another equivalent condition for an affine K-group scheme to be algebraic. We say that an object X in a tensor category $\mathscr{C}$ is a *tensor generator of $\mathscr{C}$* if every object in $\mathscr{C}$ is isomorphic to a subquotient of a finite direct sum of an object of the form $X^{\otimes m} \otimes (X^{\vee})^{\otimes n}$.

Lemma 1.2.13 ([DMOS82, Lem. II.2.20])**.** *An affine K-group scheme G is algebraic if and only if there exists a object X in $\mathscr{R}\!ep_K(G)$ that is a tensor generator for $\mathscr{R}\!ep_K(G)$.*

Moreover, a homomorphism $f : G \to G'$ of affine K-group schemes induces a functor $\omega^f : \mathscr{R}\!ep_K(G') \to \mathscr{R}\!ep_K(G)$ such that $\omega^G \circ \omega^f = \omega^{G'}$ by sending a representation $\rho : G' \to \mathrm{GL}(V)$ to $\rho \circ f : G \to G' \to \mathrm{GL}(V)$. Then one has the following result of Deligne:

Lemma 1.2.14 ([DMOS82, Cor. 2.9])**.** *Let G and G' be affine K-group schemes and let $F : \mathscr{R}\!ep_K(G') \to \mathscr{R}\!ep_K(G)$ be a tensor functor such that $\omega^G \circ F = \omega^{G'}$. Then there is a unique homomorphism $f : G \to G'$ of affine K-group schemes such that $F \cong \omega^f$.*

We later construct a homomorphism between the Galois group and Hodge-Pink group that is an isomorphism if it is a closed immersion and faithfully flat. The following proposition gives us equivalent conditions for this that we will make use of.

Proposition 1.2.15 ([DMOS82, Prop. II.2.21])**.** *Let $f : G \to G'$ be a homomorphism of affine K-group schemes and $\omega^f : \mathscr{R}\!ep_K(G') \to \mathscr{R}\!ep_K(G)$ be defined as above.*

(i) f is faithfully flat if and only if

  (a) ω^f is fully faithful and

  (b) each subobject of $\omega^f(X')$ is isomorphic to the image of a subobject of an object X' in $\mathscr{R}\!ep_K(G')$.

(ii) f is a closed immersion if and only if there exists an object X' in $\mathscr{R}\!ep_K(G')$ for every object X of $\mathscr{R}\!ep_K(G)$ such that X is isomorphic to a subquotient of $\omega^f(X')$.

Corollary 1.2.16. *A homomorphism $f : G \to G'$ of K-group schemes is an isomorphism if and only if the induced functor $\omega^f : \mathscr{R}\!ep_K(G') \to \mathscr{R}\!ep_K(G)$ is fully faithful and essentially surjective and thus an equivalence of categories.*

1.3 Rigid analytic geometry

We give a short introduction to rigid analytic geometry adjusted to our purpose to establish notation and explain the rigid analytic analog of Serre's GAGA principle. In order to get back from a sub-Q-Hodge-Pink structure to a sub-dual Anderson A-motive, we need to define F-modules which live on Tate's rigid analytic spaces. Roughly speaking we find through those F-modules an analytic inclusion between rigid sheaves and from the rigid analytic GAGA principle we obtain an algebraic inclusion of algebraic sheaves. So let us explain the basic ideas of classical rigid geometry; a full account of the material can for example be found in the standard references [BGR84] and [FvdP04].

One important aspect of rigid analytic geometry is the study of globally convergent power series expansions called *analytic functions* over complete fields with a non-Archimedian absolute value $|\cdot|$. Whereas one may study holomorphic functions in complex analysis that are analytic, the study of "analytic functions" faces problems in non-Archimedian analysis. For example the topology of a non-Archimedian valued field is totally disconnected and if a function has locally convergent power series expansions in a neighborhood of each point of its domain, it need not have a globally convergent power series expansion. What today one calls *classical rigid geometry* is the theory of rigid (analytic) K-spaces that was developed by Tate in [Tat71].

We suppose that K is a complete field with a non-Archimedian absolute value $|\cdot|$ and let $\overline{K}$ be an algebraic closure on which one has the unique extension of $|\cdot|$ that is complete on each finite field extension K'/K with $K \subseteq K' \subseteq \overline{K}$. One shows that a formal power series

$$f = \sum_{i\in\mathbb{N}^n} \alpha_i t^i = \sum_{i\in\mathbb{N}^n} \alpha_{i_1\ldots i_n} t_1^{i_1} \cdot \ldots \cdot t_n^{i_n} \in K[\![t_1,\ldots,t_n]\!]$$

converges on the *unit disk in* $\overline{K}$

$$\mathfrak{D}_{\overline{K}} := \{(x_1,\ldots,x_n) \in \overline{K} : \ |x_i| \leq 1 \text{ for } i = 1,\ldots,n\}$$

if and only if $\lim_{|i|\to\infty} |\alpha_i| = 0$. This leads to the definition of the K-algebra consisting of the convergent power series on $\mathfrak{D}_{\overline{K}}$

$$\mathbb{T} := K\langle t_1,\ldots,t_n\rangle := \{\sum_{i\in\mathbb{N}^n} \alpha_i t^i \in K[\![t_1,\ldots,t_n]\!] : \lim_{|i|\to\infty} |\alpha_i| = 0\},$$

called the *Tate algebra of restricted power series.* Note that elements of $\mathbb{T}$ may be interpreted as functions $\mathfrak{D}_{\overline{K}} \to \overline{K}$. A K-algebra A is called an *affinoid K-algebra* if there is an epimorphism $\phi : \mathbb{T} \to A$ and one may regard an element in A as a function on $\operatorname{Max} A := \{\text{maximal ideals in } A\}$. We may define closed subsets $V(\mathfrak{a})$ for ideals $\mathfrak{a} \in A$ and thus the Zariski topology in the usual way on $\operatorname{Max} A$. Similar to the definition of affine K-schemes we then define an *affinoid rigid space* $\operatorname{Sp} A$ to be the set $\operatorname{Sp} A := \operatorname{Max} A$ together with its ring of functions A. One may associate a *presheaf of affinoid functions* $\mathcal{O}_{\operatorname{Sp} A}$ to an affinoid K-space that does not satisfy sheaf properties since the topology of K is totally disconnected. Instead one works with a Grothendieck topology on $\operatorname{Sp} A$.

Corresponding to the notion of K-schemes in algebraic geometry, a *rigid (analytic) K-space* is roughly speaking a pair $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ such that $\mathfrak{X}$ is a topological space with a Grothendieck topology that admits an admissible covering by affinoid K-spaces and $\mathcal{O}_{\mathfrak{X}}$ is a sheaf of K-algebras on it (cf. [FvdP04, Def. 4.3.1]). Let us now define the other rigid analytic K-spaces that we need throughout the thesis. We let $\theta \in K$ with $|\theta| > 1$, $n > 0$, and define the affinoid K-algebras

$$\begin{aligned}
K\langle \frac{t}{\theta^n} \rangle &:= \{\sum_{i=0}^{\infty} \alpha_i t^i \in K[\![t]\!] : \lim_{i\to\infty} |\alpha_i \theta^{ni}| = 0\}, \\
\mathbb{E} := K\{t\} &:= \bigcap_{n\to\infty} K\langle \frac{t}{\theta^n} \rangle \\
&= \{\sum_{i=0}^{\infty} \alpha_i t^i \in K[\![t]\!] : \lim_{i\to\infty} \frac{\log|\alpha_i|}{i} = -\infty\},
\end{aligned}$$

whose elements converge globally on the disk centered at $t = 0$ with radius $|\theta|^n$

$$\mathfrak{D}(\theta^n)_{\overline{K}} := \operatorname{Sp} K\langle \frac{t}{\theta^n} \rangle = \{x \in \overline{K} : |x| \leq |\theta|^n\},$$

and on all of $\overline{K}$

$$\mathfrak{D}(\infty)_{\overline{K}} := \operatorname{Sp} K\{t\} = \{x \in \overline{K}\},$$

respectively. Further, set $z := \frac{1}{t}$ and $\zeta := \frac{1}{\theta}$. For $n' \geq n > 0$, we consider the affinoid K-algebras

$$\begin{aligned}
K\langle \frac{z}{\zeta^n} \rangle &:= \{\sum_{i=0}^{\infty} \alpha_i z^i \in K[\![z]\!] : \lim_{i\to\infty} |\alpha_i \zeta^{ni}| = 0\}, \\
K\langle \frac{z}{\zeta^n}, \frac{\zeta^{n'}}{z} \rangle &:= \{\sum_{i=-\infty}^{\infty} \alpha_i z^i : \lim_{i\to\infty} |\alpha_i \zeta^{ni}| = 0,\ \lim_{i\to-\infty} |\alpha_i \zeta^{n'i}| = 0\}, \\
K\langle \frac{z}{\zeta^n}, z^{-1}\} &:= \bigcap_{n'\to\infty} K\langle \frac{z}{\zeta^n}, \frac{\zeta^{n'}}{z} \rangle \\
&= \{\sum_{i=-\infty}^{\infty} \alpha_i z^i : \lim_{i\to\pm\infty} |\alpha_i \zeta^{n'i}| = 0 \text{ for all } n' \geq n\}, \\
K\{z, z^{-1}\} &:= \bigcap_{n\to 0} K\langle \frac{z}{\zeta^n}, z^{-1}\} \\
&= \{\sum_{i=-\infty}^{\infty} \alpha_i z^i : \lim_{i\to\pm\infty} |\alpha_i \zeta^{ni}| = 0 \text{ for all } n > 0\},
\end{aligned}$$

whose elements converge globally on

$$\begin{aligned}
\mathfrak{D}(\zeta^n)^{\infty}_{\overline{K}} &:= \operatorname{Sp} K\langle \frac{z}{\zeta^n}\rangle = \{x \in \overline{K} : |x| \le |\zeta|^n\} \\
&= \text{the disk centered at } \infty \text{ with radius } |\zeta|^n, \\
\mathfrak{D}(\zeta^n, \zeta^{n'})^{\infty}_{\overline{K}} &:= \operatorname{Sp} K\langle \frac{z}{\zeta^n}, \frac{\zeta^{n'}}{z}\rangle = \{x \in \overline{K} : |\zeta|^{n'} \le |x| \le |\zeta|^n\} \\
&= \text{the annulus centered at } \infty \text{ with inner radius } |\zeta|^{n'} \text{ and outer radius } |\zeta|^n, \\
\dot{\mathfrak{D}}(\zeta^n)^{\infty}_{\overline{K}} &:= \operatorname{Sp} K\langle \frac{z}{\zeta^n}, z^{-1}\} = \{x \in \overline{K} : 0 < |x| \le |\zeta|^n\} \\
&= \text{the } \textit{punctured disk} \text{ centered at } \infty \text{ with radius } |\zeta|^n, \\
\dot{\mathfrak{D}}^{\infty}_{\overline{K}} &:= \operatorname{Sp} K\{z, z^{-1}\} = \{x \in \overline{K} : 0 < |x| < 1\} \\
&= \text{the } \textit{punctured unit disk} \text{ centered at } \infty,
\end{aligned}$$

respectively. We define a norm $\|f\|_{\theta^n} := \max_{i\in\mathbb{Z}} |\alpha_i \theta^{ni}|$ for a convergent Laurent series f on $\mathfrak{A}(\theta^n, \theta^n)_{\overline{K}}$ so that $||f||_{\theta^n} < \infty$ holds by definition.

The functions in $\mathbb{E}$ converging on all of $\overline{K}$ are called *entire functions* [Gos96, Def. 2.12]. Note that our definitions of $\mathbb{E}$ and $\dot{\mathfrak{D}}^{\infty}_{\overline{K}}$ are equivalent to the ones given in [Pap08, §2.2.4] and [HP04, §1], respectively.

Furthermore, one may also translate the definitions of separated and proper morphisms, open and closed immersions, and coherent $\mathcal{O}_{\mathfrak{X}}$-modules from algebraic geometry.

But not only definitions are similar, there are in fact close relations between K-schemes and rigid K-spaces. It will be of importance to us that there is a functor from the category of K-schemes of locally finite type to the category of rigid K-spaces, assigning a *rigid analytification* X^{rig} to each K-scheme X of locally finite type [Bos, §1.13 Prop. 4]. To give an example, let $|\theta| > 1$. One can show that the rigid analytification $\mathbb{A}^{1,\mathrm{rig}}_K$ of the affine line $\mathbb{A}^1_K$ is constructed by the inclusions of disks around the origin with increasing radius $|\theta|^n,\ n \ge 0$,

$$\operatorname{Sp} K\langle t\rangle \hookrightarrow \operatorname{Sp} K\langle \frac{t}{\theta}\rangle \hookrightarrow \operatorname{Sp} K\langle \frac{t}{\theta^2}\rangle \hookrightarrow \dots .$$

This functor is known as the “GAGA functor” because it is analogous to a functor introduced in Serre’s paper “Géométrie algébrique et géométrie analytique” [Ser56, Déf. 2]. Moreover, a coherent $\mathcal{O}_X$-module $\mathcal{G}$ on a K-scheme X of locally finite type also admits a rigid analytification that is a coherent $\mathcal{O}_{X^{\mathrm{rig}}}$-module $\mathcal{G}^{\mathrm{rig}}$ on the rigid analytification X^{rig} of X.

Theorem 1.3.1 (“The rigid analytic GAGA principle” [Bos, §1.16 Thm. 12 and 13])**.** *Let X be a proper K-scheme.*

(i) Suppose $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules. Then there is a canonical isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{O}_{X^{\mathrm{rig}}}}(\mathcal{F}^{\mathrm{rig}}, \mathcal{G}^{\mathrm{rig}}).$$

(ii) Let $\mathcal{F}'$ be a coherent $\mathcal{O}_{X^{\mathrm{rig}}}$-module. Then there is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $\mathcal{F}^{\mathrm{rig}} = \mathcal{F}'$ and $\mathcal{F}$ is unique up to unique isomorphism.

This allows us in particular to find an algebraic coherent $\mathcal{O}_X$-submodule of an $\mathcal{O}_X$-module by constructing a rigid analytic coherent $\mathcal{O}_{X^{\mathrm{rig}}}$-submodule of its rigid analytification (cf. proof of Proposition 4.2.13).

2. THE TANNAKIAN CATEGORY $\mathscr{PT}$ OF PURE DUAL T-MOTIVES

In this chapter we shall study the function field analog of Grothendieck's Tannakian category of pure motives as constructed by Deligne in [DMOS82, §6].

Generalizing slightly the definition of dual t-motives given in [ABP04], we first define what we call dual Anderson A-motives over k. Then isogenies between dual Anderson A-motives are introduced and we prove in particular that the relation of isogeny is an equivalence relation. In the next section we carry Papanikolas's definition of pre-t-motives out to generalized Papanikolas Q-motives over k and give a fully faithful functor $\mathcal{P}$ from the category $\mathscr{DA}^I$ of dual Anderson A-motives up to isogeny to the category $\mathscr{P}$ of Papanikolas Q-motives. We see next that dual Anderson A-motives and Papanikolas Q-motives give rise to algebraic σ-sheaves on $\operatorname{Spec} k$. We define purity and tensor products of algebraic σ-sheaves, dual Anderson A-motives and Papanikolas Q-motives. Making further requirements on k, we introduce rigid analytic σ-sheaves on $\operatorname{Sp} k$. We call algebraic σ-sheaves rigid analytically trivial if their "analytification" given in the form of such a rigid analytic σ-sheaf is trivial. Similarly, we define rigid analytic triviality of dual Anderson A-motives and Papanikolas Q-motives. We then see that the functor $\mathcal{P}$ induces a fully faithful functor from the category $\mathscr{PRDA}^I$ of pure rigid analytically trivial dual Anderson A-motives up to isogeny to the category $\mathscr{PR}$ of pure rigid analytically trivial Papanikolas Q-motives that we also denote by $\mathcal{P}$ by abuse of notation. We prove that $\mathscr{PR}$ is a Tannakian category over Q.

Following Papanikolas in [Pap08], we restrict ourselves to the case $Q = \mathbb{F}_q(t)$ and define the category $\mathscr{PT}$ of pure dual t-motives over k to be the Tannakian subcategory of $\mathscr{PR}$ generated by the essential image of the functor $\mathcal{P} : \mathscr{PRDA}^I \to \mathscr{PR}$. Next we construct a Tannakian category $\mathscr{PT}'$ of pure dual t-motives over k from the category $\mathscr{PRDA}^I$ through the formal inversion of the Carlitz t-motive as done by Taelman in [Tae09]. We prove that $\mathscr{PT}$ and $\mathscr{PT}'$ are equivalent, allowing us to use both definitions equivalently. In the last section of this chapter, we consider the Tannakian subcategory generated by a pure dual t-motive $\underline{P}$ over k and call the algebraic group associated with it by Tannakian duality the *Galois group of* $\underline{P}$, in analogy with the classical motivic Galois group of a pure motive. Finally, we briefly introduce systems of σ-linear equations through which Papanikolas is able to show his transcendence result on the periods and quasi-periods of a pure dual t-motive over $\overline{Q}$.

2.1 Dual Anderson A-motives

Anderson, Brownawell and Papanikolas developed dual t-motives over the algebraic closure on the rational curve in [ABP04] and we shall carry their definition over to the general case and call the objects thus obtained dual Anderson A-motives over k, where A stands for the ring of integers of an arbitrary function field and $k/\mathbb{F}_q$ is a perfect field. Definitions and assertions concerning dual Anderson A-motives in this chapter are mostly motivated by non-dual Anderson A-motives as studied in [BH07b, Har08]. These are in the same way the generalization of the t-motives introduced by Anderson in [And86] in the case $A = \mathbb{F}_q[t]$. To motivate definitions and compare results, we also define non-dual Anderson A-motives. In order to do this, we need to make first some additional definitions.

Let $\mathrm{Frob}_{q,\operatorname{Spec} k}$ denote the absolute Frobenius endomorphism on $\operatorname{Spec} k$. We set in the non-dual case:

$$\begin{array}{lcl} F & := & \mathrm{id}_C \times \mathrm{Frob}_{q,\operatorname{Spec} k} : C_k \to C_k, \text{ is the identity on the topological space} \\ & & |C_k| \text{ and acts on } \mathcal{O}_{C_k} \text{ as } \mathrm{id}_{\mathcal{O}_C} \text{ and } \mathrm{Frob}_{q,k}, \\ F^* := F_A^* & := & (F|_{\operatorname{Spec} A_k})^* = \mathrm{id}_A \otimes \mathrm{Frob}_{q,k} : A_k \to A_k,\ f = a \otimes b \mapsto F^*(f) = a \otimes b^{q^1}. \end{array}$$

In the dual setting we write similarly:

$$\begin{array}{lcl} \varsigma & := & \mathrm{id}_C \times \mathrm{Frob}_{q,\operatorname{Spec} k}^{-1} : C_k \to C_k, \text{ is the identity on the topological space} \\ & & |C_k| \text{ and acts on } \mathcal{O}_{C_k} \text{ as } \mathrm{id}_{\mathcal{O}_C} \text{ and } \mathrm{Frob}_{q,k}^{-1}, \\ \varsigma^* := \varsigma_A^* & = & (\varsigma|_{\operatorname{Spec} A_k})^* = \mathrm{id}_A \otimes \mathrm{Frob}_{q,k}^{-1} : A_k \to A_k,\ f = a \otimes b \mapsto \varsigma^*(f) = a \otimes b^{q^{-1}}. \end{array}$$

We have now collected the necessary ingredients to define non-dual and dual Anderson A-motives.

Definition 2.1.1. Let $(k, \gamma : A \to k)$ be an A-field and $r, d \in \mathbb{N}$.

(i) An *Anderson A-motive of rank r, characteristic γ and dimension d over k* is a pair $\underline{\mathrm{M}} = (\mathrm{M}, \tau_{\mathrm{M}})$, where M is a locally free A_k-module of rank r and $\tau_{\mathrm{M}} : F^*\mathrm{M} \to \mathrm{M}$ is an injective A_k-homomorphism such that

   (a) M is finitely generated over $k[\tau]$ where $\tau = \tau_{\mathrm{M}} \circ F_M^* : \mathrm{M} \to \mathrm{M}$ is the F^*-linear map induced by τ_{M},

   (b) $\dim_k \operatorname{coker} \tau_{\mathrm{M}} = d$ and

   (c) $(a \otimes 1 - 1 \otimes \gamma(a))^d = 0$ on $\operatorname{coker} \tau_{\mathrm{M}}$.

   We call $\epsilon := \ker \gamma$ the characteristic point of M.

(ii) A *morphism* $f : (\mathrm{M}, \tau_{\mathrm{M}}) \to (\mathrm{M}', \tau_{\mathrm{M}'})$ of Anderson A-motives over k is an A_k-homomorphism $f : \mathrm{M} \to \mathrm{M}'$ such that

$$\tau_{\mathrm{M}'} \circ F^* f = f \circ \tau_{\mathrm{M}}.$$

We denote the category of Anderson A-motives of positive rank and dimension over k by $\mathscr{A}$ and the set of morphisms between Anderson A-motives over k by $\mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$.

Remark 2.1.2. The field k does not need to be perfect in the definition of Anderson A-motives and Anderson A-modules. But k must be perfect in the definition of dual Anderson A-motives which will be in particular free A_k-modules over $k[\sigma]$, so we stick to assuming k is a perfect field in all definitions.

As done before, we call Anderson A-motives *non-dual Anderson A motives* whenever we want to emphasize that we are in the non-dual setting. The definition of a dual Anderson A-motive reads similarly to the one of a non-dual Anderson A-motive:

Definition 2.1.3. Let $(k, \gamma : A \to k)$ be an A-field and $r, d \in \mathbb{N}$.

(i) A *dual Anderson A-motive of rank r, dimension d and characteristic γ over k* is a pair $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$, where M is a locally free A_k-module of rank r and $\sigma_{\mathrm{M}} : \varsigma^*\mathrm{M} \to \mathrm{M}$ is an injective A_k-homomorphism such that

(a) M is finitely generated over $k[\sigma]$ where $\sigma = \sigma_{\mathrm{M}} \circ \varsigma^*_{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ is the ς^*-linear map induced by σ_{M},

(b) $\dim_k \operatorname{coker} \sigma_{\mathrm{M}} = d$ and

(c) $(a \otimes 1 - 1 \otimes \gamma(a))^d = 0$ on $\operatorname{coker} \sigma_{\mathrm{M}}$.

We call $\epsilon := \ker \gamma$ the characteristic point of M.

(ii) A *morphism* $f : (\mathrm{M}, \sigma_{\mathrm{M}}) \to (\mathrm{N}, \sigma_{\mathrm{N}})$ of dual Anderson A-motives over k is an A_k-homomorphism $f : \mathrm{M} \to \mathrm{N}$ such that

$$\sigma_{\mathrm{N}} \circ \varsigma^* f = f \circ \sigma_{\mathrm{M}}.$$

We denote the category of dual Anderson A-motives of positive rank and dimension over k by $\mathscr{D}\mathscr{A}_+$ and the set of morphisms between Anderson A-motives over k by $\operatorname{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$.

The following basic example corresponds to the Carlitz module, which was invented by Carlitz.

Example 2.1.4. Let $A = \mathbb{F}_q[t]$ and set $\theta := \gamma(t)$. Then the *dual Carlitz t-motive* $\underline{\mathrm{C}} = (\mathrm{C}, \sigma_{\mathrm{C}})$ *over k* consists of $\mathrm{C} = A_k = k[t]$ and the $k[t]$-homomorphism $\sigma_{\mathrm{C}} : \varsigma^*\mathrm{C} \to \mathrm{C}$ given by

$$\varsigma^*\mathrm{c} \mapsto (t - \theta)\varsigma^*\mathrm{c} \quad \text{for all } \mathrm{c} \in \mathrm{C}.$$

Note that the locally free A_k-module M underlying a dual Anderson A-motive of dimension d over k is a module over $A_k[\sigma]$ that is the non-commutative polynomial ring defined by the rule

$$\sigma(a \otimes \beta) = (a \otimes \beta^{(-1)})\sigma$$

for all $a \otimes \beta \in A_k$. Note that $A_k[\sigma] = k[t;\sigma]$ if $A = \mathbb{F}_q[t]$. Furthermore, we have $\mathrm{M} \cong \mathrm{Mat}_{1\times d}(k[\sigma])$ by the following Lemma.

Lemma 2.1.5 (Cf. [Har08, Lemma 2.1.5])**.** *Let (k, γ) be an A-field,* M *a finitely generated A_k-module and $\sigma_{\mathrm{M}} : \varsigma^*\mathrm{M} \to \mathrm{M}$ an A_k-homomorphism such that* M *is finitely generated over $k[\sigma]$, where $\sigma := \sigma_{\mathrm{M}} \circ \varsigma^*_{\mathrm{M}} : \mathrm{M} \to \mathrm{M}$ is the ς^*-linear map induced by σ_{M}. Further, write $d := \dim_k \operatorname{coker} \sigma_{\mathrm{M}} = \dim_k \operatorname{coker} \sigma$. Then the following are equivalent:*

(i) M *is a locally free A_k-module,*

(ii) M *is a torsion free A_k-module,*

(iii) M *is a torsion free $k[\sigma]$-module,*

(iv) M *is a free $k[\sigma]$-module of rank d.*

Proof. The arguments used in the proof of [Har08, Lemma 2.1.5] carry over to our case. □

We end this section with the example of a special type of (dual) Anderson A-motives, called (dual) Drinfeld $\mathbb{F}_q[t]$-motives that will serve us as an example throughout the thesis. In order to do this, we make some further definitions.

Consider a (dual) Anderson A-module $\underline{\mathrm{M}}$ of rank r, characteristic γ and dimension d over k. By using the ring homomorphism $i^* : \mathbb{F}_q[t] \hookrightarrow A$ that makes A into a free $\mathbb{F}_q[t]$-module of rank $\tilde{r} = \deg i$, we then have $\mathrm{M} \cong \mathrm{Mat}_{1\times r'}(k[t])$, where M is the locally free A_k-module underlying $\underline{\mathrm{M}}$ and $r' = r \cdot \tilde{r}$. We say that the vector

$$\mathbf{m} = \begin{pmatrix} \mathrm{m}_1 \\ \vdots \\ \mathrm{m}_{r'} \end{pmatrix} \in \mathrm{Mat}_{r'\times 1}(\mathrm{M})$$

is a $k[t]$-*basis for* M if $\mathrm{m}_1, \ldots, \mathrm{m}_{r'} \in \mathrm{M}$ form a $k[t]$-basis for M, providing an isomorphism

$$\mathrm{Mat}_{1\times r'}(k[t]) \overset{\cdot\mathbf{m}}{\to} \mathrm{M}.$$

Suppose $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is a dual Anderson A-motive. Then there is a unique matrix $\Phi_{\mathbf{m}} \in \mathrm{Mat}_{r'\times r'}(k[t])$ such that

$$\begin{pmatrix} \sigma_{\mathrm{M}}\left(\varsigma_{\mathrm{M}}^*(\mathrm{m}_1)\right) \\ \vdots \\ \sigma_{\mathrm{M}}\left(\varsigma_{\mathrm{M}}^*(\mathrm{m}_{r'})\right) \end{pmatrix} = \Phi_{\mathbf{m}}\mathbf{m}.$$

We say that $\Phi_{\mathbf{m}}$ *represents* σ_{M} *with respect to the basis* $\mathbf{m}$. By applying the elementary divisor theorem we find matrices $U,\ V \in \mathrm{GL}_{r'}(k[t])$ such that

$$U\Phi_{\mathbf{m}}V = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_{r'} \end{pmatrix} \quad \text{and} \quad \operatorname{coker}\sigma_{\mathrm{M}} \cong \operatorname{coker}\Phi_{\mathbf{m}} \cong \bigoplus_{i=1}^{r'} k[t]/(d_i)$$

with elementary divisors $d_i \in k[t]$ and $d_i|d_{i+1}$ for $1 \leq i < r'$. Moreover, we set $\theta := \gamma(i^*(t))$ so that $(t-\theta)^d \operatorname{coker}\Phi_{\mathbf{m}} = 0$ in $\mathrm{Mat}_{1\times r'}(k[t])$ and hence $(t-\theta)^d \in d_i k[t]$ for $i = 1, \ldots, r'$. Therefore $d_i|(t-\theta)^d$ and there exist $\alpha_i \in k^\times$ and $e_i \in \mathbb{N}$ so that

$$d_i = \alpha_i(t-\theta)^{e_i}, \quad \operatorname{coker}\sigma_{\mathrm{M}} \cong \bigoplus_{i=1}^{r'} k[t]/\alpha_i(t-\theta)^{e_i} \quad \text{and} \quad d = \dim_k \operatorname{coker}\sigma_{\mathrm{M}} = \sum_{i=1}^{r'} e_i.$$

Furthermore, we have $\det U\Phi_{\mathbf{m}}V = \alpha(t-\theta)^d$ with $\alpha := \prod_{i=1}^{r'}\alpha_i \in k^\times$ and because $\det(UV)^{-1} \in (k[t])^\times = k^\times$

$$\det\Phi_{\mathbf{m}} = \alpha_{\mathbf{m}}(t-\theta)^d \quad \text{with } \alpha_{\mathbf{m}} := (\det UV)^{-1}\alpha \in k^\times. \tag{2.1}$$

Example 2.1.6 ((Dual) Drinfeld $\mathbb{F}_q[t]$-motives)**.** Let $A := \mathbb{F}_q[t]$ and (k, γ) be an A-field so that $A_k = k[t]$ and $A_k[\sigma] = k[t;\sigma]$. Write $\theta := \gamma(t)$.

(i) We set $\mathrm{M} := \mathrm{Mat}_{1\times r}(k[t]) \cong k[t]\cdot 1 \oplus k[t]\cdot\sigma \oplus \ldots \oplus k[t]\cdot\sigma^{r-1}$. Then M is a $k[t]$-module of rank r, $\mathbf{m} = (1, \sigma, \ldots, \sigma^{r-1})^{\mathrm{tr}}$ a $k[t]$-basis for M and $\varsigma^*\mathrm{M} = \mathrm{M} \otimes_{k[t],\varsigma^*} k[t] \cong \mathrm{Mat}_{1\times r}(k[t])$. With respect to the basis $\mathbf{m}$, we let $\sigma_{\mathrm{M}} : \varsigma^*\mathrm{M} \to \mathrm{M}$ be the A_k-homomorphism represented by

$$\Phi_{\mathbf{m}} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\alpha_r^{(-r)} & -\alpha_1^{(-1)}/\alpha_r^{(-r)} & \cdots & -\alpha_{r-1}^{(-(r-1))}/\alpha_r^{(-r)} \end{pmatrix}$$

with $\alpha_r \in k^\times$, $\alpha_i \in k$ for $i = 1, \ldots, r-1$. Thus $\det\Phi_{\mathbf{m}} = (-1)^{r-1}/\alpha_r^{(-r)}(t-\theta)$ and $\underline{\mathrm{M}} := (\mathrm{M}, \sigma_{\mathrm{M}})$ defines a dual Anderson A-motive of rank r and dimension 1 over k.

(ii) Similarly, we define an Anderson A-motive $(\mathrm{M}, \tau_{\mathrm{M}})$ of rank r and dimension 1 over k with $\mathrm{M} := \mathrm{Mat}_{1\times r}(k[t]) \cong k[t]\cdot 1 \oplus k[t]\cdot\tau \oplus \ldots \oplus k[t]\cdot \tau^{r-1}$, $F^*\mathrm{M} \cong \mathrm{Mat}_{1\times r}(k[t])$ and a $k[t]$-basis $\mathsf{m} = (1, \tau, \ldots, \tau^{r-1})^{\mathrm{tr}}$ for M. With respect to m, we let $\tau_{\mathrm{M}} : F^*\mathrm{M} \to \mathrm{M}$ be the A_k-homomorphism represented by

$$\Phi_{\mathsf{m}} := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\alpha_r & -\alpha_1/\alpha_r & \cdots & -\alpha_{r-1}/\alpha_r \end{pmatrix}$$

with $\alpha_r \in k^\times$, $\alpha_i \in k$ for $i = 1, \ldots, r-1$ and $\det \Phi_{\mathsf{m}} = (-1)^{r-1}/\alpha_r(t-\theta)$.

We will see in Section 4.1.2 that such (dual) Anderson A-motives of rank r and dimnsion 1 arise from Drinfeld $\mathbb{F}_q[t]$-modules over k that were once introduced by Drinfeld. Having this in mind, we call them *(dual) Drinfeld $\mathbb{F}_q[t]$-motives of rank r over k*. We note that the dual Carlitz t-motive over k is a dual Drinfeld $\mathbb{F}_q[t]$-motive of rank 1 over k.

2.2 Isogenies

In this section, we study isogenies between (dual) Anderson A-motives. We will see that the relation of isogeny is an equivalence relation (Corollary 2.2.6), so that we may define the category $\mathscr{D}\mathscr{A}^I$ of dual Anderson A-motives up to isogeny. Morphisms in this category are the quasi-morphisms between dual Anderson A-motives.

Definitions and propositions made in this section are carried over from the ones for non-dual Anderson A-motives as given in [Har08]. Hence, we only need to prove the assertions for dual Anderson A-motives. The proofs in the non-dual case proceed similarly.

Definition 2.2.1. (i) We call a morphism f of (dual) Anderson A-motives $\underline{\mathrm{M}}$ and $\underline{\mathrm{M}}'$ over k an *isogeny* if the underlying A_k-homomorphism is injective and $\operatorname{coker} f$ is a vector space of finite dimension over k. An isogeny f is then said to be *separable* if $\sigma_{\operatorname{coker} f}$ is bijective and *inseparable* otherwise.

(ii) We say that two (dual) Anderson A-motives $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ are *isogenious* if there is an isogeny $f \in \mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$.

$$\begin{tikzcd} 0 \arrow[r] & \varsigma^*\mathrm{M} \arrow[r, "\varsigma^* f"] \arrow[d, "\sigma_{\mathrm{M}}"] & \varsigma^*\mathrm{N} \arrow[r] \arrow[d, "\sigma_{\mathrm{N}}"] & \operatorname{coker}\varsigma^* f = \varsigma^*\operatorname{coker} f \arrow[r] \arrow[d, "\sigma_{\operatorname{coker} f}"] & 0 \\ 0 \arrow[r] & \mathrm{M} \arrow[r, "f"] & \mathrm{N} \arrow[r] & \operatorname{coker} f \arrow[r] & 0 \end{tikzcd}$$

We want to show that if two (dual) Anderson A-motives $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ are isogenous then their ranks and dimensions must be equal.

Lemma 2.2.2 ([Har08, Lem. 2.3.7]). *If $f : \mathrm{M} \to \mathrm{N}$ is a homomorphism of locally free A_k-modules of finite rank, then the following assertions are equivalent:*

(i) f is injective and $\operatorname{coker} f$ *is a finite dimensional k-vector space,*

(ii) f is injective and $\operatorname{rank}_{A_k}\mathrm{M} = \operatorname{rank}_{A_k}\mathrm{N}$,

(iii) $\operatorname{coker} f$ *is a finite dimensional k-vector space and* $\operatorname{rank}_{A_k}\mathrm{M} = \operatorname{rank}_{A_k}\mathrm{N}$,

(iv) $f \otimes \mathrm{id}_{Q_k} : \mathrm{N} \otimes_{A_k} Q_k \to \mathrm{M} \otimes_{A_k} Q_k$ *is an isomorphism of Q_k-vector spaces with* $Q_k = \mathrm{Quot}(A_k)$*.*

Thus the ranks of isogenous (dual) Anderson A-motives are equal. By the next proposition, we find that the same holds for their dimensions.

Proposition 2.2.3 (Cf. [Har08, Prop. 2.3.8])**.** *Let $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ be an isogeny between (dual) Anderson A-motives. Then* $\dim \underline{\mathrm{M}} = \dim \underline{\mathrm{N}}$*.*

Proof. The following diagram is commutative with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker \sigma_{\mathrm{coker}\, f} \arrow[d] & \\
0 \arrow[r] & \varsigma^*\mathrm{M} \arrow[r, "\varsigma^* f"] \arrow[d, "\sigma_{\mathrm{N}}"] & \varsigma^*\mathrm{N} \arrow[r] \arrow[d, "\sigma_{\mathrm{M}}"] & \varsigma^* \operatorname{coker} f \arrow[r] \arrow[d, "\sigma_{\mathrm{coker}\, f}"] & 0 \\
0 \arrow[r] & \mathrm{M} \arrow[r, "f"] \arrow[d] & \mathrm{N} \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} \sigma_{\mathrm{M}} \arrow[r] \arrow[d] & \operatorname{coker} \sigma_{\mathrm{N}} \arrow[r] \arrow[d] & \operatorname{coker} \sigma_{\mathrm{coker}\, f} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Observe that $\dim_k \varsigma^* \operatorname{coker} f = \dim_k \operatorname{coker} f \otimes_{k,\mathrm{Frob}_q^{-1}} k = \dim_k \operatorname{coker} f$ so that by exactness of the the most right column:

$$\begin{aligned}
0 &= \dim_k \ker \sigma_{\mathrm{coker}\, f} - \dim_k \varsigma^* \operatorname{coker} f + \dim_k \operatorname{coker} f - \dim_k \operatorname{coker} \sigma_{\mathrm{coker}\, f} \\
&= \dim_k \ker \sigma_{\mathrm{coker}\, f} - \dim_k \operatorname{coker} \sigma_{\mathrm{coker}\, f}.
\end{aligned}$$

Moreover, there is an exact sequence by the Snake Lemma

$$0 \to \ker \sigma_{\mathrm{coker}\, f} \to \operatorname{coker} \sigma_{\mathrm{M}} \to \operatorname{coker} \sigma_{\mathrm{N}} \to \operatorname{coker} \sigma_{\mathrm{coker}\, f} \to 0$$

so that, as desired,

$$\begin{aligned}
0 &= \dim_k \ker \sigma_{\mathrm{coker}\, f} - \dim_k \operatorname{coker} \sigma_{\mathrm{M}} + \dim_k \operatorname{coker} \sigma_{\mathrm{N}} - \dim_k \operatorname{coker} \sigma_{\mathrm{coker}\, f} \\
&= \dim_k \operatorname{coker} \sigma_{\mathrm{N}} - \dim_k \operatorname{coker} \sigma_{\mathrm{M}} \\
&= \dim \underline{\mathrm{N}} - \dim \underline{\mathrm{M}}.
\end{aligned}$$

□

We now want to show that the relation of isogeny is an equivalence relation to be able to define the category $\mathscr{D}\mathscr{A}^I$ of dual Andeson A-motives up to isogeny. It is clearly reflexive and transitive and it will follow from Corollary 2.2.5 that the relation of isogeny is also symmetric as desired. In order to prove this, we need to do some preparatory work.

Proposition 2.2.4 (Cf. [Har08, Prop. 2.4.7])**.** *Let $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ be (dual) Anderson A-motives over k and $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ a homomorphism of dual Anderson A-motives. Then f is an isogeny if and only if $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ is injective and* $\operatorname{coker} f$ *is annihilated by a non-zero $a \in A$.*

Proof. Suppose $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is a dual Anderson A-motive of rank $r_{\underline{\mathrm{M}}}$ and dimension $d_{\underline{\mathrm{M}}}$ and $\underline{\mathrm{N}}$ of rank $r_{\underline{\mathrm{N}}}$ and dimension $d_{\underline{\mathrm{N}}}$. We let $\Phi_{\mathbf{m}} \in \mathrm{Mat}_{r'_{\underline{\mathrm{M}}} \times r'_{\underline{\mathrm{M}}}}(k[t])$ and $\Phi_{\mathbf{n}} \in \mathrm{Mat}_{r'_{\underline{\mathrm{N}}} \times r'_{\underline{\mathrm{N}}}}(k[t])$ represent σ_{M} and σ_{M} with respect to $k[t]$-bases $\mathbf{m} \in \mathrm{Mat}_{r'_{\underline{\mathrm{M}}} \times 1}(\mathrm{M})$ and $\mathbf{n} \in \mathrm{Mat}_{r'_{\underline{\mathrm{N}}} \times 1}(\mathrm{N})$ respectively, so that $\det \Phi_{\mathbf{m}} = \alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}}$ and $\det \Phi_{\mathbf{n}} = \alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}$.
"$\Rightarrow$" Assume that $f \in \mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$ is an isogeny and hence $r_{\underline{\mathrm{M}}} = r_{\underline{\mathrm{N}}}$ and $d_{\underline{\mathrm{M}}} = d_{\underline{\mathrm{N}}}$ hold. We write $r := r_{\underline{\mathrm{M}}}$, $r' := r \cdot \tilde{r}$ and $d := d_{\underline{\mathrm{M}}}$. Then f is given by a matrix $F \in \mathrm{Mat}_{r' \times r'}(k[t])$ with respect to the bases $\mathbf{m}$ and $\mathbf{n}$ so that $f \circ \sigma_{\mathrm{M}} = \sigma_{\mathrm{N}} \circ \varsigma^* f$ corresponds to $\Phi_{\mathbf{m}} \cdot F = F^{(-1)} \cdot \Phi_{\mathbf{n}}$.

We have $\alpha_{\mathbf{m}}(t-\theta)^d \cdot \det F = \det F^{(-1)} \cdot \alpha_{\mathbf{n}}(t-\theta)^d$ and by defining $\lambda \in \overline{k}$ by $\lambda^{\frac{1}{q}-1} := \frac{\alpha_{\mathbf{n}}}{\alpha_{\mathbf{m}}}$ and $a := \lambda \cdot \det F$, we obtain

$$a = \lambda \cdot \det F = \lambda^{\frac{1}{q}} \cdot \frac{\alpha_{\mathbf{m}}}{\alpha_{\mathbf{n}}} \cdot \det F = \lambda^{\frac{1}{q}} \cdot \det F^{(-1)} = \varsigma^*(a)$$

so that $0 \neq a \in (\overline{k}[t])^{\varsigma} = \mathbb{F}_q[t]$. By the elementary divisor theorem for F we find that $0 = a \cdot \operatorname{coker} F \cong (a \otimes 1) \operatorname{coker} f$ as desired.

"$\Leftarrow$" By Lemma 2.2.2 it remains to show that $\operatorname{coker} f$ is a finite dimensional vector space over k. We have an isomorphism $\mathrm{Mat}_{r'_{\underline{\mathrm{M}}} \times 1}(k[t]) \overset{\cdot \mathbf{m}}{\to} \mathrm{M}$ and thus by assumption a surjective map $\mathrm{Mat}_{r'_{\underline{\mathrm{M}}} \times 1}(k[t]/(a)) \twoheadrightarrow \operatorname{coker} f$. Clearly $\mathrm{Mat}_{r'_{\underline{\mathrm{M}}} \times 1}(k[t]/(a))$ is a finite dimensional vector space over k, so that the assertion follows. □

Corollary 2.2.5 (Cf. [Har08, Cor. 2.4.8])**.** *Let $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ be an isogeny between (dual) Anderson A-motives $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ over k. Then there is a non-zero $a \in A$ and an $\hat{f} \in \mathrm{Hom}_k(\underline{\mathrm{N}}, \underline{\mathrm{M}})$ such that*

$$f \circ \hat{f} = a \cdot \mathrm{id}_{\underline{\mathrm{N}}} \quad \textit{and} \quad \hat{f} \circ f = a \cdot \mathrm{id}_{\underline{\mathrm{M}}}\,.$$

We call $\hat{f}$ a dual isogeny *of f.*

Proof. Write $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ and $\underline{\mathrm{N}} = (\mathrm{N}, \sigma_{\mathrm{N}})$. By Proposition 2.2.4 there is a non-zero $a \in A$ such that multiplication by a is the zero map on $\operatorname{coker} f$. Hence, there is an $\hat{f} \in \mathrm{Hom}_{A_k}(\mathrm{N}, \mathrm{M})$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{M} \arrow[r, "f"] \arrow[d, "a\cdot\mathrm{id}_{\mathrm{M}}"'] & \mathrm{N} \arrow[r] \arrow[d, "a\cdot\mathrm{id}_{\mathrm{N}}"] \arrow[dl, "\hat{f}"'] & \operatorname{coker} f \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{M} \arrow[r, "f"] & \mathrm{N} \arrow[r] & \operatorname{coker} f \arrow[r] & 0
\end{tikzcd}$$

Moreover, we have by injectivity of f that $\hat{f} \in \mathrm{Hom}_k(\underline{\mathrm{N}}, \underline{\mathrm{M}})$ because

$$\begin{aligned}
f \circ (\hat{f} \circ \sigma_{\mathrm{N}}) &= (a \cdot \mathrm{id}_{\mathrm{N}}) \circ \sigma_{\mathrm{N}} = \sigma_{\mathrm{N}} \circ (a \cdot \mathrm{id}_{\varsigma^*\mathrm{N}}) \overset{a \in A}{=} \sigma_{\mathrm{N}} \circ (\varsigma^*(a) \cdot \mathrm{id}_{\varsigma^*\mathrm{N}}) = \sigma_{\mathrm{N}} \circ \varsigma^* f \circ \varsigma^* \hat{f} \\
&= f \circ (\sigma_{\mathrm{M}} \circ \varsigma^* \hat{f}),
\end{aligned}$$

where we have used that σ_{M} is an A_k-homomorphism and that $f \in \mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$. □

Thus the relation of isogeny is also symmetric as desired.

Corollary 2.2.6 (Cf. [Har08, Cor. 2.4.9])**.** *The relation of isogeny is an equivalence relation for (dual) Anderson A-motives.*

This allows us to define the category of dual Anderson A-motives up to isogeny.

Definition 2.2.7 (Cf. [Har08, Def. 2.4.11])**.** Let $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ be two (dual) Anderson A-motives over k. We set

$$\begin{array}{lll} \mathrm{QHom}_k(\underline{\mathrm{M}},\underline{\mathrm{N}}) & := \mathrm{Hom}_k(\underline{\mathrm{M}},\underline{\mathrm{N}}) \otimes_A Q & \text{the } Q\text{-vector space of } \textit{quasi-morphisms} \text{ and} \\ \mathrm{QEnd}_k(\underline{\mathrm{M}}) & := \mathrm{End}_k(\underline{\mathrm{M}}) \otimes_A Q & \text{the } Q\text{-algebra of } \textit{quasi-endomorphisms.} \end{array}$$

(i) We define the *category $\mathscr{D}\mathscr{A}^I$ of dual Anderson A-motives over k up to isogeny* as follows:

- Objects of $\mathscr{D}\mathscr{A}^I$: dual Anderson A-motives over k;
- Morphisms of $\mathscr{D}\mathscr{A}^I$: The quasi-morphisms in $\mathrm{QHom}_k(\underline{\mathrm{M}},\underline{\mathrm{N}})$.

(i) We define the *category $\mathscr{D}\mathscr{A}_+^I$ of dual Anderson A-motives of positive rank and dimension over k up to isogeny* as follows:

- Objects of $\mathscr{D}\mathscr{A}_+^I$: dual Anderson A-motives of positive rank and dimension over k;
- Morphisms of $\mathscr{D}\mathscr{A}_+^I$: The quasi-morphisms in $\mathrm{QHom}_k(\underline{\mathrm{M}},\underline{\mathrm{N}})$.

Definition 2.2.8. Let $\underline{\mathrm{M}} = (\mathrm{M},\sigma_{\mathrm{M}})$, $\underline{\mathrm{M}}' = (\mathrm{M}',\sigma_{\mathrm{M}'})$ and $\underline{\mathrm{M}}'' = (\mathrm{M}'',\sigma_{\mathrm{M}''})$ be dual Anderson A-motives over k.

(i) A *short exact sequence of dual Anderson A-motives over k in $\mathscr{D}\mathscr{A}_+$*

$$0 \longrightarrow \underline{\mathrm{M}}' \longrightarrow \underline{\mathrm{M}} \longrightarrow \underline{\mathrm{M}}'' \longrightarrow 0$$

is a sequence of dual Anderson A-motives such that the underlying sequence of locally free A_k-modules is exact.

(ii) A *short exact sequence of dual Anderson A-motives over k in $\mathscr{D}\mathscr{A}^I$*

$$0 \longrightarrow \underline{\mathrm{M}}' \xrightarrow{f} \underline{\mathrm{M}} \xrightarrow{g} \underline{\mathrm{M}}'' \longrightarrow 0$$

consists of quasi-morphisms $f : \underline{\mathrm{M}}' \hookrightarrow \underline{\mathrm{M}}$ and $g : \underline{\mathrm{M}} \twoheadrightarrow \underline{\mathrm{M}}''$ such that for some $a, b \in A$ with

$$af \in \mathrm{Hom}_k(\underline{\mathrm{M}}',\underline{\mathrm{M}}) \quad \text{and} \quad bg \in \mathrm{Hom}_k(\underline{\mathrm{M}},\underline{\mathrm{M}}''),$$

$\mathrm{M}''/\operatorname{im} bg$ is a torsion module and $\underline{\mathrm{M}}'$ is isogenous to the dual Anderson A-motive $\underline{\ker bg}$[1].

Lemma 2.2.9. *The category $\mathscr{D}\mathscr{A}^I$ of dual Anderson A-motives over k up to isogeny is abelian.*

Proof. Consider a quasi-morphism $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ of dual Anderson A-motives $\underline{\mathrm{M}} = (\mathrm{M},\sigma_{\mathrm{M}})$ and $\underline{\mathrm{N}} = (\mathrm{N},\sigma_{\mathrm{N}})$ over k. Let $(\operatorname{im} f)^{\mathrm{sat}} \subset \mathrm{N}$ be the saturation of $\operatorname{im} f$ so that $\mathrm{N}/(\operatorname{im} f)^{\mathrm{sat}}$ is a locally free A_k-module. Then $\underline{\ker(\mathrm{N} \to \mathrm{N}/(\operatorname{im} f)^{\mathrm{sat}})}$[1] and $\underline{\mathrm{M}}$ are isogenous through the natural inclusion $\mathrm{M} \hookrightarrow \ker(\mathrm{N} \to \mathrm{N}/(\operatorname{im} f)^{\mathrm{sat}})$. Hence, the cokernel of f is given by the dual Anderson A-motive $\underline{\operatorname{coker} f}$ consisting of the locally free A_k-module $\mathrm{N}/(\operatorname{im} f)^{\mathrm{sat}}$ and the A_k-homomorphism $\varsigma^*\overline{\mathrm{N}/(\operatorname{im} f)}^{\mathrm{sat}} \to \mathrm{N}/(\operatorname{im} f)^{\mathrm{sat}}$ induced by σ_{N}. Moreover, pick an $a \in A$ such that $af \in \mathrm{Hom}_k(\underline{\mathrm{M}},\underline{\mathrm{N}})$. Then the kernel of f in $\mathscr{D}\mathscr{A}^I$ is given by the dual Anderson A-motive $\underline{\ker f} := \underline{\ker af}$[1] over k. We conclude that kernels and cokernels exist in $\mathscr{D}\mathscr{A}^I$. Furthermore, the cokernel and kernel of an isomorphism vanish, whence the assertion. □

[1] For an $h \in \mathrm{Hom}_k(\underline{\mathrm{M}}_1,\underline{\mathrm{M}}_2)$, we define the dual Anderson A-motive $\underline{\ker h}$ over k to be the locally free A_k-module $\ker h$ and the A_k-homomorphism $\varsigma^* \ker h \to \ker h$ induced by σ_{M_1}, where $\underline{\mathrm{M}}_1 = (\mathrm{M}_1,\sigma_{\mathrm{M}_1})$.

In the next subsection, we give a functor $\mathcal{P}$ from the category of dual Anderson A-motives up to isogeny to the category of Papanikolas Q-motives. In order to show that this functor is fully faithful, we state two propositions on the quasi-endomorphisms and quasi-morphisms of dual Anderson A-motives.

Proposition 2.2.10 (Cf. [Har08, Prop. 2.4.12])**.** *Let $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ be two (dual) Anderson A-motives over k, $f \in \mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$ and $\hat{f} \in \mathrm{Hom}_k(\underline{\mathrm{N}}, \underline{\mathrm{M}})$ be isogenies such that $f \circ \hat{f} = a \cdot \mathrm{id}_{\underline{\mathrm{N}}}$ and $\hat{f} \circ f = a \cdot \mathrm{id}_{\underline{\mathrm{M}}}$ for a non-zero $a \in A$. Then*

$$\phi : \mathrm{QEnd}_k(\underline{\mathrm{M}}) \xrightarrow{\sim} \mathrm{QEnd}_k(\underline{\mathrm{N}}), \quad g \mapsto \frac{1}{a} \cdot \left(f \circ g \circ \hat{f}\right)$$

is an isomorphism of Q-algebras.

Proof. We see that ϕ is Q-linear since

$$\phi(\alpha \cdot g) = \frac{1}{a} \cdot \left(f \circ (\alpha \cdot g) \circ \hat{f}\right) = \alpha \cdot \frac{1}{a} \cdot \left(f \circ g \circ \hat{f}\right) = \alpha \cdot \phi(g) \quad \text{for all } \alpha \in Q.$$

It is further a Q-algebra homomorphism because

$$\begin{aligned}\phi(g \circ g') &= \frac{1}{a} \cdot \left(f \circ (g \circ g') \circ \hat{f}\right) = \frac{1}{a} \cdot \left(f \circ g \circ \left(\frac{1}{a}(\hat{f} \circ f)\right) \circ g' \circ \hat{f}\right) \\ &= \frac{1}{a}\left(f \circ g \circ \hat{f}\right) \circ \frac{1}{a}\left(f \circ g' \circ \hat{f}\right) = \phi(g) \circ \phi(g)\end{aligned}$$

for all $g,\ g' \in \mathrm{QEnd}_k(\underline{\mathrm{M}})$. Moreover, ϕ is an isomorphism since it has an inverse, which is given by $\phi^{-1}(g) = \frac{1}{a} \cdot \hat{f} g \cdot f$ for all $g \in \mathrm{QEnd}_k(\underline{\mathrm{M}})$. □

We will see that functor $\mathcal{P}$ is fully faithful with the help of the following:

Proposition 2.2.11 (Cf. [Har08, Prop. 2.4.13])**.** *Let $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ and $\underline{\mathrm{N}} = (\mathrm{N}, \sigma_{\mathrm{N}})$ be two dual Anderson A-motives over k. Consider the Q_k-vector spaces $P := \mathrm{M} \otimes_{A_k} Q_k$ and $R := \mathrm{N} \otimes_{A_k} Q_k$ together with the induced Q_k-isomorphisms*

$$\sigma_P := \sigma_{\mathrm{M}} \otimes \sigma_{\mathrm{id}_{Q_k}} : \varsigma_Q^* P \to P, \quad \text{and} \quad \sigma_R := \sigma_{\mathrm{N}} \otimes \sigma_{\mathrm{id}_{Q_k}} : \varsigma_Q^* R \to R.$$

We define

$$\begin{aligned}\phi : \mathrm{QHom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}}) &\to \{f : P \to R : \ f \circ \sigma_P = \sigma_R \circ \varsigma^* f\}, \\ f \otimes x &\mapsto xf.\end{aligned}$$

Then ϕ is an isomorphism of Q-algebras if $\underline{\mathrm{M}} = \underline{\mathrm{N}}$, and of Q-vector spaces otherwise.

Proof. Clearly, ϕ is Q-linear and injective since $\phi(f \otimes x) = 0$ implies $f(\mathrm{m}) = 0$ for all $\mathrm{m} \in \mathrm{M}$ because $\mathrm{N} \subset \mathrm{N} \otimes_{A_k} Q_k$. Suppose $f : P \to R$ is a Q_k-homomorphism such that $f \circ \sigma_P = \sigma_R \circ \varsigma^* f$. We want to find an element $f' \otimes b \in \mathrm{QHom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$ with $\phi(f' \otimes b \in \mathrm{QHom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})) = f$ to show surjectivity of ϕ.

In order to do this, we denote the ranks of $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ by $r_{\underline{\mathrm{M}}}$ and $r_{\underline{\mathrm{N}}}$ respectively, and their dimensions by $d_{\underline{\mathrm{M}}}$ and $d_{\underline{\mathrm{N}}}$ respectively. Let $\mathbf{\Phi_m} \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{M}}}(k[t])$ and $\mathbf{\Phi_n} \in \mathrm{Mat}_{r'_{\mathrm{N}} \times r'_{\mathrm{N}}}(k[t])$ represent σ_{M} and σ_{M} with respect to $k[t]$-bases $\mathbf{m} \in \mathrm{Mat}_{r'_{\mathrm{M}} \times 1}(\mathrm{M})$ and $\mathbf{n} \in \mathrm{Mat}_{r'_{\mathrm{N}} \times 1}(\mathrm{N})$ for M and N respectively, so that $\det \mathbf{\Phi_m} = \alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}}$ and $\det \mathbf{\Phi_n} = \alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}$.

Define the adjoint matrices

$$\begin{aligned}\Phi_{\mathbf{m}}^{*} &:= \det(\Phi_{\mathbf{m}}) \cdot \Phi_{\mathbf{m}}^{-1} = \alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}} \cdot \Phi_{\mathbf{m}}^{-1} &\in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{M}}}(k[t]),\\ \Phi_{\mathbf{n}}^{*} &:= \det(\Phi_{\mathbf{n}}) \cdot \Phi_{\mathbf{n}}^{-1} = \alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}} \cdot \Phi_{\mathbf{n}}^{-1} &\in \mathrm{Mat}_{r'_{\mathrm{N}} \times r'_{\mathrm{N}}}(k[t]).\end{aligned}$$

Recall that the finite ring homomorphism $i^* : \mathbb{F}_q[t] \hookrightarrow A$ induces a homomorphism $i^* : \mathbb{F}_q(t) \hookrightarrow Q$ that makes Q into a free $\mathbb{F}_q(t)$-module of rank $\tilde{r} = \deg i$. By definition, the entries of $\mathbf{m}$ and $\mathbf{n}$ form $k[t]$-bases of M and N respectively, which we can extend to $k(t)$-bases of P and R respectively. We denote the corresponding vectors by $\mathbf{p} \in \mathrm{Mat}_{r'_{\mathrm{M}} \times 1}(P)$ and $\mathbf{q} \in \mathrm{Mat}_{r'_{\mathrm{N}} \times 1}(R)$ so that there are isomorphisms

$$\mathrm{Mat}_{1 \times r'_{\mathrm{M}}}(k(t)) \xrightarrow{\cdot \mathbf{p}} P \quad \text{and} \quad \mathrm{Mat}_{1 \times r'_{\mathrm{N}}}(k(t)) \xrightarrow{\cdot \mathbf{q}} R.$$

To find the desired quasi-morphism $f' \otimes b \in \mathrm{QHom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$, we use the fact that f is given by a unique matrix $F \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{N}}}(k(t))$ with respect to the bases $\mathbf{p}$ and $\mathbf{q}$ that satisfies $\Phi_{\mathbf{m}} \cdot F = F^{(-1)} \cdot \Phi_{\mathbf{n}}$. Consider the ideals in $\overline{k}[t]$:

$$\begin{aligned} I &:= \{\tilde{f} \in \overline{k}[t] : \tilde{f}F \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{N}}}(k[t])\} \text{ and}\\ I^{\varsigma^*} &:= \{\varsigma^*(\tilde{f}) \in \overline{k}[t] : \tilde{f} \in I\}.\end{aligned}$$

We will define an element $a \in \mathbb{F}_q[t] \subseteq A$, so that the homomorphism $f' \in \mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}})$ given by $aF \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{N}}}(k[t])$ satisfies $\phi(f' \otimes b) = f$ with $b := \frac{1}{a} \in Q$.

In oder to find such an a, we claim that

(i) If $\tilde{f} \in I$, then $\alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}\tilde{f} \in I^{\varsigma^*}$,

(ii) If $\tilde{f} \in I^{\varsigma^*}$, then $\alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}}\tilde{f} \in I$.

To see (i), suppose $\tilde{f} \in I$ and define $g := (\varsigma^*)^{-1}(\alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}\tilde{f})$ so that $\alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}\tilde{f} \in I^{\varsigma^*}$ if and only if $g \in I$. This means, we need to show that $gF \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{N}}}(k[t])$. This holds because

$$\begin{aligned}\varsigma^*(g)F^{(-1)} &= \alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}\tilde{f} \cdot F^{(-1)} = \alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}\tilde{f} \cdot F^{(-1)} \cdot \Phi_{\mathbf{n}}\Phi_{\mathbf{n}}^{-1}\\ &= \Phi_{\mathbf{m}} \cdot \tilde{f}F \cdot \Phi_{\mathbf{n}}^{*} \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{N}}}(k[t]).\end{aligned}$$

For (ii), suppose $\tilde{f} \in I^{\varsigma^*}$ so that we have

$$\alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}}\tilde{f} \cdot F = \alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}} \cdot \Phi_{\mathbf{m}}^{-1}\Phi_{\mathbf{m}} \cdot \tilde{f}F = \Phi_{\mathbf{m}}^{*} \cdot \varsigma^*(\tilde{f})F^{(-1)} \cdot \Phi_{\mathbf{n}} \in \mathrm{Mat}_{r'_{\mathrm{M}} \times r'_{\mathrm{N}}}(k[t]),$$

and the claim follows by definition of I.

Since $\overline{k}[t]$ is a principal ideal domain, there is an $\tilde{f} \in I$ such that $(\tilde{f}) = I$ and $(\varsigma^*(\tilde{f})) = I^{\varsigma^*}$. Moreover, there are by the previous claims $g, \tilde{g} \in \overline{k}[t]$ such that

$$\alpha_{\mathbf{n}}(t-\theta)^{d_{\underline{\mathrm{N}}}}\tilde{f} = g \cdot \varsigma^*(\tilde{f}) \in I^{\varsigma^*} \quad \text{and} \quad \alpha_{\mathbf{m}}(t-\theta)^{d_{\underline{\mathrm{M}}}}\varsigma^*(\tilde{f}) = \tilde{g} \cdot \tilde{f} \in I.$$

Since $g\tilde{g} \cdot \tilde{f} = \alpha_{\mathbf{n}}\alpha_{\mathbf{m}} \cdot (t-\theta)^{d_{\underline{\mathrm{N}}}+d_{\underline{\mathrm{M}}}}\tilde{f}$, $\deg_t \tilde{f} = \deg_t \varsigma^*(\tilde{f})$, and by factoriality of $\overline{k}[t]$ we obtain $g = \alpha(t-\theta)^{d_{\underline{\mathrm{N}}}}$ for an $\alpha \in k^{\times}$.

Note that then $\tilde{f} = \alpha\varsigma^*(\tilde{f})$ and define $\tilde{\alpha} \in \overline{k}$ by requiring that $\tilde{\alpha}^{\frac{1}{q}-1} = \alpha$. Then

$$\varsigma^*(\tilde{\alpha}\tilde{f}) = \tilde{\alpha}\tilde{\alpha}^{\frac{1}{q}-1}\varsigma^*(\tilde{f}) = \tilde{\alpha}\tilde{f} \in \mathbb{F}_q[t],$$

so that $a := \tilde{\alpha}\tilde{f} \in A$ has the desired properties. □

Corollary 2.2.12 (Cf. [Har08, Cor. 2.4.10]). *Let $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ be an isogeny between (dual) Anderson A-motives $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ of characteristic γ over k. If the characteristic point $\epsilon = \ker\gamma = (0)$; that is, if k is of generic characteristic, then f is separable.*

Proof. The following diagram commutes and has exact rows and columns:

$$\begin{tikzcd}
0 \arrow[r] & \varsigma^*\mathrm{M} \arrow[r, "\varsigma^* f"] \arrow[d, "\sigma_{\mathrm{N}}"] & \varsigma^*\mathrm{N} \arrow[r] \arrow[d, "\sigma_{\mathrm{M}}"] & \varsigma^* \operatorname{coker} f \arrow[r] \arrow[d, "\sigma_{\operatorname{coker} f}"] & 0 \\
0 \arrow[r] & \mathrm{M} \arrow[r, "f"] \arrow[d] & \mathrm{N} \arrow[r] \arrow[d] & \operatorname{coker} f \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}\sigma_{\mathrm{M}} \arrow[r] & \operatorname{coker}\sigma_{\mathrm{N}} \arrow[r] \arrow[d] & \operatorname{coker}\sigma_{\operatorname{coker} f} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We want to show that $\operatorname{coker}\sigma_{\operatorname{coker} f} = (0)$, which means that $\sigma_{\operatorname{coker} f}$ is surjective.

By Proposition 2.2.4 there is a non-zero $a \in A$ such that $(a \otimes 1) \operatorname{coker} f = 0$ and since $\operatorname{coker} f \twoheadrightarrow \operatorname{coker}\sigma_{\operatorname{coker} f}$ also $(a \otimes 1) \operatorname{coker}\sigma_{\operatorname{coker} f} = 0$. We choose an $n \in \mathbb{N}$ such that $q^n \geq \dim \underline{\mathrm{N}}$ and hence $(a^{q^n} \otimes 1 - 1 \otimes \gamma(a)^{q^n}) = (a \otimes 1 - 1 \otimes \gamma(a))^{q^n} \operatorname{coker}\sigma_{\mathrm{N}} = 0$. We have $(a^{q^n} \otimes 1) \operatorname{coker}\sigma_{\operatorname{coker} f} = 0$ and by assumption $\gamma(a) \neq 0$ so that

$$0 = (a^{q^n} \otimes 1 - 1 \otimes \gamma(a)^{q^n}) \cdot \operatorname{coker}\sigma_{\operatorname{coker} f} = -(1 \otimes \gamma(a)^{q^n}) \operatorname{coker}\sigma_{\operatorname{coker} f},$$

and therefore $\operatorname{coker}\sigma_{\operatorname{coker} f} = (0)$ holds. As a map between k-vector spaces of the same dimension, $\sigma_{\operatorname{coker} f}$ must be bijective, so that f is a separable isogeny. □

2.3 Papanikolas Q-motives

In order to have a Q-linear theory, Papanikolas introduced the category of pre-t-motives together with a functor from the category of dual t-motives to the category of pre-t-motives and proved that the category of rigid analytically trivial pre-t-motives is a Tannakian category over Q [Pap08, Thm. 3.3.15]. This justifies regarding them as the analog of the (semi-simple) Tannakian category of motives over $\mathbb{Q}$.[2] We consider

$$\varsigma_Q^* = \text{ the endomorphism of } Q_k = \mathrm{Quot}(A_k) \text{ induced by } \varsigma^* : A_k \to A_k,$$

and state the generalized definition of pre-t-motives for arbitrary function fields as follows:

Definition 2.3.1. Let $r \in \mathbb{N}$ be a non-negative integer.

(i) A *(dual) Papanikolas Q-motive of rank r over k* is a pair $\underline{P} = (P, \sigma_P)$ where P is a Q_k-vector space of dimension r and $\sigma_P : \varsigma_Q^* P \to P$ a Q_k-isomorphism.

(ii) A *morphism* $f : (P_1, \sigma_{P_1}) \to (P_2, \sigma_{P_2})$ of Papanikolas Q-motives over k is a Q_k-homomorphism $f : P_1 \to P_2$ such that the following diagram commutes

$$\begin{tikzcd}
\varsigma_Q^* P_1 \arrow[r, "\sigma_{P_1}"] \arrow[d, "\varsigma_Q^* f"'] & P_1 \arrow[d, "f"] \\
\varsigma_Q^* P_2 \arrow[r, "\sigma_{P_2}"] & P_2.
\end{tikzcd} \tag{2.2}$$

[2] See [DMOS82, Prop. 6.5].

We denote the category of Papanikolas Q-motives by $\mathscr{P}$ and the set of morphisms between Papanikolas Q-motives $\underline{P}_1$ and $\underline{P}_2$ by $\mathrm{Hom}_{\mathscr{P}}(\underline{P}_1, \underline{P}_2)$.

Definition 2.3.2. A *short exact sequence of Papanikolas Q-motives over k*

$$0 \longrightarrow (P', \sigma_{P'}) \longrightarrow (P, \sigma_P) \longrightarrow (P'', \sigma_{P''}) \longrightarrow 0$$

is a sequence of Papanikolas Q-motives such that the underlying sequence of Q_k-vector spaces is exact.

By Lemma 1.1.3 we may replace the Q_k-isomorphism $\sigma_P : \varsigma_Q^* P \to P$ underlying a Papanikolas Q-motive (P, σ_P) with a bijective ς_Q^*-linear map $\sigma : P \to P$. Then the Q_k-vector space P is a module over $Q_k[\sigma, \sigma^{-1}]$ that is the non-commutative polynomial ring defined by the rule

$$\sigma\mu = \varsigma_Q^*(\mu)\sigma = \mu^{(-1)}\sigma$$

for all $\mu \in Q_k$. Thus kernels and cokernels of morphisms of Papanikolas Q-motives are the ordinary group-theoretic kernels and cokernels in the category of $Q_k[\sigma, \sigma^{-1}]$-modules and exist for all morphisms in $\mathscr{P}$. Clearly any morphism with vanishing kernel and cokernel is an isomorphism, so that $\mathscr{P}$ is an abelian category. The set of morphisms between Papanikolas Q-motives is naturally a Q-vector space. By the next proposition, we find in particular that it is finite dimensional over Q.

Proposition 2.3.3 (Cf. [And86, Thm. 2])**.** *Let $\underline{P}_1 = (P_1, \sigma_{P_1})$ and $\underline{P}_2 = (P_2, \sigma_{P_2})$ be Papanikolas Q-motives of respective ranks r and r'. Then the evident map*

$$\mathrm{Hom}_{\mathscr{P}}(\underline{P}_1, \underline{P}_2) \otimes_{\mathbb{F}_q} k \longrightarrow \mathrm{Hom}_{Q_k}(P_1, P_2)$$

is injective.

Proof. For the sake of contradiction, we assume that the lemma is false. Hence, we may choose a smallest positive integer n such that there are $\mathbb{F}_q$-linearly independent morphisms $f_i : \underline{P}_1 \to \underline{P}_2$, $1 \le i \le n$, of Papanikolas Q-motives and $\beta_1, \dots, \beta_n \in k$ with

$$\sum_{i=1}^{n} \beta_i f_i = 0 \quad \text{in } \mathrm{Hom}_{Q_k}(P_1, P_2). \tag{2.3}$$

Without loss of generality we may assume $\beta_1 = 1$. By the condition $\sigma_{P_2} \circ \varsigma_Q^* f_i = f_i \circ \sigma_{P_1}$, we have

$$\sum_{i=1}^{n} \beta_i^{(1)} f_i(\sigma_{P_1}(\varsigma_Q^* p)) = 0 \quad \text{for all } \varsigma_Q^* p \in \varsigma_Q^* P_1.$$

Since $\sigma_{P_1}(\varsigma_Q^* P_1) \cong P_1$, we find

$$\sum_{i=1}^{n} \beta_i^{(-1)} f_i = 0 \quad \text{in } \mathrm{Hom}_{Q_k}(P_1, P_2). \tag{2.4}$$

Subtracting (2.4) from (2.3) yields

$$\sum_{i=2}^{n} (\beta_i - \beta_i^{(-1)}) f_i = 0 \quad \text{in } \mathrm{Hom}_{Q_k}(P_1, P_2).$$

By the minimality of n, $\beta_i - \beta_i^{(-1)} = 0$ must hold, and thus $\beta_i \in \mathbb{F}_q$ for all i . Since $f_1, \dots, f_r$ are $\mathbb{F}_q$-linearly independent, we deduce $\beta_i = 0$ for all i. This contradicts the assumption $\beta_1 = 1$. $\square$

In order to make $\mathscr{P}$ into a rigid abelian Q-linear tensor category, we now introduce tensor products of Papanikolas Q-motives. One sees directly that the category of Papanikolas Q-motives is closed under the formation of tensor products.

Definition 2.3.4 (Cf. [Pap08, §3.2.4]). (i) Let $\underline{P}_1 = (P_1, \sigma_{P_1})$ and $\underline{P}_2 = (P_2, \sigma_{P_2})$ be Papanikolas Q-motives over k. The *tensor product of* $\underline{P}_1$ *and* $\underline{P}_2$ is the Papanikolas Q-motive that consists of the finite dimensional Q_k-vector space $P_1 \otimes_{Q_k} P_2$ and the Q_k-isomorphism

$$\sigma_{P_1 \otimes_{Q_k} P_2} := \sigma_{P_1} \otimes \sigma_{P_2}.$$

(ii) Let $\underline{\mathbb{1}}_{Q_k} = (\mathbb{1}_{Q_k}, \sigma_{\mathbb{1}_{Q_k}})$ be the Papanikolas Q-motive over k consisting of the Q_k-vector space $\mathbb{1}_{Q_k} := Q_k$ together with the natural isomorphism $\sigma_{\mathbb{1}_{Q_k}} : \varsigma_Q^* \mathbb{1}_{Q_k} \xrightarrow{\sim} \mathbb{1}_{Q_k}$.

Obviously, $\underline{\mathbb{1}}_{Q_k}$ is an identity object for tensor products in $\mathscr{P}$. Motivated by the commutative diagram (2.2), we define inner hom and duals of Papanikolas Q-motives.

(i) We define the *inner hom* $\mathcal{H}om(\underline{P}_1, \underline{P}_2)$ to be the finite dimensional Q_k-vector space $\mathrm{Hom}_{Q_k}(P_1, P_2)$ together with the Q_k-isomorphism

$$\begin{aligned} \sigma_{\mathcal{H}om(\underline{P}_1, \underline{P}_2)} : \varsigma_Q^* \mathrm{Hom}_{Q_k}(P_1, P_2) &\to \mathrm{Hom}_{Q_k}(P_1, P_2), \\ \varsigma_Q^* f &\mapsto \sigma_{P_2} \circ \varsigma_Q^* f \circ \sigma_{P_1}^{-1}. \end{aligned}$$

(ii) We define the *dual* $\underline{P}_1^\vee$ *of* $\underline{P}_1$ to be the Papanikolas Q-motive $\underline{P}_1^\vee := \mathcal{H}om(\underline{P}_1, \underline{\mathbb{1}}_{Q_k})$ over k.

By the next proposition we find that $\mathcal{H}om(\underline{P}_1, \underline{P}_2)$ plays indeed the role of an inner hom in $\mathscr{P}$ that is compatible with tensor products.

Proposition 2.3.5. *(i) Let $\underline{P}_1 = (P_1, \sigma_{P_1})$, $\underline{P}_2 = (P_2, \sigma_{P_2})$ and $\underline{P}_3 = (P_3, \sigma_{P_3})$ be Papanikolas Q-motives over k. The inner hom satisfies the adjunction formula*

$$\mathrm{Hom}_{\mathscr{P}}(\underline{P}_1 \otimes \underline{P}_2, \underline{P}_3) \cong \mathrm{Hom}_{\mathscr{P}}(\underline{P}_1, \mathcal{H}om(\underline{P}_2, \underline{P}_3)).$$

(ii) Consider finite families $(\underline{P}_i)_{i \in I}$ and $(\underline{P}'_i)_{i \in I}$ of Papanikolas Q-motives over k. Then there is a isomorphism

$$t_{\otimes_{i\in I}\underline{P}_i, \otimes_{i\in I}\underline{P}'_i} : \otimes_{i\in I} \mathcal{H}om(\underline{P}_i, \underline{P}'_i) \xrightarrow{\sim} \mathcal{H}om(\otimes_{i\in I}\underline{P}_i, \otimes_{i\in I}\underline{P}'_i).$$

(iii) Every Papanikolas Q-motive $\underline{P}$ over k is reflexive.

Proof. The natural map

$$\begin{aligned} \mathrm{Hom}_{Q_k}(P_1 \otimes P_2, P_3) &\to \mathrm{Hom}_{Q_k}(P_1, \mathrm{Hom}_{Q_k}(P_2, P_3)), \\ f &\mapsto \tilde{f}, \end{aligned}$$

with $\tilde{f}(p) := (q \mapsto f(p \otimes q))$ is an isomorphism of Q_k-vector spaces. Moreover, $f \in \mathrm{Hom}_{\mathscr{P}}(\underline{P}_1 \otimes \underline{P}_2, \underline{P}_3)$ if and only if

$$\sigma_{P_3} \circ \varsigma_Q^* f(\varsigma_Q^* p \otimes \varsigma_Q^* q) = f\left(\sigma_{P_1}(\varsigma_Q^* p) \otimes \sigma_{P_2}(\varsigma_Q^* q)\right) \tag{2.5}$$

for all $\varsigma_Q^* p \in \varsigma_Q^* P_1$ and $\varsigma_Q^* q \in \varsigma_Q^* P_2$. Similarly $\tilde{f} \in \mathrm{Hom}_{\mathscr{P}}(\underline{P}_1, \mathcal{H}om(\underline{P}_2, \underline{P}_3))$ is equivalent to

$$\sigma_{P_3} \circ \varsigma_Q^* \tilde{f}(\varsigma_Q^* p) \circ \sigma_{P_2}^{-1}(q') = \tilde{f}\left(\sigma_{P_1}(\varsigma_Q^* p)\right)(q') \tag{2.6}$$

for all $\varsigma_Q^* p \in \varsigma_Q^* P_1$ and $q' \in P_2$. We may write (2.6) alternatively as

$$\sigma_{P_3} \circ \varsigma_Q^* \tilde{f}(\varsigma_Q^* p)(\varsigma^* q) = \tilde{f}\left(\sigma_{P_1}(\varsigma_Q^* p)\right)\left(\sigma_{P_2}(\varsigma_Q^* q)\right),$$

for all $\varsigma_Q^* p \in \varsigma_Q^* P_1$ and $\varsigma_Q^* q \in \varsigma_Q^* P_2$. By definition of $\tilde{f}$ we find that the equations (2.5) and (2.6) are equivalent, thus proving the adjunction formula. As we have seen in Section 1.2, the adjunction formula provides the existence of the morphisms $t_{\otimes_{i\in I}\underline{P}_i, \otimes_{i\in I}\underline{P}'_i}$ and $i_{\underline{P}} : \underline{P} \to (\underline{P}^\vee)^\vee$. By definition, these morphisms of dual Papanikolas Q-motives are isomorphisms since the underlying morphisms of Q_k-vector spaces are known to be bijective. $\square$

Thus we may adapt [Pap08, Thm. 3.2.13] to the general case and state

Theorem 2.3.6. *The category $\mathscr{P}$ of Papanikolas Q-motives is a rigid abelian Q-linear tensor category.*

Recall that the finite ring homomorphism $i^* : \mathbb{F}_q[t] \hookrightarrow A$ induces a homomorphism $i^* : \mathbb{F}_q(t) \hookrightarrow Q$ that makes Q into a free $\mathbb{F}_q(t)$-module of rank $\tilde{r} = \deg i$. If (P, σ_P) is a Papanikolas Q-motive we have then $P \cong \mathrm{Mat}_{1\times r'}(k(t))$ where $r' = r \cdot \tilde{r}$.

As for a dual Anderson A-motive, we call

$$\mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_{r'} \end{pmatrix} \in \mathrm{Mat}_{r'\times 1}(P)$$

a *$(k(t)$-$)$basis for P* if $p_1, \ldots, p_{r'} \in P$ form a $k(t)$-basis for P. This provides an isomorphism

$$\mathrm{Mat}_{1\times r'}(k(t)) \xrightarrow{\cdot\mathbf{p}} P.$$

Then there is a unique matrix $\Phi_{\mathbf{p}} \in \mathrm{GL}_{r'}(k(t))$ such that

$$\left(\sigma_P\left(\varsigma_{Q,P}^*(p_1)\right), \ldots, \sigma_P\left(\varsigma_{Q,P}^*(p_{r'})\right)\right)^{\mathrm{tr}} = \Phi_{\mathbf{p}} \cdot \mathbf{p},$$

where $\varsigma_{Q,P}^* : P \to \varsigma_Q^* P$ is the ς_Q^*-linear map given by $p \mapsto p \otimes 1$. We say that $\Phi_{\mathbf{p}}$ *represents σ_P with respect to the basis* $\mathbf{p}$.

Lemma 2.3.7. *(i) Let $\mathbf{p} = (p_1, \ldots, p_{r_1\tilde{r}})^{\mathrm{tr}}$ and $\mathbf{q} = (q_1, \ldots, q_{r_2\tilde{r}})^{\mathrm{tr}}$ be $k(t)$-bases for Papanikolas Q-motives (P_1, σ_{P_1}) and (P_2, σ_{P_2}) of rank r_1 and r_2, respectively. Then the Kronecker product*

$$\Phi_{\mathbf{p}\otimes\mathbf{q}} = \Phi_{\mathbf{p}} \otimes \Phi_{\mathbf{q}}$$

represents $\sigma_{P_1\otimes P_2}$ with respect to the $k(t)$-basis

$$\mathbf{p}\otimes\mathbf{q} := (p_1\otimes q_1, p_1\otimes q_2, \ldots, p_{r_1\tilde{r}}\otimes q_{r_2\tilde{r}})^{\mathrm{tr}} \qquad \textit{for } P_1\otimes P_2.$$

(ii) Consider a Papanikolas Q-motive $\underline{P} = (P, \sigma_P)$ and its dual $\underline{P}^\vee = (P^\vee, \sigma_{P^\vee})$. Let $\mathbf{p} = (p_1, \ldots, p_r)^{\mathrm{tr}}$ be a basis for P and $\mathbf{p}^\vee = (p_1^\vee, \ldots, p_r^\vee)^{\mathrm{tr}}$ be the dual basis for the dual vector space $P^\vee$ of P. Then $\Phi_{\mathbf{p}^\vee} = (\Phi_{\mathbf{p}}^{-1})^{\mathrm{tr}}$ represents $\sigma_{P^\vee}$ with respect to $\mathbf{p}^\vee$.

Proof. Part (i) is clear from the definition of the Kronecker product. To see (ii), recall that by definition

$$\sigma_{P^\vee} : \varsigma_Q^* P^\vee \to P^\vee, \quad \varsigma_Q^*\check{p} \mapsto \sigma_{\mathbb{1}_{Q_k}} \circ \varsigma_Q^*\check{p} \circ \sigma_P^{-1}.$$

The natural isomorphism $\sigma_{\mathbb{1}_{Q_k}}$ is represented by the $\tilde{r}\times\tilde{r}$-identity matrix with respect to a $k(t)$-basis of Q_k. Hence, if $\Phi_{\mathbf{p}}$ represents σ_P with respect to the basis $\mathbf{p}$, then $(\Phi_{\mathbf{p}}^{-1})^{\mathrm{tr}}$ represents $\sigma_{P^\vee}$ with respect to $\mathbf{p}^\vee$. $\square$

Finally, we want to associate a Papanikolas Q-motive with a dual Anderson A-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r over k. Note that Q_k is in particular a flat A_k-module, so that $P := \mathrm{M} \otimes_{A_k} Q_k$ is a Q_k-vector space of dimension r and $\sigma_P := \sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_k} : \varsigma_Q^* P \to P$ is an injective Q_k-homomorphism between Q_k-vector spaces of the same dimension and thus also surjective. By Proposition 2.2.11 we may state the following:

Definition 2.3.8. (i) Let $\mathcal{P} : \mathscr{D}\mathscr{A}^I \to \mathscr{P}$ be the fully faithful functor from the category of dual Anderson A-motives up to isogeny to the category of Papanikolas Q-motives that sends an Anderson A-motive $\underline{\mathrm{M}}$ of rank r over k to its associated Papanikolas Q-motive $\mathcal{P}(\underline{\mathrm{M}}) := (\mathrm{M} \otimes_{A_k} Q_k, \sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_k})$ of rank r over k.

(ii) When $A = \mathbb{F}_q[t]$, we define the *dual Lefschetz t-motive* $\underline{L} = (L, \sigma_L)$ *over* k to be the Papanikolas Q-motive associated with the dual Carlitz t-motive $\underline{\mathrm{C}}$ over k.

Note that $\underline{L} \otimes \underline{L}^\vee \cong \mathbb{1}_{Q_k}$. Hence, the dual Lefschetz t-motive is invertible and the functor $(\underline{P} \mapsto \underline{P} \otimes \underline{L}) : \mathscr{P} \to \mathscr{P}$ is an equivalence of categories.

Corollary 2.3.9. *Let $\underline{P}_1$ and $\underline{P}_2$ be Papanikolas Q-motives over k and $n \in \mathbb{N}$. Then the natural map*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathscr{P}}(\underline{P}_1, \underline{P}_2) & \to & \mathrm{Hom}_{\mathscr{P}}(\underline{P}_1 \otimes \underline{L}^n, \underline{P}_2 \otimes \underline{L}^n), \\ f & \mapsto & f \otimes \mathrm{id}_{L^n}, \end{array}$$

is an isomorphism.

2.4 Algebraic σ-sheaves, purity and tensor products

Grothendieck's motives are either pure or mixed objects. In this section, we first define algebraic σ-sheaves and through these pure dual Anderson A-motives in analogy with pure motives. This is also necessary in order to assign later a pure Q-Hodge-Pink structure to a pure dual Anderson A-motive. Due to space consideration, we do not pursue the aspect of mixed dual Anderson A-motives in this thesis.

The purity condition adds additional information at $\infty_k := \infty \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\, k$. A dual Anderson A-motive over k will give rise to a locally free algebraic σ-sheaf over A on $\mathrm{Spec}\, k$ and the purity condition provides an extension to a locally free sheaf on all of C_k. Similarly, a Papanikolas Q-motive over k defines a smooth algebraic σ-sheaf over Q on $\mathrm{Spec}\, k$. In the next section, we then define rigid analytic triviality of dual Anderson A-motives in terms of the corresponding algebraic σ-sheaves.

Suited to these purposes, we let the "coefficient ring" $\mathcal{C}$ in the definition of σ-sheaves over $\mathcal{C}$ on $\mathrm{Spec}\, k$ be either A or Q. Moreover, we denote the pullback of a coherent sheaf $\mathcal{F}$ on $\mathrm{Spec}\, \mathcal{C} \times_{\mathbb{F}_q} \mathrm{Spec}\, k$ along $\mathrm{id} \times \mathrm{Frob}_{q, \mathrm{Spec}\, k}^{-1}$ by $\varsigma_{\mathcal{C}}^* \mathcal{F}$.

Definition 2.4.1 (Cf. [BH07a, Def. 1.1])**.** Let $\mathcal{C}$ be either A or Q and $\mathcal{S} := \mathrm{Spec}\, \mathcal{C}$ its spectrum.

(i) An *(algebraic) σ-sheaf over $\mathcal{C}$ on* $\mathrm{Spec}\, k$ is a pair $\underline{\mathcal{F}} := (\mathcal{F}, \sigma_{\mathcal{F}})$ consisting of a coherent sheaf $\mathcal{F}$ on $\mathcal{S} \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\, k$ and an $\mathcal{O}_{\mathcal{S} \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\, k}$-module homomorphism $\sigma_{\mathcal{F}} : \varsigma_{\mathcal{C}}^* \mathcal{F} \to \mathcal{F}$. We call $\underline{\mathcal{F}}$ *locally free of rank* r if $\mathcal{F}$ is locally free of rank r on $\mathcal{S} \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\, k$ and *smooth* if $\sigma_{\mathcal{F}}$ is an $\mathcal{O}_{\mathcal{S} \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}}$-module isomorphism.

(ii) A *homomorphism of (algebraic) σ-sheaves* $(\mathcal{F}, \sigma_{\mathcal{F}})$ and $(\mathcal{G}, \sigma_{\mathcal{G}})$ is an $\mathcal{O}_{\mathcal{S} \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\, k}$-module homomorphism $f : \mathcal{F} \to \mathcal{G}$ such that $\sigma_{\mathcal{G}} \circ \varsigma_{\mathcal{C}}^* f = f \circ \sigma_{\mathcal{F}}$.

(iii) We define the tensor product $\underline{\mathcal{F}} \otimes \underline{\mathcal{G}}$ of two σ-sheaves $\underline{\mathcal{F}} = (\mathcal{F}, \sigma_{\mathcal{F}})$ and $\underline{\mathcal{G}} = (\mathcal{G}, \sigma_{\mathcal{G}})$ over $\mathcal{C}$ on $\operatorname{Spec} k$ to be the coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{G}$ on $\mathcal{S} \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} k$ together with the $\mathcal{O}_{\mathcal{S} \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} k}$-module homomorphism $\sigma_{\underline{\mathcal{F}} \otimes \underline{\mathcal{G}}} := \sigma_{\mathcal{F}} \otimes \sigma_{\mathcal{G}}$.

We denote the category of algebraic σ-sheaves over $\mathcal{C}$ on $\operatorname{Spec} k$ by $\operatorname{Coh}_\sigma(\operatorname{Spec} k, \mathcal{C})$.

Remark 2.4.2. Within this and the next section, we follow widely the notation of [BH07a]. Böckle and Hartl define algebraic σ-sheaves on X where X is a k-scheme locally of finite type to study rigid analytic triviality of families of non-dual Anderson A-motives. We could carry this over to our dual setting, but do not want to pursue this here and therefore restrict ourselves to the case $X = \operatorname{Spec} k$.

As desired, we see that a dual Anderson A-motive $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r over k defines a locally free σ-sheaf of rank r over A on $\operatorname{Spec} k$ that we denote by $\underline{\mathcal{F}}_{\mathrm{M}} = (\mathcal{F}_{\mathrm{M}}, \sigma_{\mathcal{F}_{\mathrm{M}}})$. Similarly, we write $\underline{\mathcal{F}}_P = (\mathcal{F}_P, \sigma_{\mathcal{F}_P})$ for the smooth locally free σ-sheaf of rank r over Q on $\operatorname{Spec} k$ corresponding to a Papanikolas Q-motive (P, σ_P) of rank r over k.

Purity and tensor products

In order to define purity of dual Anderson A-motives, we fix some further notation. Let D be a divisor on C. We denote the invertible sheaf on all of C_k whose sections φ have divisor $(\varphi) \geq -D$ by $\mathcal{O}_{C_k}(D)$. For a coherent sheaf $\mathcal{M}$ on C_k we set $\mathcal{M}(D) := \mathcal{M} \otimes_{\mathcal{O}_{C_k}} \mathcal{O}_{C_k}(D)$. Moreover, we let z be a uniformizing parameter of $A_{\infty,k}$.

Definition 2.4.3 (Cf. [BH09, Def. 1.1])**.** Let l, n be integers with $n > 0$.

(i) A locally free σ-sheaf $\underline{\mathcal{F}} = (\mathcal{F}, \sigma_{\mathcal{F}})$ of rank r over A on $\operatorname{Spec} k$ is called *pure of weight* $\frac{l}{n}$ if $\mathcal{F}$ admits an extension to a locally free sheaf $\mathcal{M}$ of rank r on C_k such that $\mathcal{M}|_{\operatorname{Spec} A_k} = \mathcal{F}$ and the $\mathcal{O}_{\operatorname{Spec} A_k}$-module homomorphism

$$\sigma_{\mathcal{F}}^n := \sigma_{\mathcal{F}} \circ \varsigma_A^* \sigma_{\mathcal{F}} \circ \ldots \circ (\varsigma_A^*)^{n-1} \sigma_{\mathcal{F}} : (\varsigma_A^*)^n \mathcal{F} \to \mathcal{F}$$

induces an isomorphism

$$(\varsigma^*)^n \mathcal{M}_{\infty_k} \xrightarrow{\sim} \mathcal{M}(l \cdot \infty_k)_{\infty_k} \tag{2.7}$$

of the stalks of $\mathcal{M}$ at ∞_k. We call $\operatorname{wt}(\underline{\mathcal{F}}) := \frac{l}{n}$ the *weight of* $\underline{\mathcal{F}}$.

(ii) We call a dual Anderson A-motive $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ over k *pure of weight* $\frac{l}{n}$ if there is a free $A_{\infty,k}$-module $W_{\underline{\mathrm{M}}} \subseteq \mathrm{M} \otimes_{A_k} Q_{\infty,k}$ of rank r such that

$$z^l (\sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_{\infty,k}})^n ((\varsigma_{Q_\infty}^*)^n W_{\underline{\mathrm{M}}}) \xrightarrow{\sim} W_{\underline{\mathrm{M}}}. \tag{2.8}$$

The weight of $\underline{\mathrm{M}}$ is denoted by $\operatorname{wt}(\underline{\mathrm{M}}) := \frac{l}{n}$.

(iii) A Papanikolas Q-motive $\underline{P} = (P, \sigma_P)$ of rank r over k is called *pure of weight* $\frac{l}{n}$ if there is a free $A_{\infty,k}$-module $W_{\underline{P}} \subseteq P \otimes_{Q_k} Q_{\infty,k}$ of rank r such that

$$z^l (\sigma_P \otimes \mathrm{id}_{Q_{\infty,k}})^n ((\varsigma_{Q_\infty}^*)^n W_{\underline{P}}) \xrightarrow{\sim} W_{\underline{P}}. \tag{2.9}$$

The weight of $\underline{P}$ is denoted by $\operatorname{wt}(\underline{P}) := \frac{l}{n}$.

Corollary 2.4.4. *A dual Anderson A-motive $\underline{\mathrm{M}}$ over k is pure of weight $\frac{l}{n}$ if and only if its associated Papanikolas Q-motive $\mathcal{P}(\underline{\mathrm{M}})$ over k is pure of weight $\frac{l}{n}$.*

As the next proposition shows, every dual Drinfeld A-motive of rank r over k is an example for a pure dual Anderson A-motive.

Proposition 2.4.5 (Cf. [And86, Prop. 4.1.1]). *If $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is a dual Drinfeld A-module of rank r over k, then $\underline{\mathrm{M}}$ is pure of weight $\frac{1}{r}$.*

Proof. Using the inclusion $i^* : \mathbb{F}_q[t] \to A$, we choose a non-constant $a \in A$ such that $Q_{\infty,k} = A \otimes_{\mathbb{F}_q[a]} k((\frac{1}{a}))$. Thus we may assume $A = \mathbb{F}_q[t]$ and $z = \frac{1}{t}$. Let $\mathrm{m} \in \mathrm{M}$ be a $k[\sigma]$-basis of M and $(1, \sigma, \ldots, \sigma^{r-1})^{\mathrm{tr}}$ the usual $k[t]$-basis for M. Then there are unique $\alpha_i \in k$, $i = 0, \ldots, r-1$, such that

$$t\mathrm{m} = \left(\sum_{i=0}^{r-1} \alpha_i \sigma^i \right) \mathrm{m}.$$

For all $j \gg 0$, we define the $k[\![z]\!]$-module W_j of rank r by putting

$$W_j := \langle \mathrm{m}, \sigma\mathrm{m}, \ldots, \sigma^{j(r-1)}\mathrm{m} \rangle_{k[\![z]\!]} \subset \mathrm{M} \otimes_{k[t]} k(\!(z)\!).$$

We then find

$$tW_j = tW_j + W_j = W_{j+1} = \sigma^r W_j + W_j = \sigma^r W_j,$$

and conclude that $\underline{\mathrm{M}}$ is indeed pure of weight $\frac{1}{r}$. □

Lemma 2.4.6. *A dual Anderson A-motive $\underline{\mathrm{M}}$ over k is pure of weight $\frac{l}{n}$ if and only if its associated locally free σ-sheaf $(\mathcal{F}_{\mathrm{M}}, \sigma_{\mathcal{F}_{\mathrm{M}}})$ over A on $\operatorname{Spec} k$ is pure of weight $\frac{l}{n}$.*

Proof. For a proof that the conditions (2.7) and (2.8) are equivalent in the non-dual setting, see [LRS93, Thm. 3.17]. This proof could be adapted to dual Anderson A-motives. Due to page limit, we only sketch a proof.

Suppose first that $\mathcal{F}_{\mathrm{M}}$ admits an extension to a locally free sheaf $\mathcal{M}$ on C_k that satisfies the purity condition (2.7). Let $W_{\underline{\mathrm{M}}}$ be the completion of $\mathcal{M}_{\infty_k}$ at ∞_k. Then $W_{\underline{\mathrm{M}}} \subseteq \mathrm{M} \otimes_{A_k} Q_{\infty,k}$ is a free $A_{\infty,k}$-module of rank r and in order to even out the pole of degree l at ∞_k we multiply by z^l so that $z^l (\sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_{\infty,k}})^n \left((\varsigma^*_{Q_\infty})^n (\mathrm{M} \otimes_{A_k} Q_{\infty,k}) \right) \to \mathrm{M} \otimes_{A_k} Q_{\infty,k}$ satisfies as desired

$$z^l (\sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_{\infty,k}})^n \left((\varsigma^*_{Q_\infty})^n W_{\underline{\mathrm{M}}} \right) \xrightarrow{\sim} W_{\underline{\mathrm{M}}}.$$

Conversely, suppose that there is a free $A_{\infty,k}$-module $W_{\underline{\mathrm{M}}} \subseteq \mathrm{M} \otimes_{A_k} Q_{\infty,k}$ of rank r such that (2.8) holds. Then we get the desired locally free sheaf $\mathcal{M}$ on C_k by the construction described in [Gos96, §6]; see [Gos96, Rem. 6.2.14]. The purity condition at ∞_k must hold because of the data given by $W_{\underline{\mathrm{M}}}$. □

In order to show that the weight of a pure dual Anderson A-motive of rank r and dimension d in fact equals $\frac{d}{r}$, we need the following:

Lemma 2.4.7. *Let $\mathcal{G}$ be a coherent sheaf on C_k. Then $\deg \varsigma^* \mathcal{G} = \deg \mathcal{G}$.*

Proof. By [BH09, Lem. 1.3] we know that $\deg \varsigma^* \mathcal{G} = \deg F^*(\varsigma^* \mathcal{G}) = \deg \mathcal{G}$. □

Corollary 2.4.8 (Cf. [BH09, Prop. 1.2]). *Let $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ be a pure dual Anderson A-motive of rank r and dimension d over k. Then $\mathrm{wt}(\underline{\mathrm{M}}) = \frac{d}{r}$. In particular, each dual Carlitz A-motive is pure of weight 1.*

Proof. Suppose $\mathrm{wt}(\underline{\mathrm{M}}) = \frac{l}{n}$ and let $\mathcal{M}$ be the locally free coherent sheaf on C_k that satisfies the purity condition (2.7). Then

$$l \cdot r = \deg \mathcal{M}(l \cdot \infty) - \deg \mathcal{M} \overset{\text{Lem. 2.4.7}}{=} \deg \mathcal{M}(l \cdot \infty) - \deg(\varsigma^*)^n \mathcal{M} = \dim_k \operatorname{coker} \sigma_{\mathcal{M}}^n = d \cdot n. \quad \square$$

Finally, we want to show that two isogenous dual Anderson A-motives are either both pure or none of them. This allows us to define the category of pure dual Anderson A-motives up to isogeny.

Proposition 2.4.9 (Cf. [Har08, Prop. 2.4.6])**.** *Let $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ be an isogeny between two dual Anderson A-motives $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ and $\underline{\mathrm{N}} = (\mathrm{N}, \sigma_{\mathrm{N}})$ over k. Then $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ are dual Anderson A-motives of the same rank and dimension over k and $\underline{\mathrm{M}}$ is pure of weight $\frac{l}{n}$ if and only if $\underline{\mathrm{N}}$ is pure of the same weight.*

Proof. Suppose $\underline{\mathrm{M}}$ has rank r and is pure of weight $\frac{l}{n}$. Then there is an $A_{\infty,k}$-module $W_{\underline{\mathrm{M}}} \subseteq \mathrm{M} \otimes_{A_k} Q_{\infty,k}$ of rank r satisfying the purity condition (2.8).

By Lemma 2.2.2 and Proposition 2.2.3 it remains to show that $\underline{\mathrm{M}}$ is pure if and only if $\underline{\mathrm{N}}$ is pure. Notice that $Q_{\infty,k}$ is a flat A_k-module and

$$f \otimes \mathrm{id}_{Q_{\infty,k}} : \mathrm{M} \otimes_{A_k} Q_{\infty,k} \to \mathrm{N} \otimes_{A_k} Q_{\infty,k}, \quad (\sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_{\infty,k}})^n : (\varsigma_{Q_\infty}^*)^n (\mathrm{M} \otimes_{A_k} Q_{\infty,k}) \to \mathrm{M} \otimes_{A_k} Q_{\infty,k}$$
$$\text{and} \quad (\sigma_{\mathrm{N}} \otimes \mathrm{id}_{Q_{\infty,k}})^n : (\varsigma_{Q_\infty}^*)^n (\mathrm{N} \otimes_{A_k} Q_{\infty,k}) \to \mathrm{N} \otimes_{A_k} Q_{\infty,k}$$

are injective $Q_{\infty,k}$-homomorphisms between free $Q_{\infty,k}$-modules of the same rank so that they must be isomorphisms.

We put $W_{\underline{\mathrm{N}}} := (f \otimes \mathrm{id}_{Q_{\infty,k}})(W_{\underline{\mathrm{M}}}) \subseteq \mathrm{N} \otimes_{A_k} Q_{\infty,k}$ that is an $A_{\infty,k}$-module of rank r.

Then the following diagram commutes

$$\begin{tikzcd}
W_{\underline{\mathrm{M}}} \arrow[r, hook] & \mathrm{M} \otimes_{A_k} Q_{\infty,k} \arrow[r, "f \otimes \mathrm{id}_{Q_{\infty,k}}", "\sim"'] & \mathrm{N} \otimes_{A_k} Q_{\infty,k} & W_{\underline{\mathrm{N}}} \arrow[l, hook'] \\
(\varsigma_{Q_\infty}^*)^n W_{\underline{\mathrm{M}}} \arrow[u] \arrow[r, hook] & (\varsigma_{Q_\infty}^*)^n (\mathrm{M} \otimes_{A_k} Q_{\infty,k}) \arrow[u, "(\sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_{\infty,k}})^n"] \arrow[r, "(\varsigma_{Q_\infty}^*)^n (f \otimes \mathrm{id}_{Q_{\infty,k}})", "\sim"'] & (\varsigma_{Q_\infty}^*)^n (\mathrm{N} \otimes_{A_k} Q_{\infty,k}) \arrow[u, "(\sigma_{\mathrm{N}} \otimes \mathrm{id}_{Q_{\infty,k}})^n"'] & (\varsigma_{Q_\infty}^*)^n W_{\underline{\mathrm{N}}}. \arrow[u] \arrow[l, hook']
\end{tikzcd}$$

Thus $z^l (\sigma_{\mathrm{N}} \otimes \mathrm{id}_{Q_{\infty,k}})^n ((\varsigma_{Q_\infty}^*)^n W_{\underline{\mathrm{N}}}) \cong W_{\underline{\mathrm{N}}}$ by purity of $\underline{\mathrm{M}}$ and we conclude that $\underline{\mathrm{N}}$ is pure of the same weight. For the converse take $W_{\underline{\mathrm{M}}} := (f \otimes \mathrm{id}_{Q_{\infty,k}})^{-1}(W_{\underline{\mathrm{N}}})$ so that the same diagram shows that $\underline{\mathrm{M}}$ is pure of weight $\frac{l}{n}$ if the same holds for $\underline{\mathrm{N}}$. $\square$

Definition 2.4.10. We define the following strictly full subcategories by restriction:

(i) the *category $\mathscr{P}\mathscr{D}\mathscr{A}^I \subset \mathscr{D}\mathscr{A}^I$ of pure dual Anderson A-motives up to isogeny* by restriction,

(ii) the *category $\mathscr{P}\mathscr{D}\mathscr{A}_+^I \subset \mathscr{D}\mathscr{A}_+^I$ of pure dual Anderson A-motives of positive rank and dimension up to isogeny* by restriction,

(iii) the *category $\mathscr{P}\mathscr{P} \subset \mathscr{P}$ of pure Papanikolas Q-motives* by restriction.

Recall that the tensor product of Papanikolas Q-motives (P_1, σ_{P_1}) and (P_2, σ_{P_2}) is the Papanikolas Q-motive

$$(P_1 \otimes_{Q_k} P_2, \sigma_{P_1} \otimes \sigma_{P_2}).$$

For dual Anderson A-motives $(\mathrm{M}_1, \sigma_{\mathrm{M}_1})$ and $(\mathrm{M}_2, \sigma_{\mathrm{M}_2})$, it is not clear that the pair $(\mathrm{M} \otimes_{A_k} \mathrm{N}, \sigma_{\mathrm{M} \otimes_{A_k} \mathrm{N}})$ defines a dual Anderson A-motive. We show this for pure dual Anderson A-motives with the help of the additional information given by purity.

Proposition 2.4.11 (Cf. [And86, Prop. 1.11.1]). *Consider pure dual Anderson A-motives $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ of rank r and s, respectively. Then $\underline{\mathrm{M}} \otimes \underline{\mathrm{N}} := (\mathrm{M} \otimes_{A_k} \mathrm{N}, \sigma_{\mathrm{M}\otimes_{A_k}\mathrm{N}})$ is a pure dual Anderson A-motive of rank rs and weight* $\mathrm{wt}(\underline{\mathrm{M}}) + \mathrm{wt}(\underline{\mathrm{N}})$.

Proof. Suppose $\underline{\mathrm{M}}$ and $\underline{\mathrm{N}}$ are pure of weight $\frac{l_1}{n_1}$ and $\frac{l_2}{n_2}$, respectively. Then there are $A_{\infty,k}$-modules

$$W_{\underline{\mathrm{M}}} \subseteq \mathrm{M} \otimes_{A_k} Q_{\infty,k} \quad \text{and} \quad W_{\underline{\mathrm{N}}} \subseteq \mathrm{N} \otimes_{A_k} Q_{\infty,k}$$

of rank r and s, respectively, such that

$$z^{l_1}(\sigma_{\mathrm{M}} \otimes \mathrm{id}_{Q_{\infty,k}})^{n_1}\left((\varsigma^*_{Q_\infty})^{n_1} W_{\underline{\mathrm{M}}}\right) \xrightarrow{\sim} W_{\underline{\mathrm{M}}} \quad \text{and} \quad z^{l_2}(\sigma_{\mathrm{N}} \otimes \mathrm{id}_{Q_{\infty,k}})^{n_2}\left((\varsigma^*_{Q_\infty})^{n_2} W_{\underline{\mathrm{N}}}\right) \xrightarrow{\sim} W_{\underline{\mathrm{N}}}.$$

The A_k-homomorphism $\sigma_{\mathrm{M}_1\otimes_{A_k}\mathrm{M}_2} := \sigma_{\mathrm{M}} \otimes \sigma_{\mathrm{N}}$ induces a ς^*-linear map that we denote by σ as usual. By definition, $\mathrm{M}_1 \otimes_{A_k} \mathrm{M}_2$ is a locally free A_k-module and the cokernel of the injective A_k-homomorphism $\sigma_{\mathrm{M}_1\otimes_{A_k}\mathrm{M}_2}$ is killed by a power of the ideal $J := (a \otimes 1 - 1 \otimes \gamma(a) | a \in A)$. Hence, it remains only to check that $\mathrm{M} \otimes_{A_k} \mathrm{N}$ is finitely generated over $k[\sigma]$ and is pure of weight $\mathrm{wt}(\underline{\mathrm{M}}) + \mathrm{wt}(\underline{\mathrm{N}})$. The $A_{\infty,k}$-module

$$W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} := W_{\underline{\mathrm{M}}} \otimes_{A_{\infty,k}} W_{\underline{\mathrm{N}}} \subset (\mathrm{M} \otimes_{A_k} \mathrm{N}) \otimes_{A_k} Q_{\infty,k}$$

has rank $r_1 r_2$ and satisfies

$$z^{l_1 n_2 + l_2 n_1}\left(\sigma \otimes \mathrm{id}_{Q_{\infty,k}}\right)^{n_1 n_2} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} \xrightarrow{\sim} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}}.$$

We define an increasing filtration $W_0 \subset W_1 \subset W_2 \subset \ldots$ by setting

$$W_j := (\mathrm{M} \otimes_{A_k} \mathrm{N}) \cap z^{-(j+N_0)(l_1 n_2 + l_2 n_1)} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} \quad \text{for } j \geq 0,$$

where $N_0 \in \mathbb{N}^{>0}$ is sufficiently large so that

$$\mathrm{M} \otimes_{A_k} \mathrm{N} + z^{-N_0} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} = (\mathrm{M} \otimes_{A_k} \mathrm{N}) \otimes_{A_k} Q_{\infty,k}. \tag{2.10}$$

Then there are evident maps

$$\begin{aligned} W_{j+1}/W_j &\xrightarrow{\sim} z^{-(j+N_0+1)(l_1 n_2 + l_2 n_1)} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} / z^{-(j+N_0)(l_1 n_2 + l_2 n_1)} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} \\ &\xrightarrow{\sim} (\sigma \otimes \mathrm{id}_{Q_{\infty,k}})^{(j+N_0+1)(n_1 n_2)} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} / (\sigma \otimes \mathrm{id}_{Q_{\infty,k}})^{(j+N_0)(n_1 n_2)} W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}}. \end{aligned}$$

We conclude that

$$z^{-(l_1 n_2 + l_2 n_1)} W_j + W_j = W_{j+1} = W_j + \sigma^{n_1 n_2} W_j \tag{2.11}$$

holds for all $j \geq 0$. Because $W_{\underline{\mathrm{M}}\otimes\underline{\mathrm{N}}} \cong k[\![z]\!]^{\oplus r_1 r_2}$, the W_j are finite dimensional over k. Moreover, $\mathrm{M} \otimes_{A_k} \mathrm{N} = \cup_{j\geq 0} W_j$, so that $\mathrm{M} \otimes_{A_k} \mathrm{N}$ is finitely generated over $k[\sigma]$ by (2.11) and the assertion follows. □

Remark 2.4.12 (Cf. [Tae09, Thm. 5.3.1]). In case k is algebraically closed, Laumon proves that every locally free coherent $Q_{\infty,k}$-module W of finite rank together with an isomorphism $\sigma_W : \varsigma^*_{Q_\infty} W \xrightarrow{\sim} W$ admits a classification; that is, a decomposition into so-called "building blocks" of different weights [Lau96, Thm. 2.4.5/App. B]. The weights of an arbitrary dual Anderson A-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$ over k are defined to be these weights occurring in the classification of $\mathrm{M} \otimes_{A_k} Q_{\infty,k}$. As all of them must be positive, it is possible to show the preceding proposition for arbitrary dual Anderson A-motives in a similar way (for a proof of this when $A = \mathbb{F}_q[t]$, see also [ABP, Thm. 1.5.10]).

Definition 2.4.13. (i) The *tensor product* $\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{M}}_2$ of two pure dual Anderson A-motives $\underline{\mathrm{M}}_1$ and $\underline{\mathrm{M}}_2$ is defined to be the dual Anderson A-motive $(\mathrm{M}_1 \otimes_{A_k} \mathrm{M}_2, \sigma_{\mathrm{M}_1} \otimes \sigma_{\mathrm{M}_2})$.

(ii) We let $\underline{\mathbb{1}}_{A_k} = (\mathbb{1}_{A_k}, \sigma_{\mathbb{1}_{A_k}})$ be the dual Anderson A-motive over k consisting of the A_k-module $\mathbb{1}_{A_k} := A_k$ together with the natural injection $\sigma_{\mathbb{1}_{A_k}} : \varsigma^* \mathbb{1}_{A_k} \xrightarrow{\sim} \mathbb{1}_{A_k}$.

Clearly $\underline{\mathbb{1}}_{A_k}$ plays the role of a unit object in $\mathscr{PDA}^I$. In order to see that $\mathscr{PDA}^I$ and $\mathscr{PP}$ are abelian Q-linear tensor categories, we show that kernels and cokernels exist in $\mathscr{PDA}^I$ and $\mathscr{PP}$.

Proposition 2.4.14. *(i) Consider a short exact sequence of Papanikolas Q-motives*

$$0 \longrightarrow \underline{P}' \longrightarrow \underline{P} \longrightarrow \underline{P}'' \longrightarrow 0.$$

If $\underline{P} = (P, \sigma_P)$ is pure, then $\underline{P}' = (P', \sigma_{P'})$ and $\underline{P}'' = (P'', \sigma_{P''})$ are also pure.

(ii) Consider a short exact sequence of dual Anderson A-motives in $\mathscr{DA}^I$

$$0 \longrightarrow \underline{\mathrm{M}}' \longrightarrow \underline{\mathrm{M}} \longrightarrow \underline{\mathrm{M}}'' \longrightarrow 0.$$

If $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is pure, then $\underline{\mathrm{M}}' = (\mathrm{M}', \sigma_{\mathrm{M}'})$ and $\underline{\mathrm{M}}'' = (\mathrm{M}'', \sigma_{\mathrm{M}''})$ are also pure.

Proof. To see (i), suppose that $W_{\underline{P}}$ is the $Q_{\infty,k}$-module given by the purity of $\underline{P}$. We set

$$W_{\underline{P}'} := W_{\underline{P}} \cap P' \otimes_{Q_k} Q_{\infty,k} \subset P \otimes_{Q_k} Q_{\infty,k} \quad \text{and} \quad W_{\underline{P}''} := W_{\underline{P}} / W_{\underline{P}'}$$

so that we obtain a short exact sequence

$$0 \longrightarrow W_{\underline{P}'} \longrightarrow W_{\underline{P}} \longrightarrow W_{\underline{P}''} \longrightarrow 0.$$

The isomorphism $z^l(\sigma_P \otimes \mathrm{id}_{Q_{\infty,k}})^n((\varsigma^*_{Q_\infty})^n W_{\underline{P}}) \xrightarrow{\sim} W_{\underline{P}}$ induces isomorphisms

$$z^l(\sigma_{P'} \otimes \mathrm{id}_{Q_{\infty,k}})^n((\varsigma^*_{Q_\infty})^n W_{\underline{P}'}) \xrightarrow{\sim} W_{\underline{P}'} \quad \text{and} \quad z^l(\sigma_{P''} \otimes \mathrm{id}_{Q_{\infty,k}})^n((\varsigma^*_{Q_\infty})^n W_{\underline{P}''}) \xrightarrow{\sim} W_{\underline{P}''},$$

whence the assertion holds.

The short exact sequence of dual Anderson A-motives in $\mathscr{DA}^I$ in (i) induces a short exact sequence of Papanikolas Q-motives

$$0 \longrightarrow \mathcal{P}(\underline{\mathrm{M}}') \longrightarrow \mathcal{P}(\underline{\mathrm{M}}) \longrightarrow \mathcal{P}(\underline{\mathrm{M}}'') \longrightarrow 0.$$

Since a dual Anderson A-motive is pure if and only if its associated Papanikolas Q-motive is pure, we may deduce (ii) from (i). $\square$

Corollary 2.4.15. *The category $\mathscr{PDA}^I$ and $\mathscr{PP}$ are abelian Q-linear tensor categories.*

The category $\mathscr{PDA}^I$ is still not exactly what we are looking for. In the next section, we study rigid analytic triviality of dual Anderson A-motives. We see next that we may define the strictly full subcategory of pure rigid analytically trivial dual Anderson A-motives up to isogeny by restriction.

2.5 Rigid analytic σ-sheaves and rigid analytic triviality

We want to introduce rigid analytic triviality of dual Anderson A-motives and Papanikolas Q-motives. This allows us to define the fiber functor underlying the Tannakian category of pure dual t-motives in the next section.

We first define the "rigid analytification" of algebraic σ-sheaves corresponding to dual Anderson A-motives, which we call rigid analytic σ-sheaves. An algebraic σ-sheaf is said to be rigid analytically trivial if its associated rigid analytic σ-sheaf is trivial. We then call a dual Anderson A-motive rigid analytically trivial if its corresponding algebraic σ-sheaf is rigid analytically trivial. Similarly, we define rigid analytic triviality of Papanikolas Q-motives. We see that a dual Anderson A-motive is rigid analytically trivial if and only if its associated Papanikolas Q-motive is rigid analytically trivial. Henceforth, there is a well-defined functor $\mathscr{PRDA}^I \to \mathscr{PR}$ from the category of pure rigid-analytic trivial dual Anderson A-motives up to isogeny to the category of pure rigid analytically trivial Papanikolas Q-motives that we also denote by $\mathcal{P}$ by abuse of notation.

As needed for the definition of rigid analytic σ-sheaves on $\operatorname{Sp} k$ and hence the study of rigid analytic triviality we assume for the rest of this chapter that k is perfect and a complete subfield of $\mathbb{C}_\infty$ that contains Q_∞. In fact, k and $\mathbb{C}_\infty$ are *complete extensions* of Q_∞ since $|\cdot|_\infty$ extends canonically to k and $\mathbb{C}_\infty$, respectively (cf. [Gos96, §2]). We denote these extensions also by $|\cdot|_\infty$. Note that such a field k is automatically an A-field (k, γ) via the inclusion $\gamma : A \hookrightarrow Q \hookrightarrow Q_\infty \hookrightarrow k$ so that k has generic characteristic.

Furthermore, we define the following k-algebras and rigid k-spaces:

1. In order to "rigidify" algebraic σ-sheaves over A on $\operatorname{Spec} k$:

$$\mathfrak{A}(\infty) \quad := \quad (\operatorname{Spec} A_k)^{\mathrm{rig}} = \operatorname{Sp}(A_k\{t\})$$

   whose coordinate ring is the ring of entire functions on $\mathfrak{A}(\infty)$

$$A(\infty) \quad := \quad A_k\{t\} := A \otimes_{\mathbb{F}_q[t]} k\{t\}.$$

2. To study triviality of rigid analytic σ-sheaves over $A(\infty)$ and rigid analytic triviality of dual Anderson A-motives over k:

$$\mathfrak{A}(1) \quad := \quad \operatorname{Sp}(A \otimes_{\mathbb{F}_q[t]} k\langle t\rangle) = \operatorname{Sp} A \times_{\operatorname{Sp} k} \mathfrak{D}_{\overline{k}}$$

   whose coordinate rings is

$$A(1) \quad := \quad A_k\langle t\rangle := A \otimes_{\mathbb{F}_q[t]} \mathbb{T} \quad \text{with } \mathbb{T} := k\langle t\rangle.$$

3. To define rigid analytic triviality of Papanikolas Q-motives over k:

$$Q(1) \quad := \quad \operatorname{Quot}(A(1)).$$

We will refer to $A(\infty)$ and $A(1)$ in the following as the *coefficient rings* for rigid analytic σ-sheaves. If $\mathfrak{C}$ is one of them and $\mathfrak{S}$ its spectrum we denote the pullback of $\operatorname{Frob}_{q,\operatorname{Sp} k}^{-1}$ along $\mathfrak{S} \to \operatorname{Sp} k$ by $\varsigma_{\mathfrak{C}} : \mathfrak{S} \to \mathfrak{S}$.

Definition 2.5.1 (Cf. [BH07a, Def. 1.2]). Let $\mathfrak{C}$ be one of the coefficient rings listed above and $\mathfrak{S} := \operatorname{Sp} \mathfrak{C}$.

(i) A *rigid (analytic) σ-sheaf over $\mathfrak{C}$ on* $\operatorname{Sp} k$ is a pair $\underline{\widetilde{\mathcal{F}}} := (\widetilde{\mathcal{F}}, \sigma_{\widetilde{\mathcal{F}}})$, where $\widetilde{\mathcal{F}}$ is a coherent sheaf of $\mathcal{O}_{\mathfrak{S}}$-modules on $\mathfrak{S}$ and an $\mathcal{O}_{\mathfrak{S}}$-module homomorphism $\sigma_{\widetilde{\mathcal{F}}} : \varsigma_{\mathfrak{C}}\widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$.

We say that $\underline{\widetilde{\mathcal{F}}}$ is *locally free of rank* r if $\widetilde{\mathcal{F}}$ is locally free of rank r on $\mathfrak{S}$ and *smooth* if $\sigma_{\widetilde{\mathcal{F}}}$ is an isomorphism.

(ii) A *homomorphism of rigid (analytic) σ-sheaves* $(\widetilde{\mathcal{F}}, \sigma_{\widetilde{\mathcal{F}}})$ and $(\widetilde{\mathcal{G}}, \sigma_{\widetilde{\mathcal{G}}})$ is an $\mathcal{O}_{\mathfrak{S}}$-module homomorphism $f : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ such that $\sigma_{\widetilde{\mathcal{G}}} \circ \varsigma_{\mathfrak{C}}^* f = f \circ \sigma_{\widetilde{\mathcal{F}}}$.

(iii) We define the tensor product $\underline{\widetilde{\mathcal{F}}} \otimes \underline{\widetilde{\mathcal{G}}}$ of two σ-sheaves $\underline{\widetilde{\mathcal{F}}} = (\widetilde{\mathcal{F}}, \sigma_{\widetilde{\mathcal{F}}})$ and $\underline{\widetilde{\mathcal{G}}} = (\widetilde{\mathcal{G}}, \sigma_{\widetilde{\mathcal{G}}})$ over $\mathfrak{C}$ on $\operatorname{Sp} k$ to be the coherent sheaf of $\mathcal{O}_{\mathfrak{S}}$-modules $\widetilde{\mathcal{F}} \otimes_{\mathcal{O}_{\operatorname{Sp}\mathfrak{C}}} \widetilde{\mathcal{G}}$ on $\mathfrak{S}$ together with the $\mathcal{O}_{\mathfrak{S}}$-module homomorphism $\sigma_{\underline{\widetilde{\mathcal{F}}} \otimes \underline{\widetilde{\mathcal{G}}}} := \sigma_{\widetilde{\mathcal{F}}} \otimes \sigma_{\widetilde{\mathcal{G}}}$.

We denote the category of rigid analytic σ-sheaves over $\mathfrak{C}$ on $\operatorname{Sp} k$ by $\widetilde{\operatorname{Coh}}_\sigma(\operatorname{Sp} k, \mathfrak{C})$.

If $\underline{\mathcal{F}} = (\mathcal{F}, \sigma_{\mathcal{F}})$ is an algebraic σ-sheaf over A on $\operatorname{Spec} k$, we may naturally associate a rigid analytic σ-sheaf $\underline{\mathcal{F}}^{\mathrm{rig}} := (\mathcal{F}^{\mathrm{rig}}, \sigma_{\mathcal{F}^{\mathrm{rig}}})$ over $A(\infty)$ on $\operatorname{Sp} k$ with $\underline{\mathcal{F}}$ where $\mathcal{F}^{\mathrm{rig}}$ is the pullback along the morphism

$$\operatorname{Sp}(A(\infty)) = (\operatorname{Spec} A_k)^{\mathrm{rig}} \to \operatorname{Spec} A_k$$

and $\sigma_{\mathcal{F}^{\mathrm{rig}}}$ is the homomorphism

$$\sigma_{\mathcal{F}^{\mathrm{rig}}} : \varsigma_{A(\infty)}^* \mathcal{F}^{\mathrm{rig}} \to \mathcal{F}^{\mathrm{rig}}$$

induced by $\sigma_{\mathcal{F}}$. If $\underline{\mathcal{F}}$ is locally free of rank r, the same is true for $\underline{\mathcal{F}}^{\mathrm{rig}}$.

So if $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is a dual Anderson A-motive over k, we obtain its "analytification" in the form of a rigid σ-sheaf $(\mathcal{F}_{\mathrm{M}}^{\mathrm{rig}}, \sigma_{\mathcal{F}_{\mathrm{M}}^{\mathrm{rig}}})$ over $A(\infty)$ on $\operatorname{Sp} k$. Taking global sections yields the $A(\infty)$-module $\mathrm{M}(\infty) := \mathrm{M} \otimes_{A_k} A(\infty) = \mathrm{M} \otimes_{A_k} k\{t\}$ and the induced $A(\infty)$-module homomorphism

$$\sigma_{\mathrm{M}(\infty)} = \sigma_{\mathrm{M}} \otimes \mathrm{id} : \varsigma_{A(\infty)}^* \mathrm{M}(\infty) \to \mathrm{M}(\infty).$$

We give now the example of the "most trivial" rigid σ-sheaves over $\mathfrak{C}$.

Example 2.5.2 (Cf. [BH07a, Ex. 1.3])**.** Let $\mathfrak{C}$ be one of the coefficient rings. Then the simplest rigid σ-sheaf in $\widetilde{\operatorname{Coh}}_\sigma(\operatorname{Sp} k, \mathfrak{C})$ is the rigid σ-sheaf $\underline{\mathbb{1}}_{\mathfrak{C}} := (\mathbb{1}_{\mathfrak{C}}, \sigma_{\mathbb{1}_{\mathfrak{C}}})$ over $\mathfrak{C}$ on $\operatorname{Sp} k$ where $\mathbb{1}_{\mathfrak{C}}$ is the structure sheaf $\mathcal{O}_{\operatorname{Sp}\mathfrak{C}}$ and $\sigma_{\mathbb{1}_{\mathfrak{C}}}$ is the natural isomorphism

$$\varsigma_{\mathfrak{C}}^* \mathcal{O}_{\operatorname{Sp}\mathfrak{C}} \xrightarrow{\sim} \mathcal{O}_{\operatorname{Sp}\mathfrak{C}}.$$

This means the induced $\mathfrak{C}$-module homomorphism on global sections is

$$\varsigma_{\mathfrak{C}}^* \mathfrak{C} \xrightarrow{\sim} \mathfrak{C}.$$

In order to define triviality of rigid σ-sheaves over $A(\infty)$, we assign σ-sheaves over $A(1)$ to them by applying "change of coefficients".

Definition 2.5.3 (Change of coefficients)**.** Let $\underline{\widetilde{\mathcal{F}}} = (\widetilde{\mathcal{F}}, \sigma_{\widetilde{\mathcal{F}}})$ be a rigid σ-sheaf over $A(\infty)$ on $\operatorname{Spec} k$. We may restrict coefficients from $A(\infty)$ to $A(1)$ so that we obtain a rigid σ-sheaf $\underline{\widetilde{\mathcal{F}}} \otimes_{A(\infty)} A(1) = (\widetilde{\mathcal{F}} \otimes_{A(\infty)} A(1), \sigma_{\widetilde{\mathcal{F}} \otimes_{A(\infty)} A(1)})$ over $A(1)$ on $\operatorname{Sp} k$ with $\widetilde{\mathcal{F}} \otimes_{A(\infty)} A(1) = \widetilde{\mathcal{F}}|_{\mathfrak{A}(1)}$ and the well-defined $\mathcal{O}_{\mathfrak{A}(1)}$-module homomorphism $\sigma_{\widetilde{\mathcal{F}} \otimes_{A(\infty)} A(1)} : \varsigma_{A(1)}^* (\widetilde{\mathcal{F}} \otimes_{A(\infty)} A(1)) \to \widetilde{\mathcal{F}} \otimes_{A(\infty)} A(1)$ induced by $\sigma_{\widetilde{\mathcal{F}}}$.

Note that for a dual Anderson A-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$ the associated rigid σ-sheaf $\underline{\mathcal{F}}_{\mathrm{M}}^{\mathrm{rig}} \otimes_{\mathfrak{A}(\infty)} A(1)$ over $A(1)$ on $\mathrm{Sp}(k)$ arises from a pair $(\mathrm{M}(1), \sigma_{\mathrm{M}(1)})$, which consists of the $A(1)$-module

$$\mathrm{M}(1) := (\mathrm{M} \otimes_{A_k} A(\infty)) \otimes_{A(\infty)} A(1) = \mathrm{M} \otimes_{A_k} A(1)$$

and the $A(1)$-homomorphism

$$\sigma_{\mathrm{M}(1)} : \varsigma_{A(1)}^* \mathrm{M}(1) \to \mathrm{M}(1), \quad \varsigma_{A(1)}^*(\mathrm{m} \otimes f) \mapsto \sigma_{\mathrm{M}}(\varsigma^*(\mathrm{m})) \otimes f \quad \text{for } \mathrm{m} \in \mathrm{M},\ f \in A(1).$$

Definition 2.5.4 (Cf. [BH07a, Def. 1.6])**.** Denote the global sections of a rigid σ-sheaf $\underline{\widetilde{\mathcal{F}}} = (\widetilde{\mathcal{F}}, \sigma_{\widetilde{\mathcal{F}}})$ over $\mathfrak{C}$ on $\mathrm{Sp}\, k$ by $M_{\underline{\widetilde{\mathcal{F}}}} := \Gamma(\mathrm{Sp}\,\mathfrak{C}, \widetilde{\mathcal{F}})$ and the corresponding $\mathfrak{C}$-module homomorphism by $\sigma_{M_{\underline{\widetilde{\mathcal{F}}}}}$. We define the *$\sigma$-invariants* $\widetilde{\mathcal{F}}^\sigma(k)$ of $\underline{\widetilde{\mathcal{F}}}$ to be

$$M_{\underline{\widetilde{\mathcal{F}}}}^\sigma := \{m \in M_{\underline{\widetilde{\mathcal{F}}}} \mid \sigma_{M_{\underline{\widetilde{\mathcal{F}}}}}(\varsigma_{\mathfrak{C}}^* m) = m\}.$$

In fact, $M_{\underline{\widetilde{\mathcal{F}}}}^\sigma$ is a module over $\mathbb{1}_{\mathfrak{C}}^\sigma(k) = \{m \in \mathfrak{C} :\ \varsigma_{\mathfrak{C}}^* m = m\}$. Observe that $\mathbb{1}_{\mathfrak{C}}^\sigma(k) = \mathfrak{C}^\sigma = A$ if $\mathfrak{C}$ is $A(1)$.

Let $\underline{\widetilde{\mathcal{F}}} = (\widetilde{\mathcal{F}}, \sigma_{\widetilde{\mathcal{F}}})$ be a locally free rigid σ-sheaf over $A(1)$ on $\mathrm{Sp}\, k$. Define the natural map

$$\phi : \widetilde{\mathcal{F}}^\sigma \otimes_{\mathcal{C}} \underline{\mathbb{1}}_{\mathcal{C}(1)} \hookrightarrow \underline{\widetilde{\mathcal{F}}},$$

which is given by $\mathrm{m} \otimes f \mapsto f\mathrm{m}$ on global sections (cf. [BH07a, Lem. 4.2]).

Definition 2.5.5 (Cf. [BH07a, Def. 4.1])**.** (i) We call a locally free rigid σ-sheaf $\underline{\widetilde{\mathcal{F}}}$ over $A(1)$ on $\mathrm{Sp}\, k$ *trivial* if

$$\widetilde{\mathcal{F}}^\sigma \otimes_A \underline{\mathbb{1}}_{A(1)} \cong \underline{\widetilde{\mathcal{F}}}.$$

(ii) We call a locally free rigid σ-sheaf $\underline{\widetilde{\mathcal{F}}}$ over $A(\infty)$ on $\mathrm{Sp}\, k$ *trivial* if the σ-sheaf $\underline{\mathcal{F}} \otimes_{\mathcal{C}(\infty)} \mathcal{C}(1)$ over $A(1)$ arising by change of coefficients is trivial.

(iii) We call a locally free algebraic σ-sheaf $(\mathcal{F}, \sigma_{\mathcal{F}})$ *rigid analytically trivial* if its "rigid analytification" $(\mathcal{F}^{\mathrm{rig}}, \sigma_{\mathcal{F}^{\mathrm{rig}}})$ is trivial.

(iv) Consider the natural $A(1)$-homomorphism

$$\sigma_{\mathrm{M}(1)^\sigma \otimes_A A(1)} : \varsigma_{A(1)}^* \left(\mathrm{M}(1)^\sigma \otimes_A A(1)\right) \xrightarrow{\sim} \mathrm{M}(1)^\sigma \otimes_A A(1).$$

We call a dual Anderson A-motive $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r over k *rigid analytically trivial* if its associated rigid σ-sheaf $(\mathcal{F}_{\mathrm{M}}^{\mathrm{rig}}, \sigma_{\mathcal{F}_{\mathrm{M}}^{\mathrm{rig}}})$ over $\mathrm{Sp}\, k$ is trivial, so if

$$\phi : \begin{pmatrix} \mathrm{M}(1)^\sigma \otimes_A A(1) \\ \sigma_{\mathrm{M}(1)^\sigma \otimes_A A(1)} \end{pmatrix} \to \begin{pmatrix} \mathrm{M}(1) = \mathrm{M} \otimes_{A_k} A(1) \\ \sigma_{\mathrm{M}(1)} \end{pmatrix}$$

with $\phi(\mathrm{m} \otimes f) := f\mathrm{m}$ is an isomorphism.

Motivated by this, we make the analogous definitions for Papanikolas Q-motives.

Definition 2.5.6. Consider a Papanikolas Q-motive (P, σ_P) over k and the map $\varsigma_{Q(1)}^* : Q(1) \to Q(1)$ induced by $\varsigma_{A(1)}^* : A(1) \to A(1)$.

(i) We define the $Q(1)$-module $P(1) := P \otimes_{Q_k} Q(1)$ together with the $Q(1)$-homomorphism

$$\sigma_{P(1)} : \varsigma_{Q(1)}^* P(1) \xrightarrow{\sim} P(1)$$

induced by σ_P.

(ii) The *σ-invariants* $P(1)^\sigma$ of $\underline{P}$ are defined to be the Q-module

$$P(1)^\sigma := \{p \in P(1) \mid \sigma_{P(1)}(\varsigma_{Q(1)}^* p) = p\}.$$

(iii) We call a dual Papanikolas Q-motive (P, σ_P) over k *rigid analytically trivial* if the natural map

$$\phi : P(1)^\sigma \otimes_Q Q(1) \to P(1), \quad p \otimes g \mapsto gp,$$

is an isomorphism.

Definition 2.5.7. We define the following strictly full subcategories by restriction:

(i) the *category $\mathscr{PRDA}^I \subset \mathscr{DA}^I$ of pure rigid analytically trivial dual Anderson A-motives up to isogeny*,

(ii) the *category $\mathscr{PRDA}^I_+ \subset \mathscr{DA}^I_+$ of pure rigid analytically trivial dual Anderson A-motives of positive rank and dimension up to isogeny*,

(iii) the *category $\mathscr{R} \subset \mathscr{P}$ of rigid analytically trivial Papanikolas Q-motives*,

(iv) the *category $\mathscr{PR} \subset \mathscr{R}$ of pure rigid analytically trivial Papanikolas Q-motives.*

We want to show that we may restrict the functor $\mathcal{P} : \mathscr{DA}^I \to \mathscr{P}$ to a functor from the category $\mathscr{PRDA}^I$ to the category $\mathscr{R}$. In order to do this, we need to show that this is well-defined. This will follow from the last assertion of the next Proposition.

Using the the inclusions $i^* : \mathbb{F}_q[t] \to A$ and $i^* : \mathbb{F}_q(t) \hookrightarrow Q$, we see that

$$\mathrm{M}(1) \cong \mathrm{Mat}_{1\times r'}(\mathbb{T}) \quad \text{and} \quad P(1) \cong \mathrm{Mat}_{1\times r'}(\mathbb{L}) \quad \text{with } \mathbb{L} := \mathrm{Quot}(\mathbb{T}),$$

for a dual Anderson A-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r over k and Papanikolas Q-motive (P, σ_P) of rank r over k. We may then state the following:

Proposition 2.5.8 (Cf. [Pap08, Prop. 3.3.9 and Prop. 3.4.7]). *(i) Let $\underline{P} = (P, \sigma_P)$ be a Papanikolas Q-motive of rank r over k and $\Phi_{\mathbf{p}}$ represent σ_P with respect to a $k(t)$-basis $\mathbf{p}$ of P.*

(a) $\underline{P}$ is rigid analytically trivial if and only if there is a matrix $\Psi_{\mathbf{p}} \in \mathrm{GL}_{r'}(\mathbb{L})$ such that

$$\Psi_{\mathbf{p}}^{(-1)} = \Phi_{\mathbf{p}} \Psi_{\mathbf{p}}.$$

We call $\Psi_{\mathbf{p}}$ a rigid analytic trivialization of $\Phi_{\mathbf{p}}$.

(b) If $\Psi_{\mathbf{p}}$ is a rigid analytic trivialization of $\Phi_{\mathbf{p}}$, then the entries of $\Psi_{\mathbf{p}}^{-1}\mathbf{p}$ form an $\mathbb{F}_q(t)$-basis for $P(1)^\sigma$.

(c) Suppose $\underline{P}$ is rigid analytically trivial, $\Phi_{\mathbf{p}} \in \mathrm{Mat}_{r'\times r'}(k[t])$ and $\det(\Phi_{\mathbf{p}}) = \alpha(t-\theta)^e$ for some $e \geq 0$ and $\alpha \in k^\times$. Then there is a rigid analytic trivialization $\Psi_{\mathbf{p}}$ of $\Phi_{\mathbf{p}}$ such that $\Psi_{\mathbf{p}} \in \mathrm{GL}_r(\mathbb{T})$.

(ii) Let $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ be a dual Anderson A-motive of rank r over k and $\underline{P}$ its associated Papanikolas Q-motive. Suppose $\Phi_{\mathbf{m}}$ represents σ_{M} with respect to a $k[t]$-basis $\mathbf{m} \in \mathrm{Mat}_{r' \times 1}(\mathrm{M})$ for M.

(a) $\underline{\mathrm{M}}$ is rigid analytically trivial if and only if there is a matrix $\Psi_{\mathbf{m}} \in \mathrm{GL}_{r'}(\mathbb{T})$ such that

$$\Psi_{\mathbf{m}}^{(-1)} = \Phi_{\mathbf{m}} \Psi_{\mathbf{m}}.$$

We call $\Psi_{\mathbf{m}}$ a rigid analytic trivialization of $\Phi_{\mathbf{m}}$.

(b) If $\Psi_{\mathbf{m}} \in \mathrm{GL}_{r'}(\mathbb{T})$ is a rigid analytic trivialization of $\Phi_{\mathbf{m}}$, then the entries of $\Psi_{\mathbf{m}}^{-1}\mathbf{m}$ form an $\mathbb{F}_q[t]$-basis for $\mathrm{M}(1)^{\sigma}$.

(c) $\underline{\mathrm{M}}$ is rigid analytically trivial if and only if $\underline{P}$ is rigid analytically trivial.

Corollary 2.5.9 (Cf. [Pap08, Theorem 3.4.9])**.** *The functor $\mathscr{PRDA}^I \to \mathscr{PR}$ that maps a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ to its associated Papanikolas Q-motive $\mathcal{P}(\underline{\mathrm{M}})$ is well-defined and fully faithful. By abuse of notation we denote the restriction also by $\mathcal{P}$. Similarly, we write $\mathcal{P} : \mathscr{PRDA}_+^I \to \mathscr{PR}$.*

In order to show that an Anderson A-module is uniformizable in Section 4.1.2 if and only if its associated dual Anderson A-motive is rigid analytically trivial, we need to state a further equivalent condition for a dual Anderson A-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$ to be rigid analytically trivial. We define the *a-adic completion* of M to be

$$\mathrm{M}_a := \varprojlim \mathrm{M}/a^n\mathrm{M} = \mathrm{M} \otimes_{A_k} \varprojlim (A/a^n \otimes_{\mathbb{F}_q} k)$$

(cf. [ABP, §1.8.6]). By viewing M as a free $k[t]$-module of rank r' via the ring homomorphism $i^* : \mathbb{F}_q[t] \to A$, $t \mapsto a$, we find

$$\mathrm{M}_a = \mathrm{M} \otimes_{A_k} (A \otimes_{\mathbb{F}_q[t]} k[\![t]\!]) \cong \mathrm{Mat}_{1 \times r'}(k[\![t]\!]).$$

Let $\sigma : \mathrm{M} \to \mathrm{M}$ denote the ς^*-linear map induced by σ_{M}. Note that σ induces a $\varsigma^*_{A/(a)}$-linear map $\mathrm{M}_a \to \mathrm{M}_a$, which we also denote by σ by abuse of notation. We define the *σ-invariants* M_a^{σ} *of* M_a to be

$$\mathrm{M}_a^{\sigma} := \{\mathrm{m} \in \mathrm{M}_a \mid \sigma(\mathrm{m}) = \mathrm{m}\}.$$

Moreover, we call the elements in $\mathrm{M}(1) \subseteq \mathrm{M}_a$ *convergent* and $\mathrm{m} \in \mathrm{M}_a$ an M-*cycle* if m is convergent and σ-invariant, that is, if

$$\mathrm{m} \in \mathrm{M}(1) \cap \mathrm{M}_a^{\sigma}.$$

Since $\mathrm{M}(1) \cap \mathrm{M}_a^{\sigma} = \mathrm{M}(1)^{\sigma}$ we have the following consequence:

Corollary 2.5.10 (Cf. [ABP, §1.8.7])**.** *A dual Anderson A-module $(\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r over k is rigid analytically trivial if and only if the natural map*

$$(A\text{-}module\ of\ \mathrm{M}\text{-}cycles) \otimes_A A(1) \hookrightarrow \mathrm{M}(1)$$

is bijective.

2.6 Papanikolas's definition of pure dual t-motives

Following Papanikolas in [Pap08], we show that $\mathscr{PR}$ is a neutral Tannakian category over Q with fiber functor $(\underline{P} \mapsto P(1)^{\sigma}) : \mathscr{PR} \to \mathscr{V}ec_Q$ in the sense of Definition 1.2.11. Having done this, we define the Tannakian category $\mathscr{PT}$ of pure dual t-motives as a subcategory of $\mathscr{PR}$.

Lemma 2.6.1. *Consider a Papanikolas Q-motive $\underline{P} = (P, \sigma_P)$ of rank r over k and let $\mu_1, \dots, \mu_m \in P(1)^{\sigma}$. If $\mu_1, \dots, \mu_m$ are linearly independent over Q, then they are linearly independent over $Q(1)$ in $P(1)$. In particular,*

$$\dim_Q P(1)^{\sigma} \leq r.$$

Equality holds if and only if $\underline{P}$ is rigid analytically trivial.

Proof. For the sake of contradiction, we assume that $m \geq 2$ is minimal such that $\mu_1, \dots, \mu_m$ are linearly independent over Q, but $\mu_1, \dots, \mu_m$ are not linearly independent over $Q(1)$ in $P(1)$. That is, there are $\alpha_i \in Q(1)$ such that

$$\sum_{i=1}^{m} \alpha_i \mu_i = 0.$$

Without loss of generality, we suppose $\alpha_1 = 1$. Since $\operatorname{im} \sigma_P \cong P$, we also have

$$\sum_{i=1}^{m} \alpha_i^{(-1)} \mu_i = 0$$

and thus $\sum_{i=1}^{m} (\alpha_i - \alpha_i^{(-1)}) \mu_i = \sum_{i=2}^{m} (\alpha_i - \alpha_i^{(-1)}) \mu_i = 0$. Taking $\underline{\mathrm{M}} = \underline{\mathrm{N}} = (A_k, \varsigma^* A_k \xrightarrow{\sim} A_k)$ in Proposition 2.2.11, we find $\alpha_i \in Q$. This contradicts the assumption. Thus the natural map $\phi : P(1)^{\sigma} \otimes_Q Q(1) \to P(1)$ is injective, which proves $\dim_Q P(1)^{\sigma} \leq \dim_{Q(1)} P(1) = \dim_{Q_k} P = r$. Clearly equality holds if and only if ϕ is also surjective, which means that $\underline{P}$ is rigid analytically trivial. □

Proposition 2.6.2 (Cf. [Pap08, Prop. 3.3.11])**.** *Let $\underline{P} = (P, \sigma_P)$ be a rigid analytically trivial Papanikolas Q-motive and*

$$0 \to \underline{P}' \to \underline{P} \to \underline{P}'' \to 0$$

be a short exact sequence of Papanikolas Q-motives. Then the following hold:

(i) The Papanikolas Q-motives $\underline{P}' = (P', \sigma_{P'})$ and $\underline{P}'' = (P'', \sigma_{P''})$ are also rigid analytically trivial.

(ii) The sequence $0 \to P(1)^{\sigma} \to P'(1)^{\sigma} \to P''(1)^{\sigma} \to 0$ is a short exact sequence of Q-vector spaces.

Proof. By definition, we see that the sequence $0 \to P'(1)^{\sigma} \to P(1)^{\sigma} \to P''(1)^{\sigma}$ is exact. Consider the natural map $\phi : P(1)^{\sigma} \otimes_Q Q(1) \to P''(1)^{\sigma} \otimes_Q Q(1)$ that gives rise to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & P'(1)^{\sigma} \otimes_Q Q(1) \arrow[r] \arrow[d] & P(1)^{\sigma} \otimes_Q Q(1) \arrow[r] \arrow[d, "\wr"] & \operatorname{im} \phi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'(1) \arrow[r] & P(1) \arrow[r] & P''(1) \arrow[r] & 0.
\end{tikzcd}$$

Since all three maps are injective by Lemma 2.6.1, we find that all of them must be isomorphisms. Hence, $\underline{P}'$ is rigid analytically trivial and

$$\dim_Q P''(1)^\sigma = \dim_{Q(1)} P''(1)^\sigma \otimes_Q Q(1) \geq \dim_{Q(1)} \operatorname{im}\phi = \dim_{Q(1)} P''(1) = \dim_{Q_k} P''.$$

Again by Lemma 2.6.1 we may conclude that $\underline{P}''$ is rigid analytically trivial. Finally we find that (ii) holds since

$$\dim_Q P(1)^\sigma = \dim_{Q_k} P = \dim_{Q_k} P' + \dim_{Q_k} P'' = \dim_Q P'(1)^\sigma + \dim_Q P''(1)^\sigma. \qquad \square$$

In particular, we see that kernels and cokernels exist in $\mathscr{R}$, which implies that $\mathscr{R}$ is an abelian category. By (ii), $(P \mapsto P(1)^\sigma) : \mathscr{P}\mathscr{R} \to \mathscr{V}ec_Q$ is a Q-linear tensor functor. The next proposition shows that it is indeed a fiber functor and $\mathscr{P}\mathscr{P}$ and $\mathscr{R}$ are rigid tensor subcategories of $\mathscr{P}$.

Proposition 2.6.3. *Let $\underline{P}_1 = (P_1, \sigma_{P_1})$ and $\underline{P}_2 = (P_2, \sigma_{P_2})$ be rigid analytically trivial Papanikolas Q-motives of respectively rank r_1 and r_2 over k. Consider the isomorphisms*

$$\phi_1 : P_1(1)^\sigma \otimes_Q Q(1) \xrightarrow{\sim} P_1(1) \quad \textit{and} \quad \phi_2 : P_1(1)^\sigma \otimes_Q Q(1) \xrightarrow{\sim} P_1(1).$$

(i) The natural map $\operatorname{Hom}_{\mathscr{P}}(\underline{P}_1, \underline{P}_2) \to \operatorname{Hom}_Q(P_1(1)^\sigma, P_2(1)^\sigma)$, $f \mapsto \phi_2 \circ f \circ \phi_1^{-1}|_{P_1(1)^\sigma}$, is injective;

(ii) $\underline{P}_1 \otimes \underline{P}_2$ is rigid analytically trivial, and the natural map $P_1(1)^\sigma \otimes_Q P_2(1)^\sigma \to (P_1 \otimes P_2)^\sigma$ is an isomorphism of Q-vector spaces;

(iii) $\underline{P}_1^\vee$ is rigid analytically trivial and the natural map $(P_1(1)^\sigma)^\vee \to P_1^\vee(1)^\sigma$ is an isomorphism of Q-vector spaces.

Proof. Part (i) is clear from the definition. To see (ii), we let $\mathbf{p}$ and $\mathbf{q}$ be $k(t)$-bases for $\underline{P}_1$ and $\underline{P}_2$, respectively. By Lemma 2.3.7 the Kronecker product $\Phi_{\mathbf{p}\otimes\mathbf{q}} = \Phi_{\mathbf{p}} \otimes \Phi_{\mathbf{q}}$ represents $\sigma_{P_1 \otimes P_2}$ with respect to the $k(t)$-basis $\mathbf{p} \otimes \mathbf{q}$ for $P_1 \otimes P_2$. Because $\underline{P}_1$ and $\underline{P}_2$ are rigid analytically trivial, there are matrices $\Psi_{\mathbf{p}}$ and $\Psi_{\mathbf{q}}$ that are rigid analytic trivializations of $\Phi_{\mathbf{p}}$ and $\Phi_{\mathbf{q}}$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
\varsigma_{Q(1)}^* ((P_1 \otimes_{Q_k} P_2) \otimes_{Q_k} Q(1)) \arrow[r, "\sigma_{P_1\otimes P_2}\otimes \mathrm{id}_{Q(1)}"] \arrow[d, "\wr"] & (P_1 \otimes_{Q_k} P_2) \otimes_{Q_k} Q(1) \arrow[d, "\wr"] \\
\varsigma_{Q(1)}^* ((P_1 \otimes_{Q_k} Q(1)) \otimes_{Q(1)} (P_2 \otimes_{Q_k} Q(1))) \arrow[r, "(\sigma_{P_1}\otimes \mathrm{id}_{Q(1)})\otimes(\sigma_{P_2}\otimes \mathrm{id}_{Q(1)})"'] & (P_1 \otimes_{Q_k} Q(1)) \otimes_{Q(1)} (P_2 \otimes_{Q_k} Q(1)).
\end{tikzcd}$$

Hence, the Kronecker product $\Psi_{\mathbf{p}\otimes\mathbf{q}} := \Psi_{\mathbf{p}} \otimes \Psi_{\mathbf{q}}$ provides a rigid analytic trivialization of $\Phi_{\mathbf{p}\otimes\mathbf{q}}$, which proves the first part of (ii). To show the first part of (iii), we let $\mathbf{p}$ be a $k(t)$-basis for P. Again by Lemma 2.3.7, it follows that $\Phi_{\mathbf{p}^\vee} := (\Phi_{\mathbf{p}}^{-1})^{\mathrm{tr}}$ represents σ_P with respect to the dual basis. Then there is a matrix $\Psi_{\mathbf{p}}$, which is a rigid analytic trivialization of $\Phi_{\mathbf{p}}$. Similarly to above, we find that $\Psi_{\mathbf{p}^\vee} := (\Psi_{\mathbf{p}}^{-1})^{\mathrm{tr}}$ is a rigid analytic trivialization of $\Phi_{\mathbf{p}^\vee}$. The second parts of (ii) and (iii) follow from Proposition 2.5.8 (i). $\square$

Thus [Pap08, Thm. 3.3.15] also holds when Q is an arbitrary function field:

Theorem 2.6.4. *(i) The category $\mathscr{R}$ of rigid analytically trivial Papanikolas Q-motives is a neutral Tannakian category over Q with fiber functor $(\underline{P} \mapsto P(1)^\sigma) : \mathscr{R} \to \mathscr{V}ec_Q$.*

(ii) The category $\mathscr{PR}$ of pure rigid analytically trivial Papanikolas Q-motives is a neutral Tannakian category over Q with fiber functor $(\underline{P} \mapsto P(1)^{\sigma}) : \mathscr{PR} \to \mathscr{V}ec_Q$.

Since we later want to relate pure Q-Hodge-Pink structures and pure dual t-motives associated with *pure* rigid analytically trivial dual Anderson A-motives, we need to modify Papanikolas's definition of the category $\mathscr{T}$ correspondingly.

Definition 2.6.5 (Cf. [Pap08, §3.4.10]). (i) The *category $\mathscr{PT}$ of pure dual t-motives* is defined to be the strictly full Tannakian subcategory of $\mathscr{PR}$ generated by the essential image of the functor $\mathscr{PRDA}^I \to \mathscr{PR}$. Its fiber functor is

$$\omega : \mathscr{PT} \to \mathscr{V}ec_Q, \quad \underline{P} \mapsto P(1)^{\sigma},$$

and we call a pure Papanikolas Q-motive $\underline{P} = (P, \sigma_P) \in \mathrm{Ob}(\mathscr{PT})$ of rank r and weight $\frac{l}{n}$ over k a *pure dual Q-motive of rank r and weight $\frac{l}{n}$ over k.* If $Q = \mathbb{F}_q(t)$, then we refer to objects in $\mathscr{PT}$ as *pure dual t-motives.*

(ii) $\Gamma_{\mathscr{PT}} := \underline{\mathrm{Aut}}^{\otimes}(\omega)$ is defined to be the affine Q-group scheme given by Tannakian duality such that the category $\mathscr{R}ep_Q(\Gamma_{\mathscr{PT}})$ is equivalent to $\mathscr{PT}$.

(iii) We define an extension $\omega^{(R)} : \mathscr{PT} \to \mathscr{M}od_R$ from the category of pure dual Q-motives to the category of finitely generated left R-modules of ω by setting $\omega^{(R)}(\underline{P}) := P(1)^{\sigma} \otimes_Q R$ for an arbitrary commutative Q-algebra R so that $\Gamma_{\mathscr{PT}}(R) = \underline{\mathrm{Aut}}^{\otimes}(\omega^{(R)})$.

We observe that such a pure dual Q-motive $\underline{P} \in \mathrm{Ob}(\mathscr{PT})$ over k is constructed via direct sums, subquotients, tensor products, duals and internal Hom's from Papanikolas Q-motives over k associated with pure rigid analytically trivial dual Anderson A-motives over k.

2.7 *Taelman's equivalent definition of pure dual t-motives*

In order to define a functor to the category of Q-Hodge-Pink structures, we give an alternative description of $\mathscr{PT}$, which was elaborated by Taelman in [Tae07] and [Tae09]. The idea is to make the Carlitz t-motive invertible and to describe pure dual t-motives similarly as classical pure motives [DMOS82, II.§6]. The latter is a pair $M(i)$, that is, an effective motive M twisted with the i-fold tensor power of the Tate motive. The Tate motive $\mathbb{1}(1)$ is the inverse of the Lefschetz motive $\mathbb{1}(-1)$ whose function field counterpart is the Carlitz t-motive when $A = \mathbb{F}_q[t]$.[3] The dual Carlitz t-motive is invertible if the functor $(\underline{\mathrm{M}} \mapsto \underline{\mathrm{M}} \otimes \underline{\mathrm{C}}) : \mathscr{PRDA}^I \to \mathscr{PRDA}^I$ is an equivalence of categories. We first see that it is fully faithful by the next lemma. For the rest of this chapter, we restrict ourselves to $A = \mathbb{F}_q[t]$.

Lemma 2.7.1 (Cf. Lemma 2.3.9). *Let $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ and $\underline{\mathrm{N}} = (\mathrm{N}, \sigma_{\mathrm{N}})$ be pure dual Anderson A-motives over k and $n \in \mathbb{N}$. Then the natural map*

$$\begin{array}{rcl} \phi : \mathrm{Hom}_k(\underline{\mathrm{M}}, \underline{\mathrm{N}}) & \to & \mathrm{Hom}_k(\underline{\mathrm{M}} \otimes \underline{\mathrm{C}}^n, \underline{\mathrm{M}} \otimes \underline{\mathrm{C}}^n), \\ f & \mapsto & f \otimes \mathrm{id}, \end{array}$$

is an isomorphism.

[3] The construction described here can be generalized to arbitrary function fields by considering the different isomorphism classes of Carlitz A-motives, see [Gos96, §7.1].

Proof. Because C^n is the n-fold tensor power of A_k by definition, the natural map

$$\begin{array}{rcl}\phi : \mathrm{Hom}_{A_k}(\mathrm{M},\mathrm{N}) & \to & \mathrm{Hom}_{A_k}(\mathrm{M}\otimes \mathrm{C}^n, \mathrm{M}\otimes \mathrm{C}^n) \\ f & \mapsto & f \otimes \mathrm{id}\end{array}$$

is an isomorphism of A_k-modules. Consider an arbitrary $f\otimes \mathrm{id} \in \mathrm{Hom}_k(\underline{\mathrm{M}}\otimes\underline{\mathrm{C}}^n, \underline{\mathrm{N}}\otimes\underline{\mathrm{C}}^n)$. This means that

$$(f\otimes \mathrm{id})\circ(t-\theta)^n\sigma_{\mathrm{M}} = (t-\theta)^n\sigma_{\mathrm{N}}\circ\varsigma^*(f\otimes\mathrm{id})$$

so that we find, as desired,

$$f\circ\sigma_{\mathrm{M}} = \sigma_{\mathrm{N}}\circ\varsigma^* f.$$

□

Suppose that $\underline{\mathrm{M}}$ is a pure dual Anderson A-motive of weight $\frac{l}{n}$ over k and $i\in\mathbb{N}$ a non-negative integer. By Proposition 2.4.11, we see that the tensor product $\underline{\mathrm{M}}\otimes\underline{\mathrm{C}}^i$ of $\underline{\mathrm{M}}$ and the i-fold tensor power of the dual Carlitz t-motive $\underline{\mathrm{C}}$ over k is pure of weight $\frac{l}{n}+i$.

Definition 2.7.2. (i) A *pure dual t-motive of rank r and weight $\frac{l}{n}-i$ over k* is a pair $\underline{P} = (\underline{\mathrm{M}}, i)$, where $\underline{\mathrm{M}}$ is a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ of rank r and weight $\frac{l}{n}$ over k and $i\in\mathbb{Z}$ an integer.[4]

(ii) A *morphism* $f : (\underline{\mathrm{M}}_1, i_1)\to(\underline{\mathrm{M}}_2, i_2)$ *between pure dual t-motives* is a quasi-morphism $f\in\mathrm{QHom}_k(\underline{\mathrm{M}}_1\otimes\underline{\mathrm{C}}^{N-i_1}, \underline{\mathrm{M}}_2\otimes\underline{\mathrm{C}}^{N-i_2})$ of dual Anderson A-motives, where $N\in\mathbb{Z}$ is sufficiently large.

We denote the category of pure dual t-motives by $\mathscr{P}\mathscr{T}'$ and the set of morphisms between pure dual t-motives $\underline{P}_1$ and $\underline{P}_2$ in $\mathscr{P}\mathscr{T}'$ by $\mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\underline{P}_1, \underline{P}_2)$.

To see that these definitions are well-defined, consider pure dual t-motives $(\underline{\mathrm{M}}\otimes\underline{\mathrm{C}}, -n)$ and $(\underline{\mathrm{M}}, -n-1)$ over k for $n\in\mathbb{N}^{>0}$. Then the natural isomorphism between pure dual Anderson A-motives $(\underline{\mathrm{M}}\otimes\underline{\mathrm{C}})\otimes\underline{\mathrm{C}}^{n-1}$ and $\underline{\mathrm{M}}\otimes\underline{\mathrm{C}}^n$ provides an isomorphism

$$(\underline{\mathrm{M}}\otimes\underline{\mathrm{C}}, -n+1)\cong(\underline{\mathrm{M}}, -n). \tag{2.12}$$

We then find that $\mathcal{P}'(\underline{\mathrm{C}}^n)\cong(\underline{\mathbb{1}}_{A_k}, -n)$. The identity object is given by $(\underline{\mathbb{1}}_{A_k}, 0)$ since

$$(\mathrm{M}, i)\cong(\underline{\mathrm{M}}\otimes\underline{\mathbb{1}}_{A_k}, i+0) = (\underline{\mathrm{M}}, i)\otimes(\underline{\mathbb{1}}_{A_k}, 0)$$

For $i\in\mathbb{Z}$, we write alternatively $\underline{\mathrm{M}}(i)$ for a pure dual t-motive $(\underline{\mathrm{M}}, i)$. Motivated by classical theory, we call the pure dual t-motive $\underline{\mathbb{1}}_{A_k}(1)$ the *dual Tate t-motive*. We will see later that $\underline{\mathbb{1}}_{A_k}(1)$ corresponds indeed to the inverse of the dual Lefschetz t-motive over k (Definition 2.3.8). By Lemma 2.7.1 we may make the following

Definition 2.7.3. Let $\mathcal{P}' : \mathscr{P}\mathscr{R}\mathscr{D}\mathscr{A}^I\to\mathscr{P}\mathscr{T}'$ be the fully faithful functor that sends a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ over k to the pure dual t-motive $(\underline{\mathrm{M}}, 0)$. By abuse of notation we denote the restriction $\mathscr{P}\mathscr{R}\mathscr{D}\mathscr{A}^I_+\to\mathscr{P}\mathscr{T}'$ of $\mathcal{P}$ to $\mathscr{P}\mathscr{R}\mathscr{D}\mathscr{A}^I_+$ also by $\mathcal{P}'$.

Next we want to make $\mathscr{P}\mathscr{T}'$ into a rigid abelian Q-linear tensor category.

Definition 2.7.4. Let $(\underline{\mathrm{M}}_1, i_1)$, $(\underline{\mathrm{M}}_2, i_2)$ and $(\underline{\mathrm{M}}_3, i_3)$ be pure dual t-motives over k.

[4] Note that our definition differs from [Tae09, Def. 2.3.2] by sign. This coincides with the notation used in classical theory, see for example [DMOS82, II.§6].

(i) The *direct sum of* $(\underline{\mathrm{M}}_1, i_1)$ *and* $(\underline{\mathrm{M}}_2, i_2)$ is the pure dual t-motive

$$(\underline{\mathrm{M}}_1, i_1) \oplus (\underline{\mathrm{M}}_2, i_2) := ((\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N-i_1}) \oplus (\underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N-i_2}), N),$$

where $N \in \mathbb{Z}$ is sufficiently large.

(ii) The *tensor product of* $(\underline{\mathrm{M}}_1, i_1)$ *and* $(\underline{\mathrm{M}}_2, i_2)$ is the pure dual t-motive

$$(\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{M}}_2, i_1 + i_2).$$

(iii) A *short exact sequence of pure dual t-motives over k*

$$0 \longrightarrow (\underline{\mathrm{M}}_1, i_1) \longrightarrow (\underline{\mathrm{M}}_2, i_2) \longrightarrow (\underline{\mathrm{M}}_3, i_3) \longrightarrow 0$$

is a short exact sequence of the underlying dual Anderson A-motives in $\mathscr{D}\mathscr{A}^I$

$$0 \longrightarrow \underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N-i_1} \longrightarrow \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N-i_2} \longrightarrow \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{N-i_3} \longrightarrow 0,$$

where N is sufficiently large.

Note that $\mathscr{P}\mathscr{T}'$ is abelian since $\mathscr{D}\mathscr{A}^I$ is abelian by Lemma 2.2.9. Recall that the definition of the inner hom of Papanikolas Q-motives uses the fact that the underlying maps are Q_k-isomorphisms by definition. The next lemma helps out that the A_k-homomorphism underlying a pure dual t-motive over k does not need to be bijective.

Lemma 2.7.5 (Cf. [Tae09, Prop. 2.3])**.** *Let $\underline{\mathrm{M}}_i = (\mathrm{M}_i, \sigma_{\mathrm{M}_i})$ be a pure dual Anderson A-motive of rank r_i over k for $i = 1, 2$. For N sufficiently large, the subgroup*

$$\varsigma^* \operatorname{Hom}_{A_k}(\mathrm{M}_1, \mathrm{M}_2 \otimes \mathrm{C}^N) \subset \varsigma_Q^* \operatorname{Hom}_{Q_k}(\mathrm{M}_1 \otimes_{A_k} Q_k, \mathrm{M}_2 \otimes_{A_k} \mathrm{C}^N \otimes_{A_k} Q_k)$$

is stable under $\varsigma^* f \mapsto \sigma_{\mathrm{M}_2 \otimes \mathrm{C}^N} \circ \varsigma^* f \circ \sigma_{\mathrm{M}_1}^{-1}$.

Proof. Suppose that $\Phi_{\mathbf{m}_i} \in \operatorname{Mat}_{r_i \times r_i}(k[t])$ represents σ_{M_i} with respect to a $k[t]$-basis $\mathbf{m}_i \in \operatorname{Mat}_{r_i \times 1}(\mathrm{M}_i)$ for M_i, $i = 1, 2$. Then the matrix $(t-\theta)^N \Phi_{\mathbf{m}_2}$ represents $\sigma_{\mathrm{M}_2 \otimes_{k[t]} \mathrm{C}^N}$ with respect to the basis $\mathbf{m}_2 \otimes \mathbf{c}$, where $\mathbf{c}$ is a $k[t]$-basis for the Q_k-vector space C^N underlying the Carlitz t-motive $\underline{\mathrm{C}}$.

On matrices, the map

$$\varsigma_Q^* f \mapsto (\sigma_{\mathrm{M}_2 \otimes \mathrm{C}^N} \otimes \mathrm{id}_{Q_k}) \circ \varsigma_Q^* f \circ (\sigma_{\mathrm{M}_1} \otimes \mathrm{id}_{Q_k})^{-1}$$

is then given by

$$\begin{array}{rcl} \operatorname{Mat}_{r_2' \times r_1'}(k(t)) & \to & \operatorname{Mat}_{r_2' \times r_1'}(k(t)) \\ F & \mapsto & (t-\theta)^N \Phi_{\mathbf{m}_2} \cdot F \cdot \Phi_{\mathbf{m}_1}^{-1}. \end{array}$$

Recall that we have $\det \Phi_{\mathbf{m}_1} = \alpha (t-\theta)^d$ for some $\alpha \in k^\times$ and $d \in \mathbb{N}$. Hence, for N sufficiently large, $(t-\theta)^N \Phi_{\mathbf{m}_1}^{-1} \in \operatorname{Mat}_{r_1' \times r_1'}(k[t])$. Therefore $\operatorname{Mat}_{r_2' \times r_1'}(k[t])$ gets mapped to itself, which proves the assertion. $\square$

This allows us to define an inner hom of pure dual Anderson A-motives in some cases.

Definition 2.7.6. Let $\underline{\mathrm{M}}_1 = (\mathrm{M}_1, \sigma_{\mathrm{M}_1})$ and $\underline{\mathrm{M}}_2 = (\mathrm{M}_2, \sigma_{\mathrm{M}_2})$ be pure dual Anderson A-motives over k. For N sufficiently large, we define the *inner hom* $\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^N)$ to be the A_k-module $\mathrm{Hom}_{A_k}(\mathrm{M}_1, \mathrm{M}_2 \otimes_{A_k} \mathrm{C}^N)$ together with the A_k-module homomorphism

$$\begin{aligned} \sigma_{\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^N)} : \varsigma^* \mathrm{Hom}_{A_k}(\mathrm{M}_1, \mathrm{M}_2 \otimes_{A_k} \mathrm{C}^N) &\to \mathrm{Hom}_{A_k}(\mathrm{M}_1, \mathrm{M}_2 \otimes_{A_k} \mathrm{C}^N), \\ \varsigma^* f &\mapsto \sigma_{\mathrm{M}_2 \otimes_{A_k} \mathrm{C}^N} \circ \varsigma^* f \circ \sigma_{\mathrm{M}_1}^{-1}. \end{aligned}$$

For $N \in \mathbb{N}$, we then have a natural isomorphism

$$\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^N) \otimes \underline{\mathrm{C}} \xrightarrow{\sim} \mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+1}), \tag{2.13}$$

by which we may extend the definition of the inner hom to the category of pure dual t-motives.

Definition 2.7.7. We let $\underline{P}_1 = (\underline{\mathrm{M}}_1, i_1)$ and $\underline{P}_2 = (\underline{\mathrm{M}}_2, i_2)$ be pure dual t-motives over k and N a sufficiently large integer.

(i) The *inner hom* is given by the pure dual t-motive

$$\mathcal{H}om(\underline{P}_1, \underline{P}_2) := (\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2}), N).$$

(ii) We define the *dual* $\underline{P}_1^\vee$ *of* $\underline{P}_1$ to be pure dual t-motive

$$\underline{P}_1^\vee := \mathcal{H}om((\underline{\mathrm{M}}_1, i_1), (\underline{\mathbb{1}}_{A_k}, 0)) \cong (\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{C}}^{N+i_1}), N).$$

We note that the inner hom of pure dual t-motives $(\underline{\mathrm{M}}_1, i_1)$ and $(\underline{\mathrm{M}}_2, i_2)$ is pure of weight $(\mathrm{wt}(\underline{\mathrm{M}}_2) - i_2) - (\mathrm{wt}(\underline{\mathrm{M}}_1) - i_1)$. By the next proposition we find that $\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2)$ plays indeed the role of an inner hom in $\mathscr{P}$ that is compatible with tensor products.

Proposition 2.7.8. *(i) Let $\underline{P}_1$, $\underline{P}_2$ and $\underline{P}_3$ be pure dual t-motives over k. The inner hom satisfies the adjunction formula*

$$\mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\underline{P}_1 \otimes \underline{P}_2, \underline{P}_3) \cong \mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\underline{P}_1, \mathcal{H}om(\underline{P}_2, \underline{P}_3)). \tag{2.14}$$

(ii) Consider finite families $(\underline{P}_i)_{i\in I}$ and $(\underline{P}'_i)_{i\in I}$ of pure dual t-motives over k. Then there is an isomorphism

$$t_{\otimes_{i\in I}\underline{P}_i, \otimes_{i\in I}\underline{P}'_i} : \otimes_{i\in I} \mathcal{H}om(\underline{P}_i, \underline{P}'_i) \xrightarrow{\sim} \mathcal{H}om(\otimes_{i\in I}\underline{P}_i, \otimes_{i\in I}\underline{P}_i).$$

(iii) Every pure dual t-motive $\underline{P}$ over k is reflexive.

Proof. Suppose $\underline{P}_1 = (\underline{\mathrm{M}}_1, i_1)$, $\underline{P}_2 = (\underline{\mathrm{M}}_2, i_2)$ and $\underline{P}_3 = (\underline{\mathrm{M}}_3, i_3)$. For $N_2 \in \mathbb{Z}$ sufficiently large, we have that (2.14) is by definition

$$\mathrm{Hom}_{\mathscr{P}\mathscr{T}'}((\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{M}}_2, i_1 + i_2), (\underline{\mathrm{M}}_3, i_3)) \cong \mathrm{Hom}_{\mathscr{P}\mathscr{T}'}((\underline{\mathrm{M}}_1, i_1), (\mathcal{H}om(\underline{\mathrm{M}}_2, \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{N_2+i_2-i_3}), N_2)).$$

Choosing integers $N_1 \geq N_2$ sufficiently large, this is defined to be

$$\begin{aligned} &\mathrm{QHom}_k((\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{M}}_2) \otimes \underline{\mathrm{C}}^{(N_1+N_2)-i_1-i_2}, \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{(N_1+N_2)-i_3}) \\ &\quad \cong \mathrm{QHom}_k(\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N_1-i_1}, \mathcal{H}om(\underline{\mathrm{M}}_2, \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{N_2+i_2-i_3}) \otimes \underline{\mathrm{C}}^{N_1-N_2}). \end{aligned}$$

Using the underlying natural isomorphisms, we may write this as

$$\begin{aligned}&\mathrm{QHom}_k((\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N_1-i_1}) \otimes (\underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N_2-i_2}), \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{N_1+N_2-i_3})\\ &\quad \cong \mathrm{QHom}_k(\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N_1-i_1}, \mathcal{H}om(\underline{\mathrm{M}}_2, \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{(i_2-N_2)+(N_1+N_2-i_3)})),\end{aligned}$$

which is in turn equivalent to

$$\begin{aligned}&\mathrm{QHom}_k((\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N_1-i_1}) \otimes (\underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N_2-i_2}), \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{N_1+N_2-i_3})\\ &\quad \cong \mathrm{QHom}_k(\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N_1-i_1}, \mathcal{H}om(\underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N_2-i_2}, \underline{\mathrm{M}}_3 \otimes \underline{\mathrm{C}}^{N_1+N_2-i_3}))\end{aligned}$$

Thus it suffices to show that there is an isomorphism

$$\mathrm{Hom}_k(\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{M}}_2, \underline{\mathrm{M}}_3) \cong \mathrm{Hom}_k(\underline{\mathrm{M}}_1, \mathcal{H}om(\underline{\mathrm{M}}_2, \underline{\mathrm{M}}_3)),$$

for some pure dual Anderson A-motives $\underline{\mathrm{M}}_1$, $\underline{\mathrm{M}}_2$ and $\underline{\mathrm{M}}_3$ for which $\mathcal{H}om(\underline{\mathrm{M}}_2, \underline{\mathrm{M}}_3)$ is defined. This is done in the same fashion as in the proof of Proposition 2.3.5. (ii) and (iii) then follow similarly. □

Corollary 2.7.9. *The category $\mathscr{PT}'$ of pure dual t-motives over k is a rigid abelian Q-linear category.*

Note that we can write an arbitrary pure dual t-motive over k as

$$\underline{\mathrm{M}}(i) \cong (\underline{\mathrm{M}} \otimes \mathbb{1}_{A_k})(i) = \underline{\mathrm{M}}(0) \otimes \mathbb{1}_{A_k}(i) = \underline{\mathrm{M}}(0) \otimes \mathbb{1}_{A_k}(1)^i,$$

so that the category $\mathscr{PT}'$ of pure dual t-motives is indeed generated by pure rigid analytically trivial Anderson A-motives over k and the dual Tate t-motive over k. Alternatively, if we identify $\underline{\mathrm{M}}$ with its associated pure dual t-motive $\underline{\mathrm{M}}(0)$, we have

$$\underline{\mathrm{M}}(i) \cong \underline{\mathrm{M}}(0) \otimes \mathbb{1}_{A_k}(-1)^{-i} = \underline{\mathrm{M}}(0) \otimes \underline{\mathrm{C}}(0)^{-i},$$

that is, $\mathscr{PT}'$ is generated by pure rigid analytically trivial Anderson A-motives over k and the inverse of the Carlitz t-motive over k.

Theorem 2.7.10. *The functor $\mathcal{R} : \mathscr{PT}' \to \mathscr{PT}$ that is given by*

$$(\underline{\mathrm{M}}, i) \mapsto \mathcal{R}(\underline{\mathrm{M}}, i) := \mathcal{P}(\underline{\mathrm{M}}) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-i} = \mathcal{P}(\underline{\mathrm{M}}) \otimes \underline{L}^{-i}.$$

is an equivalence of categories and preserves ranks and weights.

We observe that the functors $\mathcal{P} : \mathscr{PRDA}^I \to \mathscr{PT}$ and $\mathcal{R} \circ \mathcal{P}' : \mathscr{PRDA}^I \to \mathscr{PT}$ are isomorphic.

Proof. Note $\mathcal{R}$ is well-defined since for any $(\underline{\mathrm{M}}, i) \in \mathscr{PT}'$

$$\mathcal{R}(\underline{\mathrm{M}}, i) = \mathcal{P}(\underline{\mathrm{M}}) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-i} \cong \mathcal{P}(\underline{\mathrm{M}} \otimes \underline{\mathrm{C}}) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-i-1} = \mathcal{R}(\underline{\mathrm{M}} \otimes \underline{\mathrm{C}}, i+1).$$

Consider $(\underline{\mathrm{M}}_1, i_1), (\underline{\mathrm{M}}_2, i_2) \in \mathrm{Ob}(\mathscr{PT}')$. By definition,

$$\mathrm{Hom}_{\mathscr{PT}'}((\underline{\mathrm{M}}_1, i_1), (\underline{\mathrm{M}}_2, i_2)) = \mathrm{QHom}_k(\underline{\mathrm{M}}_1 \otimes \underline{\mathrm{C}}^{N-i_1}, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N-i_2})$$

for N sufficiently large. This is isomorphic to

$$\mathrm{Hom}_{\mathscr{P}}(\mathcal{P}(\underline{\mathrm{M}}_1) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-i_1}, \mathcal{P}(\underline{\mathrm{M}}_2) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-i_2})$$

by Proposition 2.2.11 and Corollary 2.3.9, so that $\mathcal{R}$ is fully faithful.

We want to show that it is also essentially surjective and hence an equivalence of categories. By definition an arbitrary pure dual t-motive over k in $\mathscr{P}\mathscr{T}$ is constructed via direct sums, tensor products, duals and inner hom's and subquotients from the associated Papanikolas Q-motive $\mathcal{P}(\underline{\mathrm{M}})$ over k of a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ over k. Since the functor $\mathcal{P}$ commutes with forming direct sum and tensor product, we only need to consider the latter two cases.

At first we want to find a pure dual t-motive $(\underline{\mathrm{M}}, i)$ in $\mathscr{P}\mathscr{T}'$ that satisfies $\mathcal{R}(\underline{\mathrm{M}}, i) = \mathcal{H}om(\underline{P}_1, \underline{P}_2)$ for $\underline{P}_1, \underline{P}_2 \in \mathrm{Ob}(\mathscr{P}\mathscr{T})$. Without loss of generality we may assume that $\underline{P}_1 = \mathcal{P}(\underline{\mathrm{M}}_1)$ and $\underline{P}_2 = \mathcal{P}(\underline{\mathrm{M}}_2)$ for pure rigid analytically trivial dual Anderson A-motives. For N sufficiently large, we have

$$\begin{aligned}
\mathcal{R}(\mathcal{H}om((\underline{\mathrm{M}}_1, 0), (\underline{\mathrm{M}}_2, 0))) &= \mathcal{R}(\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^N), N) \\
&= \mathcal{P}(\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^N)) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-N} \\
&\cong \mathcal{H}om(\underline{P}_1, \underline{P}_2 \otimes \underline{L}^N) \otimes \mathcal{P}(\underline{\mathrm{C}})^{-N} \cong \mathcal{H}om(\underline{P}_1, \underline{P}_2).
\end{aligned}$$

Thus it remains to consider the case that $\underline{P}'' = (P'', \sigma''_P) \in \mathrm{Ob}(\mathscr{P}\mathscr{T})$ is a subquotient of a $\underline{P} \in \mathrm{Ob}(\mathscr{P}\mathscr{T})$. Again we may assume that $\underline{P} = (P, \sigma_P) = \mathcal{P}(\underline{\mathrm{M}})$, where $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is a pure rigid analytically trivial dual Anderson A-motive of rank r over k.

Suppose $\underline{P}'' = \underline{P}/\underline{P}'$ with $\underline{P}' = (P', \sigma_{P'}) \in \mathrm{Ob}(\mathscr{P}\mathscr{T})$ is a pure dual sub-t-motive of $\underline{P}$ of rank r' over k, that is, $P' \cong k(t)^{\oplus r'}$ since $Q_k = k(t)$. We define $\mathrm{M}' := \mathrm{M} \cap P'$ in M. Then M' is a finitely generated $k[t]$-module because M' is contained in M. For the same reason, we find that M' is also torsion free. Therefore $\mathrm{M}' \cong k[t]^{\oplus s'}$ and $\mathrm{M}' \otimes_{k[t]} k(t) = P'$ so that $s' = r'$ must hold. We define $\sigma_{\mathrm{M}'} := \sigma_{\mathrm{M}}|_{\varsigma^* \mathrm{M}'} : \varsigma^* \mathrm{M}' \to \mathrm{M}'$. By Lemma 2.1.5, Proposition 2.4.14 and Proposition 2.6.2 we see that $(\mathrm{M}', \sigma'_{\mathrm{M}})$ is a pure rigid analytically trivial dual Anderson A-motive over k such that $\mathcal{P}(\underline{\mathrm{M}}') = \underline{P}'$.

Similarly, we find that M/M' is finitely generated over $k[t]$ and $\mathrm{M}/\mathrm{M}' \cong k[t]^{\oplus r - s'}$. Moreover, M/M' is a torsion free $k[t]$-module since the submodule $\mathrm{M}' \subset \mathrm{M}$ is saturated. Hence, $\mathrm{M}/\mathrm{M}' \otimes_{k[t]} k(t) = P/P' = P''$ and $\mathrm{M}'' := \mathrm{M}/\mathrm{M}' \cong k[t]^{\oplus r - r'}$. Further σ_{M} and $\sigma_{\mathrm{M}'}$ induce an injective $k[t]$-homomorphism $\sigma_{\mathrm{M}''} : \varsigma^* \mathrm{M}'' \to \mathrm{M}''$. By the same reasons as above, we conclude that $(\mathrm{M}'', \sigma_{\mathrm{M}''})$ is a pure rigid analytically trivial dual Anderson A-motive over k with $\mathcal{P}(\underline{\mathrm{M}}'') = \underline{P}''$. □

Corollary 2.7.11. *$\mathscr{P}\mathscr{T}'$ is a neutral Tannakian category over Q with fiber functor $\omega' := \omega \circ \mathcal{R} : \mathscr{P}\mathscr{T}' \to \mathscr{V}ec_Q$, that is,*

$$\omega'(\underline{\mathrm{M}}(i)) = \omega(\mathcal{P}(\underline{\mathrm{M}}) \otimes \underline{L}^{-i}) = \omega(\mathcal{P}(\underline{\mathrm{M}})) \otimes_Q \omega(\underline{L}^{-i}).$$

Definition 2.7.12. (i) We define $\Gamma_{\mathscr{P}\mathscr{T}'} := \underline{\mathrm{Aut}}^{\otimes}(\omega')$ to be the affine group scheme given by Tannakian duality such that the category $\mathscr{R}ep_Q(\Gamma_{\mathscr{P}\mathscr{T}'})$ is equivalent to $\mathscr{P}\mathscr{T}'$.

(ii) We define an extension $\omega'^{(R)} : \mathscr{P}\mathscr{T}' \to \mathscr{M}od_R$ of ω' by setting $\omega'^{(R)}(\underline{P}) := P(1)^{\sigma} \otimes_Q R$ for an arbitrary commutative Q-algebra R. Then $\Gamma_{\mathscr{P}\mathscr{T}}(R) = \underline{\mathrm{Aut}}^{\otimes}(\omega^{(R)})$ holds.

We note that the the affine Q-group schemes $\Gamma_{\mathscr{P}\mathscr{T}}$ and $\Gamma_{\mathscr{P}\mathscr{T}'}$ are isomorphic over Q by Corollary 1.2.16.

Remark 2.7.13. In the non-dual case, one can define a *pure t-motive of rank r, dimension d and weight $\frac{l}{n} - i$* to be a pair $(\underline{\mathrm{M}}, i)$ where M is a pure rigid analytically trivial Anderson A-motive of rank r, dimension d and weight $\frac{l}{n}$ and $i \in \mathbb{Z}$ an integer. The category of such pure t-motives with quasi-morphisms as morphisms of pure t-motives is then equivalent to the category $\mathscr{PT}'$ of pure dual t-motives by [Tae09, Rem. 5.3.3]. In particular, the corresponding affine Q-group schemes given by Tannakian duality are isomorphic.

2.8 Galois groups of pure dual t-motives

In this section we consider the strictly full Tannakian subcategory generated by a pure dual t-motive and define its Galois group to be the affine group scheme given by Tannakian duality (Theorem 1.2.10). Then we explain Papanikolas's systems of difference equations, which lead to his main transcendence result.

Definition 2.8.1. Let $\mathscr{C}$ be either the Tannakian category $\mathscr{PT}$ or $\mathscr{PT}'$ with fiber functor $\omega : \mathscr{C} \to \mathscr{V}ec_Q$.

(i) We define $\langle\langle \underline{P} \rangle\rangle$ to be the strictly full Tannakian subcategory of $\mathscr{C}$ generated by a pure dual t-motive $\underline{P} \in \mathrm{Ob}(\mathscr{C})$ over k. Its fiber functor is the restricted fiber functor $\omega|_{\langle\langle \underline{P} \rangle\rangle} : \langle\langle \underline{P} \rangle\rangle \to \mathscr{V}ec_Q$ that we denote by $\omega_{\underline{P}}$.

(ii) We write $\omega_{\underline{P}}^{(R)}$ for the restriction of $\omega^{(R)} : \mathscr{C} \to \mathscr{M}od_R$ to $\langle\langle \underline{P} \rangle\rangle$ for a Q-algebra R.

If $\underline{P}$ is a pure dual t-motive, we note that $\langle\langle \underline{P} \rangle\rangle$ consists of all pure dual t-motives that are isomorphic to subquotients of objects of the form $\bigoplus_{i=1}^m \underline{P}^{k_i} \otimes \underline{P}^{-l_i}$ for various $k_i,\ l_i,\ m \in \mathbb{N}$.

Definition 2.8.2. Let $\mathscr{C}$ be either the Tannakian category $\mathscr{PT}$ or $\mathscr{PT}'$ with fiber functor $\omega : \mathscr{C} \to \mathscr{V}ec_Q$. We define the *Galois group* $\Gamma_{\underline{P}}$ *of a pure dual t-motive* $\underline{P}$ over k in $\mathscr{C}$ to be the affine Q-group scheme that represents the functor $\underline{\mathrm{Aut}}^{\otimes}(\omega_{\underline{P}})$.

By Tannakian duality, the Tannakian category $\langle\langle \underline{P} \rangle\rangle$ generated by a pure dual t-motive $\underline{P}$ is equivalent to $\mathscr{R}ep_Q(\Gamma_{\underline{P}})$ and for any Q-algebra R we have $\Gamma_{\underline{P}}(R) = \underline{\mathrm{Aut}}^{\otimes}(\omega_{\underline{P}}^{(R)})$. We then find that $\Gamma_{\underline{P}}$ is a linear algebraic group over Q by Lemma 1.2.13.

Suppose now that $(\underline{\mathrm{M}}, i) \in \mathrm{Ob}(\mathscr{PT}')$ is a pure dual t-motive over k. We observe that $\langle\langle (\underline{\mathrm{M}}, i) \rangle\rangle \to \langle\langle \mathcal{R}(\underline{\mathrm{M}}, i) \rangle\rangle$ is an equivalence of categories so that there is an isomorphism $\Gamma_{(\underline{\mathrm{M}},i)} \xrightarrow{\sim} \Gamma_{\mathcal{R}(\underline{\mathrm{M}},i)}$ by Corollary 1.2.16. In particular, $\Gamma_{\underline{\mathrm{M}}(0)} \cong \Gamma_{\mathcal{P}(\underline{\mathrm{M}})}$.

Definition 2.8.3. If $\underline{\mathrm{M}}$ is a pure rigid analytically trivial dual Anderson A-motive over k, we also call $\Gamma_{\mathcal{P}(\underline{\mathrm{M}})}$ the *Galois group of* $\underline{\mathrm{M}}$ and denote it by $\Gamma_{\underline{\mathrm{M}}}$.

Example 2.8.4 ([Pap08, Thm. 3.5.4])**.** The Galois group $\Gamma_{\underline{\mathrm{C}}}$ of the Carlitz t-motive $\underline{\mathrm{C}}$ over $\overline{Q}$ is isomorphic to $\mathbb{G}_{m,Q}$ over Q.

Papanikolas uses systems of difference equations to determine the Galois group of a pure dual t-motive in $\mathscr{PT}$ explicitly. Next we shortly sketch their construction and Papanikolas's obtained main result on the dimension of the Galois group of a pure dual t-motive over $\overline{Q}$.

Systems of σ-semilinear equations and difference Galois groups

After having defined the Tannakian category $\langle\langle \underline{P} \rangle\rangle$ over Q, Papanikolas develops a Galois theory for systems of σ-semilinear equations $\Psi^{(-1)} = \Phi\Psi$, in analogy with classical Galois theory. Consider a pure dual t-motive $\underline{P} = (P, \sigma_P)$ over $\overline{Q}$. Let $\Phi \in \mathrm{GL}_r(\overline{Q}(t))$ represent σ_P with respect to a $k(t)$-basis $\mathbf{m}$ for P and let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ be a rigid analytic trivialization of Φ. Papanikolas then associates another affine group scheme Γ_Ψ with the rigid analytic trivialization Ψ of $\underline{P}$, called the *difference Galois group of* $\underline{P}$. We shall shortly explain its construction and Papanikolas's main results on the difference Galois group and Galois group of $\underline{P}$. We consider an $r\times r$-matrix X whose entries are independent variables (X_{ij}) and define $\nu_\Psi : \overline{Q}(t)[X, 1/\det X] \to \mathbb{L}$ to be the $\overline{Q}(t)$-algebra homomorphism that maps X_{ij} to Ψ_{ij}. We put

$$\mathfrak{p}_\Psi := \ker \nu_\Psi, \quad \Sigma_\Psi := \operatorname{im} \nu_\Psi \subseteq \mathbb{L}, \quad \Lambda_\Psi := \operatorname{Quot}(\Sigma_\Psi) \subset \mathbb{L} \quad \text{and} \quad Z_\Psi := \operatorname{Spec} \Sigma_\Psi.$$

Z_Ψ is then the smallest closed subscheme of $\mathrm{GL}_r(\overline{Q}(t))$ such that $\Psi \in Z_\Psi(\mathbb{L})$.

Now set $\Psi_1, \Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{Q}(t)} \mathbb{L})$ to be the matrices such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ and $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$, and let $\tilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{Q}(t)} \mathbb{L})$. We define $\mu_\Psi : Q[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{Q}(t)} \mathbb{L}$ to be the Q-algebra homomorphism that sends X_{ij} to $\tilde{\Psi}_{ij}$. We set

$$\Delta_\Psi := \operatorname{im} \mu_\Psi \quad \text{and} \quad \Gamma_\Psi := \operatorname{Spec} \Delta_\Psi.$$

By construction, Γ_Ψ is the smallest closed subscheme of $\mathrm{GL}_{r,Q}$ such that $\tilde{\Psi} \in \Gamma_\Psi(\mathbb{L} \otimes_{\overline{Q}(t)} \mathbb{L})$. We call Γ_Ψ the *difference Galois group of* $\underline{P}$.

Theorem 2.8.5 ([Pap08, §4]). *Let $\underline{P}$ be a pure dual t-motive of rank r over $\overline{Q}$. Let $\Phi \in \mathrm{GL}_r(\overline{Q}(t))$ represent σ_P with respect to a basis $\mathbf{m}$ for P and let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ be a rigid analytic trivialization of Φ. Consider the difference Galois group Γ_Ψ of $\underline{P}$.*

(i) Γ_Ψ *is a closed Q-subgroup scheme of* $\mathrm{GL}_{r,Q}$.

(ii) Γ_Ψ *is absolutely irreducible and smooth over* $\overline{Q}$.

(iv) $\dim \Gamma_\Psi = \operatorname{tr.deg}_{\overline{Q}(t)} \Lambda_\Psi$.

(v) Γ_Ψ *is isomorphic to the Galois group $\Gamma_{\underline{P}}$ of $\underline{P}$ over Q.*

Together with a linear independence criterion (the "ABP criterion" [ABP04, Thm. 3.1.1]), Papanikolas is able to show the following main result of [Pap08].

Theorem 2.8.6 ([Pap08, Thm. 5.2.2]). *Let $\underline{P}$ be a pure dual t-motive over $\overline{Q}$ and $\Gamma_{\underline{P}}$ its Galois group. Suppose that $\Phi_{\mathbf{p}} \in \mathrm{GL}_r(k(t)) \cap \mathrm{Mat}_{r\times r}(k[t])$ represents σ_{M} with respect to a $k(t)$-basis $\mathbf{p}$ for M such that $\det \Phi_{\mathbf{p}} = \alpha(t-\theta)^s$, $\alpha \in \overline{Q}^\times$. Then there is a rigid analytic trivialization Ψ of Φ in $\mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_{r\times r}(\mathbb{E})$ and*

$$\operatorname{tr.deg}_{\overline{Q}} \overline{Q}(\Psi(\theta)_{ij} | 1 \leq i,j \leq r) = \dim \Gamma_{\underline{P}}.$$

We will later assign a Hodge-Pink group to a pure dual t-motive $\underline{P}$ over $\mathbb{C}_\infty$. We then prove the Hodge conjecture for function fields; that is, this Hodge-Pink group is isomorphic to the Galois group of $\underline{P}$ (Theorem 4.2.19). In combination with the previous theorem, we obtain Grothendieck's period conjecture or function fields (Theorem 5.0.20).

3. THE TANNAKIAN CATEGORY $\mathscr{H}odge_Q$ OF PURE Q-HODGE-PINK STRUCTURES

Pink invented mixed Hodge-Pink structures over function fields as an analog of the classical rational mixed Hodge structures who form a Tannakian category over $\mathbb{Q}$. We will see in Section 4.1.3 that the Betti homology group $H_B(\underline{E}) = \Lambda_{\underline{E}}$ of a pure uniformizable Anderson A-module $\underline{E}$ over $\overline{Q} \subset \mathbb{C}_\infty$ gives rise to a pure Q-Hodge-Pink structure, similarly as the Betti homology group $H_1(X(\mathbb{C}), \mathbb{Q})$ of an algebraic variety X over a number field contained in $\mathbb{C}$ carries a rational Hodge structure. In this chapter, we shall concentrate on the definition of mixed Hodge-Pink structures and some of their properties which will be later of interest to us, following on the whole [Pin97a].

In order to do this, recall that a *rational mixed Hodge structure* consists of a finite dimensional $\mathbb{Q}$-vector space H, an increasing filtration of H, called the *weight filtration*, and a decreasing filtration of $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$, called the *Hodge filtration*, such that the induced filtrations on graded pieces constitutes a pure Hodge structure (cf. [Del71, Déf. 2.3.1]).

Hence, we first recall some definitions concerning filtrations. Having done this, we introduce pure resp. mixed pre-Hodge-Pink structures over global and local function fields whose category is not abelian and therefore not Tannakian. Similarly to classical Hodge theory, we shall impose in the next section a semistability condition to remedy this. Semistable pre-Hodge-Pink structures are called Hodge-Pink structures whose category Pink shows to be Tannakian as desired.

Since we will later put a pure dual t-motive over $\mathbb{C}_\infty$ defined as in Section 2.8 in correspondence with a pure Q-Hodge-Pink structure, we are interested in the comparison of the Tannakian subcategory of $\mathscr{P}\mathscr{T}$ generated by a pure dual t-motive and the Tannakian subcategory of $\mathscr{H}odge_Q$ generated by a pure Q-Hodge-Pink structure. We call the associated algebraic group by Tannakian duality the *Hodge-Pink group of* $\underline{H}$. Some properties of those Hodge-Pink groups and Hodge-Pink additivity of a pure Hodge-Pink structure are discussed, which we will need later. If a pure Hodge Pink structure is Hodge-Pink additive, we may define Hodge-Pink cocharacters whose conjugates generate the Hodge-Pink group.

Throughout this chapter we assume $A = \mathbb{F}_q[t]$ as done by Pink, hence $Q = \mathrm{Quot}(A)$ is a global function field in one variable with completion $Q_\infty \cong \mathbb{F}_q(\!(z)\!)$ for a local parameter z at the place ∞. Recall further that $\mathbb{C}_\infty$ is defined to be the completion of an algebraic closure of Q_∞ and let ζ denote the image of z under the natural inclusion $\iota : Q_\infty \hookrightarrow \mathbb{C}_\infty$. By [Pin97a, Prop. 3.1] there is then a natural injective algebra homomorphism

$$Q_\infty \hookrightarrow \mathbb{C}_\infty[\![z-\zeta]\!] \quad \sum_k a_k z^k \mapsto \sum_k \iota(a_k) z^k = \sum_{l \geq 0} (z-\zeta)^l \cdot \sum_k \iota(a_k) \cdot \binom{k}{l} \cdot \zeta^{k-l}.$$

and thus also an inclusion $Q \hookrightarrow \mathbb{C}_\infty[\![z-\zeta]\!]$. Moreover, we let $\mathcal{Q}$ denote either Q or Q_∞.

3.1 Filtrations

Let K be an arbitrary field which we will later define to be either Q or Q_∞. The general definition of decreasing and increasing filtrations reads as follows.

Definition 3.1.1. Let V be a finite dimensional vector space over K.

(i) (a) A *decreasing ($\mathbb{Q}$-)filtration of* V is a collection of subspaces $F = \big(F^iV\big)_{i\in\mathbb{Q}}$ such that $F^iV \subset F^jV$ for all $i \geq j$.

(b) For each decreasing filtration F we define the $\mathbb{Q}$-graded vector space

$$\operatorname{Gr}_F V := \bigoplus_{i\in\mathbb{Q}} \operatorname{Gr}^i_F V := \bigoplus_{i\in\mathbb{Q}} (F^iV/\bigcup_{i<j} F^jV)$$

for which $\dim_K(\operatorname{Gr}_F V) \leq \dim_K V$ holds.

(c) A decreasing filtration $F = (F^iV)_{i\in\mathbb{Q}}$ is called *trivial* if $\operatorname{Gr}^0_F V \cong V$.

(d) A decreasing filtration $F = (F^iV)_{i\in\mathbb{Q}}$ is said to be *exhaustive* if $F^iV = V$ for all $i \ll 0$, and *separated* if $F^iV = 0$ for all $i \gg 0$.

(ii) (a) An *increasing ($\mathbb{Q}$-)filtration of* V is a collection of subspaces $F = (F_iV)_{i\in\mathbb{Q}}$ such that $F_iV \subset F_jV$ for all $i \leq j$.

(b) For each increasing filtration F we define the $\mathbb{Q}$-graded vector space

$$\operatorname{Gr}^F V := \bigoplus_{i\in\mathbb{Q}} \operatorname{Gr}^F_i V := \bigoplus_{i\in\mathbb{Q}} (F_iV/\bigcup_{j<i} F_jV)$$

for which $\dim_K(\operatorname{Gr}^F V) \leq \dim_K V$ holds.

(c) An increasing filtration $F = (F_iV)_{i\in\mathbb{Q}}$ is called *trivial* if $\operatorname{Gr}^F_0 V \cong V$.

(d) An increasing filtration $F = (F_iV)_{i\in\mathbb{Q}}$ is said to be *exhaustive* if $F^iV = V$ for all $i \gg 0$, and *separated* if $F^iV = 0$ for all $i \ll 0$.

Following Pink, we require both types of filtration to be separated and exhaustive, so that equality in (b) holds. In order to define a Tannakian category of Hodge-Pink structures, we also need to define induced filtrations of duals and tensors of finite dimensional vector spaces over K.

Definition 3.1.2. Let F be a decreasing filtration of a finite dimensional K-vector space V and $V' \subset V$ a subspace.

(i) We define the *induced filtration of* V' to be $F^iV' := V' \cap F^iV$ and the *induced filtration of the factor space* $V'' := V/V'$ to be $F^iV'' := (V' + F^iV)/V'$.

(ii) Consider now another finite dimensional K-vector space W with a decreasing filtration F. The *induced filtration of the tensor product* $V \otimes_K W$ is given by

$$F^i(V \otimes_K W) := \sum_{j+k=i} F^jV \otimes_K F^kW.$$

(iii) The *induced filtration of* $\operatorname{Hom}_K(V, W)$ is

$$F^i \operatorname{Hom}_K(V, W) := \left\{ \phi \in \operatorname{Hom}_K(V, W) \middle| \, \forall j : \phi(F^j V) \subset F^{i+j} W \right\}$$

and in particular for the dual space $V^\vee$

$$F^i V^\vee := \left\{ w \in V^\vee \middle| \, \forall j < i : w|_{F^{-j} V} = 0 \right\}.$$

Similarly, one defines induced filtrations if F is an increasing filtration.

Remark 3.1.3. Using (i) we get a natural filtration of any subspace V_1/V_2 with $V_2 \subset V_1 \subset V$ which we denote as

$$F|_{(V_1/V_2)} = \frac{V_2 + (V_1 \cap F^i V)}{V_2}.$$

Definition 3.1.4. Let V, W be two finite dimensional K-vector spaces.

(i) A K-linear homomorphism $\phi : V \to W$ is called *compatible with two given decreasing (resp. increasing) filtrations* F if $\phi(F^i V) \subset F^i W$ (resp. $\phi(F_i V) \subset F_i W$) for all i.

(ii) A K-linear homomorphism ϕ is *strictly compatible with the decreasing (resp. increasing) filtrations* F or *strict* if $\phi(F^i V) = \phi(V) \cap F^i W$ (resp. $\phi(F_i V) = \phi(V) \cap F_i W$) for all i.

Taking the filtered finite dimensional K-vector spaces as objects and the compatible K-linear homomorphisms as morphisms, we get a K-linear category. This category is not abelian as the morphism $\operatorname{coim}(\phi) \to \operatorname{im}(\phi)$ is an isomorphism if and only if ϕ is strict. But one cannot require all morphisms to be strict as the composite of two strict morphism need not be strict again.

Next we define some numerical invariants attached to filtrations that are needed for example to define semistability of Hodge-Pink structures.

Definition 3.1.5. Let V be a finite dimensional vector space over K.

(i) Suppose F is a decreasing filtration of V. We define the *(total) degree of* V *with respect to* F to be

$$\deg_F V := \sum_{i \in \mathbb{Q}} i \cdot \dim_K \operatorname{Gr}^i_F V.$$

In case $V \neq 0$, we may define the *average weight* $\operatorname{wt}_F V := \frac{\deg_F V}{\dim_K V}$.

(ii) Suppose F is an increasing filtration of V. We define the *(total) degree of* V *with respect to* F to be

$$\deg^F V := \sum_{i \in \mathbb{Q}} i \cdot \dim_K \operatorname{Gr}^F_i V.$$

In case $V \neq 0$, we may define the *average weight* $\operatorname{wt}^F V := \frac{\deg^F V}{\dim_K V}$.

Proposition 3.1.6 ([Pin97a, Prop. 1.4])**.** *Let V and W be finite dimensional K-vector spaces.*

(i) $\deg_F V = \deg_F V' + \deg_F V''$ *if V' is a subspace of V and $V'' := V/V'$.*

(ii) $\operatorname{wt}_F(V \otimes_K W) = \operatorname{wt}_F V + \operatorname{wt}_F W$ *if $V \neq 0 \neq W$.*

(iii) $\operatorname{wt}_F \operatorname{Hom}_K(V, W) = \operatorname{wt}_F W - \operatorname{wt}_F V$ *if $V \neq 0 \neq W$.*

(iv) $\deg_F V^\vee = -\deg_F V$.

3.2 $\mathcal{Q}$-pre-Hodge-Pink structures

Instead of the Hodge filtration underlying a classical rational mixed Hodge-structure, we require finer information in form of a *($\mathbb{C}_\infty[\![z-\zeta]\!]$-)lattice* in a finite dimensional $\mathbb{C}_\infty(\!(z-\zeta)\!)$-vector space; that is, a finitely generated $\mathbb{C}_\infty[\![z-\zeta]\!]$-submodule containing a $\mathbb{C}_\infty(\!(z-\zeta)\!)$-basis. Furthermore, using the inclusion $\mathcal{Q} \hookrightarrow \mathbb{C}_\infty[\![z-\zeta]\!]$ we may define a canonical lattice $\mathfrak{p}_H := H \otimes_\mathcal{Q} \mathbb{C}_\infty[\![z-\zeta]\!]$ for each finite-dimensional $\mathcal{Q}$-vector space H.

Definition 3.2.1. (i) A *mixed $\mathcal{Q}$-pre-Hodge-Pink structure of rank r* over $\mathbb{C}_\infty$ is a triple $\underline{H} = (H, W, \mathfrak{q}_H)$ such that

(a) H is a vector space of dimension r over $\mathcal{Q}$,

(b) the *weight filtration* $W = (W_\nu H)_{\nu\in\mathbb{Q}}$ is an increasing filtration by $\mathcal{Q}$-subspaces of H and

(b) $\mathfrak{q}_H$ is a lattice in $H \otimes_\mathcal{Q} \mathbb{C}_\infty(\!(z-\zeta)\!)$. We denote the rank r of $\underline{H}$ by $\operatorname{rank}(\underline{H})$.

A mixed $\mathcal{Q}$-pre-Hodge-Pink $(H, W, \mathfrak{q}_H)$ structure is called a *pure $\mathcal{Q}$-pre-Hodge-Pink structure of weight $\nu \in \mathbb{Q}$* if $\operatorname{Gr}^W_\nu H \cong H$.

(ii) Let $\underline{H}_1 = (H_1, W, \mathfrak{q}_{H_1})$ and $\underline{H}_2 = (H_2, W', \mathfrak{q}_{H_2})$ be mixed $\mathcal{Q}$-pre-Hodge-Pink-structures. A *morphism $\phi : \underline{H}_1 \to \underline{H}_2$ of mixed $\mathcal{Q}$-pre-Hodge-Pink structures* is a homomorphism of $\mathcal{Q}$-vector spaces $H_1 \to H_2$, which is compatible with W and W' and satisfies $\phi(\mathfrak{q}_{H_1}) \subseteq \mathfrak{q}_{H_2}$.

Additionally, ϕ is said to be *strict* if it is strictly compatible with W and W' and

$$\phi(\mathfrak{q}_{H_1}) = \mathfrak{q}_{H_2} \cap (\phi(H_1) \otimes_\mathcal{Q} \mathbb{C}_\infty(\!(z-\zeta)\!)).$$

As done before, we speak of *mixed pre-Hodge-Pink structures* if the field of definition does not require emphasis. Observe that a mixed Q-pre-Hodge-Pink structure $\underline{H} = (H, W, \mathfrak{q}_H)$ with $W = (W_\nu H)_{\nu\in\mathbb{Q}}$ defines a mixed Q_∞-pre-Hodge-Pink structure

$$\underline{H}_\infty := (H_\infty, W_\infty, \mathfrak{q}_{H_\infty}) := (H \otimes_Q Q_\infty, (W_\nu H \otimes_Q Q_\infty)_{\nu\in\mathbb{Q}}, \mathfrak{q}_H)$$

that we call the *mixed Q_∞-Hodge-Pink structure associated with $\underline{H}$*.

The lattice underlying a mixed $\mathcal{Q}$-Hodge-Pink structure allows us to assign the following decreasing filtration to a mixed $\mathcal{Q}$-Hodge-Pink structure that replaces the Hodge filtration in the classical theory of mixed Hodge structures.

Definition 3.2.2. Consider a mixed $\mathcal{Q}$-pre-Hodge-Pink structure $\underline{H} = (H, W, \mathfrak{q}_H)$ and the natural projection

$$H \to H_{\mathbb{C}_\infty} := \mathfrak{p}_H/(z-\zeta)\mathfrak{p}_H \cong H \otimes_{\mathcal{Q},\iota} \mathbb{C}_\infty.$$

We define a decreasing filtration $F = (F^i H_{\mathbb{C}_\infty})_{i\in\mathbb{Z}}$ of $H_{\mathbb{C}_\infty}$ by letting $F^i H_{\mathbb{C}_\infty}$ be the image of $\mathfrak{p}_H \cap (z-\zeta)^i \mathfrak{q}_H$ for all $i \in \mathbb{Z}$. F is called the *Hodge-Pink filtration of $\underline{H}$*.

One finds that any morphism is also compatible with the Hodge-Pink filtrations, but a strict morphism is not necessarily strictly compatible with the Hodge-Pink filtrations.

Alternatively, one may define the *Hodge-Pink weights* of a mixed $\mathcal{Q}$-pre-Hodge-Pink structure $(H, W, \mathfrak{q}_H)$ as the elementary divisors of $\mathfrak{q}_H$ relative to $\mathfrak{p}_H$. This means, if we choose integers $e_+ \geq e_-$ such that

$$(z-\zeta)^{e_+}\mathfrak{p}_H \subset \mathfrak{q}_H \subset (z-\zeta)^{e_-}\mathfrak{p}_H$$

and

$$\mathfrak{q}_H/(z-\zeta)^{e_+}\mathfrak{p}_H \cong \bigoplus_{i=1}^{n} \mathbb{C}_\infty[\![z-\zeta]\!]/(z-\zeta)^{e_+ + w_i}$$

or, alternatively,

$$(z-\zeta)^{e_-}\mathfrak{p}_H/\mathfrak{q}_H \cong \bigoplus_{i=1}^{n} \mathbb{C}_\infty[\![z-\zeta]\!]/(z-\zeta)^{e_- - w_i},$$

then the Hodge-Pink weights are the integers $w_1, \dots, w_n$, which we assume ordered $w_1 \leq \dots \leq w_n$.

Definition 3.2.3. Let $\underline{H} = (H, W, \mathfrak{q}_H)$ be a mixed $\mathcal{Q}$-pre-Hodge-structure.

(i) The total degree provided by the Hodge-Pink filtration of $\underline{H}_\infty$ can be expressed in dependence of the lattice:

$$\deg_{\mathfrak{q}}(\underline{H}) := \deg_F(H_{\mathbb{C}_\infty}) = \dim_{\mathbb{C}_\infty}\left(\frac{\mathfrak{q}_H}{\mathfrak{p}_H \cap \mathfrak{q}_H}\right) - \dim_{\mathbb{C}_\infty}\left(\frac{\mathfrak{q}_H}{\mathfrak{p}_H \cap \mathfrak{q}_H}\right).$$

(ii) For $\underline{H} \neq 0$ we define the average weight of the Hodge filtration of $\underline{H}$ in terms of $\mathfrak{q}_H$

$$\mathrm{wt}_{\mathfrak{q}}(\underline{H}) := \mathrm{wt}_F(H_{\mathbb{C}_\infty}) = \frac{\deg_{\mathfrak{q}}(\underline{H})}{\mathrm{rank}(\underline{H})}.$$

The weight filtration of $\underline{H}$ also gives us a total degree $\deg^W(\underline{H})$ and an average weight $\mathrm{wt}^W(\underline{H})$ if $H \neq 0$. We have $\mathrm{wt}^W(\underline{H}) = \nu$ if $\underline{H}_\infty$ is pure of weight ν.

Definition 3.2.4. Let $\underline{H} = (H, W, \mathfrak{q}_H)$ be a mixed $\mathcal{Q}$-pre-Hodge-Pink structure, $H' \subset H$ a subspace and $H'' := H/H'$ the factor space.

(i) A *subobject* in the category of mixed $\mathcal{Q}$-pre-Hodge-Pink structures is a morphism $\underline{H}' \to \underline{H}$ whose underlying homomorphism of $\mathcal{Q}$-vector spaces is the inclusion $H' \hookrightarrow H$. It is called *strict* if $\underline{H}' \to \underline{H}$ is strict.

(ii) We may make H' into a unique strict subobject $\underline{H}'$ and obtain a unique *strict factor object* $\underline{H}''$ in such a way that the projection $H \twoheadrightarrow H''$ extends to a strict morphism $\underline{H} \to \underline{H}''$.

A *strict exact sequence* is a sequence which is isomorphic to the the sequence $0 \to \underline{H}' \to \underline{H} \to \underline{H}/\underline{H}' \to 0$.

Putting the above constructions together, each subquotient H'/H'' of H_∞ may be equipped with a weight filtration and a lattice so that it is a natural mixed $\mathcal{Q}$-pre-Hodge-Pink structure, which depends on the subspace $H'' \subset H' \subset H$. We want to make the $\mathcal{Q}$-linear additive category of mixed $\mathcal{Q}$-pre-Hodge-Pink structures into a Tannakian category. In order to do this, one defines tensor products, inner hom and duals.

Definition 3.2.5. Let $\underline{H}_1 = (H_1, W_1, \mathfrak{q}_{H_1})$ and $\underline{H}_2 = (H_2, W_2, \mathfrak{q}_{H_2})$ be two mixed $\mathcal{Q}$-pre-Hodge-Pink structures.

(i) The *tensor product* $\underline{H}_1 \otimes \underline{H}_2$ of mixed $\mathcal{Q}$-pre-Hodge-Pink structures consists of the tensor product $H_1 \otimes_{\mathcal{Q}} H_2$ of $\mathcal{Q}$-vector spaces, the induced weight filtration and the lattice $\mathfrak{q}_{H_1} \otimes_{\mathbb{C}_\infty[\![z-\zeta]\!]} \mathfrak{q}_{H_2}$. Similarly one defines for $n \geq 1$ the *symmetric power* $\mathrm{Sym}^n \underline{H}$ and the *alternating power* $\bigwedge^n \underline{H}$.

(ii) The *inner hom* $\mathcal{H}om(\underline{H}_1, \underline{H}_2)$ consists of the $\mathcal{Q}$-vector space $\mathrm{Hom}_{\mathcal{Q}}(H_1, H_2)$, the induced weight filtration and the lattice $\mathrm{Hom}_{\mathbb{C}_\infty[\![z-\zeta]\!]}(\mathfrak{q}_{H_1}, \mathfrak{q}_{H_2})$.

(iii) The *unit object* $\mathbb{1}_{\mathcal{Q}}$ consists of $\mathcal{Q}$ itself together with the lattice $\mathfrak{p}_H$ and is pure of weight 0. The *dual* $\underline{H}^\vee$ of the mixed $\mathcal{Q}$-pre-Hodge-structure $\underline{H}$ is then $\mathcal{H}om(\underline{H}, \mathbb{1}_{\mathcal{Q}})$.

Since a strict morphism is not necessarily strictly compatible with the Hodge-Pink filtrations, the category of $\mathcal{Q}$-pre-Hodge-Pink structures is $\mathcal{Q}$-linear, but not abelian. As announced earlier, we impose next a semistability condition such that the category of semistable mixed $\mathcal{Q}$-pre-Hodge-Pink structures is Tannakian.

3.3 Semistability and $\mathcal{Q}$-Hodge-Pink structures

Consider now the equivalent assertions of the following proposition, which allow us to define pure $\mathcal{Q}$-Hodge-Pink structures whose category is Tannakian.

Proposition 3.3.1 ([Pin97a, Prop. 4.4])**.** *The following conditions on a mixed Q_∞-pre-Hodge-structure $\underline{H}_\infty$ are equivalent:*

(a) We have $\deg_{\mathfrak{q}}(\underline{H}'_\infty) \leq \deg^W(\underline{H}'_\infty)$ for each (strict) subobject $\underline{H}'_\infty$ of $\underline{H}_\infty$, with equality whenever $\underline{H}'_\infty = W_\nu \underline{H}_\infty$ for some $\nu \in \mathbb{Q}$;

(b) we have $\deg_{\mathfrak{q}}(\underline{H}''_\infty) \geq \deg^W(\underline{H}''_\infty)$ for each (strict) factor object $\underline{H}''_\infty$ of $\underline{H}_\infty$, with equality whenever $\underline{H}'_\infty = W_\nu \underline{H}_\infty$ for some $\nu \in \mathbb{Q}$.

Definition 3.3.2. (i) A mixed Q_∞-pre-Hodge-Pink structure is called *semistable* if it satisfies the equivalent conditions above. We call a semistable mixed Q_∞-pre-Hodge-structure $\underline{H}$ a *mixed Q_∞-Hodge-Pink structure* or *pure Q_∞-Hodge-Pink structure* if $\underline{H}$ is pure. We denote the subcategory of pure Q_∞-Hodge-Pink structures by $\mathscr{H}odge_{Q_\infty}$ and the set of morphisms between $\underline{H}_1, \underline{H}_2 \in \mathrm{Ob}(\mathscr{H}odge_{Q_\infty})$ by $\mathrm{Hom}_{Q_\infty}(\underline{H}_1, \underline{H}_2)$.

(ii) A mixed Q-pre-Hodge-Pink structure $\underline{H}$ is called *locally semistable* or a *mixed Q-Hodge-Pink structure* if its associated mixed Q_∞-pre-Hodge-Pink structure is semistable. Furthermore, $\underline{H}$ is called a *pure Q-Hodge-Pink structure* if $\underline{H}$ is pure and we define $\mathscr{H}odge_Q$ to be the subcategory of pure Q-Hodge-Pink structures. We write $\mathrm{Hom}_Q(\underline{H}_1, \underline{H}_2)$ for the set of morphisms between $\underline{H}_1, \underline{H}_2 \in \mathrm{Ob}(\mathscr{H}odge_Q)$.

In order to show that the category of mixed Q-Hodge-Pink structures is a Tannakian category, Pink introduces the following objects that we make use of in the proof of Theorem 5.1.2 to determine the Hodge-Pink group coming from a dual Anderson A-motive of CM-type under some conditions.

Definition 3.3.3 ([Pin97a, Def. 5.1 and Def. 5.2])**.** Let $\underline{H} = (H, W, \mathfrak{q}_H)$ be a mixed $\mathcal{Q}$-pre-Hodge-Pink structure. Denote the Frobenius endomorphism of $\mathcal{Q}$ by $\mathrm{Frob}_{q,\mathcal{Q}}$ that then induces a Frobenius endomorphism $\mathrm{Frob}_{q,R'}$ on any commutative $\mathcal{Q}$-algebra R'.

(i) The *Frobenius pullback of $\underline{H}$* is defined to be the triple

$$\mathrm{Frob}_q^* \underline{H} := (H \otimes_{\mathcal{Q},\mathrm{Frob}_{q,\mathcal{Q}}} \mathcal{Q}, W', \mathfrak{q}_H \otimes_{\mathbb{C}_\infty[\![z-\zeta]\!],\mathrm{Frob}_{q,\mathbb{C}_\infty[\![z-\zeta]\!]}} \mathbb{C}_\infty[\![z-\zeta]\!]),$$

where $W' := (W'_\nu H')_{\nu \in \mathbb{Q}}$ and $W'_\nu H' := W_{\frac{\nu}{q}} H \otimes_{\mathcal{Q},\mathrm{Frob}_{q,\mathcal{Q}}} \mathcal{Q}$ for all $\nu \in \mathbb{Q}$.

(ii) The *Frobenius pushforward of* $\underline{H}$ is defined to be the triple

$$\mathrm{Frob}_{q,*}\,\underline{H} := (H', W', \mathfrak{q}_H)$$

with $H' := H$ on which $\mathcal{Q}$ acts via $\mathrm{Frob}_{q,\mathcal{Q}}$ and $W' := (W'_\nu H')_{\nu\in\mathbb{Q}}$ where $W'_\nu H' := W_{q\nu} H'$ for all $\nu \in \mathbb{Q}$.

Consider a mixed $\mathcal{Q}$-pre-Hodge-Pink structure $\underline{H}$. Note that $\mathrm{Frob}_q^*\,\underline{H}$ and $\mathrm{Frob}_{q,*}\,\underline{H}$ are mixed $\mathcal{Q}$-pre-Hodge-Pink structures through the canonical isomorphisms

$$\mathrm{Frob}_{q,\mathcal{Q}} \otimes \mathrm{id}_{\mathcal{Q}} : \mathcal{Q} \otimes_{\mathcal{Q},\mathrm{Frob}_{q,\mathcal{Q}}} \mathcal{Q} \to \mathcal{Q}$$

and

$$\mathrm{Frob}_{q,\mathbb{C}_\infty[\![z-\zeta]\!]} \otimes \mathrm{id}_{\mathcal{Q}} : \mathbb{C}_\infty[\![z-\zeta]\!] \otimes_{\mathcal{Q},\mathrm{Frob}_{q,\mathbb{C}_\infty[\![z-\zeta]\!]}} \mathcal{Q} \to \mathbb{C}_\infty[\![z-\zeta]\!].$$

The following results on $\mathrm{Frob}_q^*\,\underline{H}$ and $\mathrm{Frob}_{q,*}\,\underline{H}$ will be important to us later.

Proposition 3.3.4 ([Pin97a, Prop. 5.4, Prop. 5.5])**.** *Let* $\underline{H} = (H, W, \mathfrak{q}_H)$ *be a mixed $\mathcal{Q}$-pre-Hodge-Pink structure.*

(i) $\mathrm{rank}(\mathrm{Frob}_q^*\,\underline{H}) = \mathrm{rank}\,\underline{H}$, $\deg^W(\mathrm{Frob}_q^*\,\underline{H}) = q\cdot\deg^W\underline{H}$ *and* $\deg_{\mathfrak{q}}(\mathrm{Frob}_q^*\,\underline{H}) = q\cdot\deg_{\mathfrak{q}}\underline{H}$,

(ii) $\mathrm{rank}(\mathrm{Frob}_{q,*}\,\underline{H}) = q\cdot\mathrm{rank}\,\underline{H}$, $\deg^W(\mathrm{Frob}_{q,*}\,\underline{H}) = \deg^W\underline{H}$ *and* $\deg_{\mathfrak{q}}(\mathrm{Frob}_{q,*}\,\underline{H}) = \deg_{\mathfrak{q}}\underline{H}$,

(iii) $\underline{H}$ *is semistable if and only if* $\mathrm{Frob}_{q,*}\,\underline{H}$ *is semistable which is the case if and only if* $\mathrm{Frob}_q^*\,\underline{H}$ *is semistable.*

3.4 The Hodge-Pink group of a pure $\mathcal{Q}$-Hodge-Pink structure

Using the Frobenius functorality of the Frobenius pullback of a mixed $\mathcal{Q}$-pre-Hodge-Pink structure (Proposition 3.3.4), Pink shows that the category of mixed $\mathcal{Q}$-Hodge-Pink structures is a neutral Tannakian category together with the fiber functor that sends a mixed $\mathcal{Q}$-Hodge-Pink structure to its underlying $\mathcal{Q}$-vector space [Pin97a, Cor. 5.7 and Thm. 9.3]. Clearly, the category $\mathscr{H}odge_{\mathcal{Q}}$ of pure $\mathcal{Q}$-Hodge-Pink structures is then also Tannakian with fiber functor

$$\varpi : \mathscr{H}odge_{\mathcal{Q}} \longrightarrow \mathscr{V}ec_{\mathcal{Q}}, \quad \underline{H} \mapsto H, \quad \text{where } \mathcal{Q} \text{ is either } Q_\infty \text{ or } Q$$

(cf. Proposition 3.1.6).

Definition 3.4.1. Let $\underline{H}$ be a pure $\mathcal{Q}$-Hodge-Pink structure. We define $\langle\!\langle \underline{H} \rangle\!\rangle$ to be the strictly full Tannakian subcategory of $\mathscr{H}odge_{\mathcal{Q}}$ generated by $\underline{H}$. We write $\varpi_{\underline{H}} : \langle\!\langle \underline{H} \rangle\!\rangle \to \mathscr{V}ec_{\mathcal{Q}}$ for the restriction of the fiber functor ϖ to $\langle\!\langle \underline{H} \rangle\!\rangle$.

As in Section 2.8 we observe that $\langle\!\langle \underline{H} \rangle\!\rangle$ consists of all pure $\mathcal{Q}$-Hodge-Pink structures in $\mathscr{H}odge_{\mathcal{Q}}$ that are isomorphic to subquotients of objects of the form $\bigoplus_{i=1}^m \underline{H}^{k_i} \otimes (\underline{H})^{-l_i}$ for various $k_i,\ l_i,\ m \in \mathbb{N}$.

Definition 3.4.2. Let $\underline{H}$ be a pure $\mathcal{Q}$-Hodge-Pink structure. We call the affine group scheme $G_{\underline{H}} := \underline{\mathrm{Aut}}^{\otimes}(\varpi_{\underline{H}})$ the *Hodge-Pink group of* $\underline{H}$.

We list now the basic properties of the Hodge-Pink group that are of interest to us later.

Proposition 3.4.3 ([Pin97a, Prop. 6.2 and Prop. 9.4])**.** *Let $\underline{H}$ be a pure $\mathcal{Q}$-Hodge-Pink structure. Its Hodge-Pink group $G_{\underline{H}}$ is connected and reduced.*

All the information of a pure $\mathcal{Q}$-Hodge-Pink structure $\underline{H}$ is contained in its underlying lattice, which we can express in terms of the Hodge-Pink group.

Proposition 3.4.4 ([Pin97a, Prop. 6.3 and Prop. 9.5])**.** *Let $\underline{H} = (H, W, \mathfrak{q}_H)$ be a pure $\mathcal{Q}$-Hodge-Pink structure and ρ be the representation of $G_{\underline{H}}$ on H.*

(i) There is an element $\gamma \in G_{\underline{H}}(\mathbb{C}_\infty((z-\zeta)))$ such that $\mathfrak{q}_H = \rho(\gamma)\mathfrak{p}_H$.

(ii) Let $\gamma \in G_{\underline{H}}(\mathbb{C}_\infty((z-\zeta)))$ as in (i), $\underline{H}' = (H', W', \mathfrak{q}_{H'})$ an object in $\langle\langle \underline{H} \rangle\rangle$ and ρ' be the associated representation of $G_{\underline{H}}$ on the underlying vector space H'. Then we have $\mathfrak{q}_{H'} = \rho'(\gamma)\mathfrak{p}_{H'}$.

Let $q \geq 1$ be a power of the characteristic of $\mathcal{Q}$. Since the Frobenius pullback of a pure $\mathcal{Q}$-Hodge-Pink structure is again a pure $\mathcal{Q}$-Hodge-Pink structure, it defines a tensor functor $\mathrm{Frob}_q^* : \mathscr{H}odge_Q \to \mathscr{H}odge_Q$.

Proposition 3.4.5 ([Pin97a, Prop. 6.4 and Prop. 9.6])**.** *Let $q \geq 1$ be a power of the characteristic of $\mathcal{Q}$ and consider a pure $\mathcal{Q}$-Hodge-Pink structure $\underline{H}$. Its Hodge-Pink group $G_{\mathrm{Frob}_q^* \underline{H}}$ is canonically isomorphic to* $\mathrm{Frob}_q^* G_{\underline{H}} := G_{\underline{H}} \times_{\mathcal{Q}, \mathrm{Frob}_{q,\mathrm{Spec}\, Q}} \mathcal{Q}$.

3.5 Polygons and Hodge-Pink additivity

We will introduce Hodge-Pink cocharacters of a pure $\mathcal{Q}$-Hodge-Pink structure in the next section since they provide additional information about its Hodge-Pink group that we need in the proof of Lemma 5.1.3. To be able to define them, we need that the functor Gr_F^i from the category $\langle\langle \underline{H} \rangle\rangle$ to the category of $\mathbb{Z}$-graded vector spaces over $\mathbb{C}_\infty$ is a faithful exact tensor functor. This does not hold in general (for a counterexample, see [Pin97a, Exmp. 6.14]), and we will define in this section the unique largest strictly full subcategory of $\mathscr{H}odge_Q$ on which Gr_F^i and F^i are exact. It is possible to make an equivalent requirement on the Hodge-Pink filtration of a pure $\mathcal{Q}$-Hodge-Pink structure based on the polygon that comes along with it (see Proposition 3.5.5).

Definition 3.5.1. (i) A *polygon* P is the graph of a piecewise linear convex function $[0, n] \to \mathbb{R}$ for an $n \in \mathbb{N}$ with starting point $(0, 0)$. The length of a subinterval of $[0, n]$ on which the polygon has a given slope $i \in \mathbb{Q}$ is assumed to be an integer and is called the *multiplicity of* i. We refer to the starting point, the endpoint and any point where the slope changes as *break points* of the polygon.

(ii) Let P and Q be polygons of functions $f, g : [0, n] \to \mathbb{R}$. We say that P *lies above* Q if $f(n) = g(n)$ and $f(x) \geq g(x)$ for all $x \in [0, n]$.

We associate to any finite dimensional $\mathbb{Q}$-graded vector space $V = \sum_{i \in \mathbb{Q}} V_i$ a unique polygon such that the multiplicity of a slope $i \in \mathbb{Q}$ matches $\dim_F V_i$. These polygons do not change under semisimplification, so they are said to be additive in short exact sequences.

Definition 3.5.2. Let $\underline{H}$ be a mixed $\mathcal{Q}$-pre-Hodge-Pink structure.

(i) The polygon associated with its weight filtration is called the *weight polygon of* $\underline{H}$.

(ii) The polygon associated with the Hodge-Pink filtration of $\underline{H}$ is called the *Hodge-Pink polygon of* $\underline{H}$.

Proposition 3.5.3. *Let* $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ *be a strict exact sequence. Then the Hodge-Pink polygon of* $\underline{H}' \oplus \underline{H}''$ *lies above that of* $\underline{H}$ *and has the same end point.*

Definition 3.5.4 ([Pin97a, Def. 7.1])**.** We denote the semisimplification of a pure $\mathcal{Q}$-Hodge-Pink structure $\underline{H}$ by $\underline{H}^{\mathrm{ss}}$.

(i) $\underline{H}$ is called *Hodge-Pink additive* if its Hodge polygon is equal to that of $\underline{H}^{\mathrm{ss}}$.

(ii) $\underline{H}$ is called *strongly Hodge-Pink additive* if every $\underline{H}' \in \mathrm{Ob}(\langle\langle \underline{H} \rangle\rangle)$ is Hodge-Pink additive.

A pure $\mathcal{Q}$-Hodge-Pink structure $\underline{H}$ is then Hodge-Pink additive if and only if the following equivalent assertions hold for any strict exact sequence $0 \to \underline{H}' \to \underline{H} \to \underline{H}'' \to 0$ in $\underline{H}$.

Proposition 3.5.5 ([Pin97a, Prop. 6.11])**.** *The following are equivalent:*

(i) The Hodge polygons of $\underline{H}$ *and* $\underline{H}' \oplus \underline{H}''$ *are equal.*

(ii) The injection $H'_{\mathbb{C}_\infty} \hookrightarrow H_{\mathbb{C}_\infty}$ *is strictly compatible with the Hodge-Pink filtrations.*

(iii) The surjection $H_{\mathbb{C}_\infty} \twoheadrightarrow H''_{\mathbb{C}_\infty}$ *is strictly compatible with the Hodge-Pink filtrations.*

(iv) We have a short exact sequence for all $i \in \mathbb{Z}$*,*

$$0 \to F^i H'_{\mathbb{C}_\infty} \to F^i H_{\mathbb{C}_\infty} \to F^i H''_{\mathbb{C}_\infty} \to 0.$$

(v) We have a short exact sequence for all $i \in \mathbb{Z}$*,*

$$0 \to \mathrm{Gr}^i_F H'_{\mathbb{C}_\infty} \to \mathrm{Gr}^i_F H_{\mathbb{C}_\infty} \to \mathrm{Gr}^i_F H''_{\mathbb{C}_\infty} \to 0.$$

In oder to show that the Hodge-Pink structure given in Lemma 5.1.3 is strongly Hodge-Pink additive, we need the following result of Pink.

Proposition 3.5.6 ([Pin97a, Prop. 7.3])**.** *Let* $\underline{H}$ *be a pure* $\mathcal{Q}$*-Hodge-Pink structure. Then* $\underline{H}$ *is strongly Hodge-Pink additive if* $G_{\underline{H}}$ *is reductive.*

3.6 Hodge-Pink Cocharacters

We first shortly review the general definition of cocharacters. They correspond to $\mathbb{Z}$-gradings of finite dimensional vector spaces, allowing us to define Hodge-Pink cocharacters for a pure $\mathcal{Q}$-Hodge-Pink structure that is strongly Hodge-Pink additive.

Definition 3.6.1. (i) Let G be an algebraic group over a field K. A homomorphism of algebraic groups $\lambda : \mathbb{G}_{m,K} \to G$ is called a *cocharacter of* G.

(ii) Let V be an algebraic representation of G and λ a cocharacter of G. The *weight space of weight* i *under* λ is the subspace

$$V_i := \{v \in V | x^i \cdot v = \lambda(x)v \text{ for all} x \in K^\times\},$$

providing thereby a natural $\mathbb{Z}$-grading $V = \sum_{i\in\mathbb{Z}} V_i$ of V.

If we fix a cocharacter $\lambda : \mathbb{G}_{m,K} \to G$, this grading is functorial in V and compatible with tensor products and duals. Conversely, given such a $\mathbb{Z}$-grading of some algebraic representation V which is functorial in V, and compatible with tensor products and duals, we can interpret this information as a K-linear tensor functor $\mathscr{R}ep_K(G) \to \mathscr{R}ep_K(\mathbb{G}_{m,K})$, so that it comes from a unique cocharacter of G (cf. [DMOS82, Exmp. II.2.30]). By going back and forth, we see that the cocharacter and its associated grading are in a 1-to-1 correspondence.

Again with the help of the Frobenius pullback of a $\mathcal{Q}$-Hodge-Pink structure, Pink is able to prove the following:

Theorem 3.6.2 ([Pin97a, Thm. 7.9 and Thm. 9.10])**.** *The strongly Hodge-Pink additive $\mathcal{Q}$-Hodge-Pink structures form a strictly full Tannakian subcategory $\mathscr{H}odge_{\mathcal{Q}}^{\mathrm{sha}}$ of $\mathscr{H}odge_{\mathcal{Q}}$.*

Let $\underline{H} \in \mathrm{Ob}(\mathscr{H}odge_{\mathcal{Q}}^{\mathrm{sha}})$ be a strongly Hodge-Pink additive $\mathcal{Q}$-Hodge-Pink structure. From Proposition 3.5.5 we deduce that the functor Gr_F from $\langle\langle \underline{H} \rangle\rangle$ to the category of $\mathbb{Z}$-graded vector spaces over $\mathbb{C}_\infty$ is a faithful exact tensor functor and denote the automorphism group of the underlying fiber functor by $G^F_{\underline{H}}$. We write $G_{\underline{H},\mathbb{C}_\infty} := G_{\underline{H}} \times_{\mathcal{Q},\iota} \mathbb{C}_\infty$. By [DMOS82, Thm. 3.2], we find that $G^F_{\underline{H}}$ is a $G_{\underline{H},\mathbb{C}_\infty}$*-torsor over* $\operatorname{Spec}\mathbb{C}_\infty$; that is, an affine scheme that is faithfully flat over $\operatorname{Spec}\mathbb{C}_\infty$, together with a morphism $G^F_{\underline{H}} \times_{\mathbb{C}_\infty} G_{\underline{H},\mathbb{C}_\infty} \to G^F_{\underline{H}}$ such that

$$G^F_{\underline{H}} \times_{\mathbb{C}_\infty} G_{\underline{H},\mathbb{C}_\infty} \to G^F_{\underline{H}} \times_{\mathbb{C}_\infty} G^F_{\underline{H}}, \quad (x,g) \mapsto (x, xg)$$

is an isomorphism. Since $G^F_{\underline{H}}$ is locally of finite type, we find in particular that $\mathbb{C}_\infty \hookrightarrow \kappa(P)$ is a finite field extension for any closed point $P \in G^F_{\underline{H}}$. Since $\mathbb{C}_\infty$ is algebraically closed, we conclude

$$G^F_{\underline{H}}(\mathbb{C}_\infty) = \{P \in G^F_{\underline{H}} \mid \kappa(P) = \mathbb{C}_\infty\} \neq \emptyset,$$

and moreover that there is an isomorphism $G^F_{\underline{H}} \cong G_{\underline{H},\mathbb{C}_\infty}$ which is canonical up to conjugation (cf. [BLR90, §6.4]). The grading means that we may interpret Gr_F as a tensor functor $\langle\langle \underline{H} \rangle\rangle \to \mathscr{R}ep_{\mathbb{C}_\infty}(\mathbb{G}_{m,\mathbb{C}_\infty})$; hence it corresponds to a unique cocharacter of $G^F_{\underline{H}}$. By the above isomorphism it corresponds to a unique conjugacy class of cocharacters of $G_{\underline{H},\mathbb{C}_\infty}$. We let $\mathcal{Q}^{\mathrm{sep}}$ denote an abstractly given separable closure of $\mathcal{Q}$. Since this conjugacy class is defined over the separable closure of $\iota(\mathcal{Q})$ in $\mathbb{C}_\infty$, there is moreover a unique $G_{\underline{H}}(\mathcal{Q}^{\mathrm{sep}}) \rtimes \mathrm{Gal}(\mathcal{Q}^{\mathrm{sep}}/\mathcal{Q})$-conjugacy class of cocharacters of $G_{\underline{H},\mathcal{Q}^{\mathrm{sep}}} := G_{\underline{H}} \times_{\mathcal{Q}} \mathcal{Q}^{\mathrm{sep}}$.

Definition 3.6.3 ([Pin97a, Def. 7.10])**.** Any cocharacter in this $G_{\underline{H}}(\mathcal{Q}^{\mathrm{sep}}) \rtimes \mathrm{Gal}(\mathcal{Q}^{\mathrm{sep}}/\mathcal{Q})$-conjugacy class is called a *Hodge-Pink cocharacter of $G_{\underline{H}}$*.

The following proposition allows us later to determine the Hodge-Pink group of a pure Q-Hodge-Pink structure that is strongly Hodge-Pink additive.

Theorem 3.6.4 ([Pin97a, Thm. 7.11 and Thm. 9.11])**.** *Let $\underline{H}$ be a pure $\mathcal{Q}$-Hodge-Pink structure in $\mathscr{H}odge_{\mathcal{Q}}^{\mathrm{sha}}$. Then the group $G_{\underline{H},\mathcal{Q}^{sep}}$ is generated by the images of all $G_{\underline{H}}(\mathcal{Q}^{sep}) \rtimes \mathrm{Gal}(\mathcal{Q}^{sep}/\mathcal{Q})$-conjugates of Hodge-Pink cocharacters.*

4. THE HODGE CONJECTURE FOR FUNCTION FIELDS

In classical algebraic number theory, the first Betti homology group $H_1(X(\mathbb{C}), \mathbb{Q})$ of an abelian variety X over $K \subset \mathbb{C}$ gives rise to a rational Hodge structure. One considers the Tannakian category over $\mathbb{Q}$ generated by this Hodge structure and the Tate twist $\mathbb{Q}(1)$, and defines the Hodge group of X to be the corresponding affine group scheme by Tannakian duality. Similarly, the motive $h_1(X)$ assigned to X generates a Tannakian category over $\mathbb{Q}$ and the motivic Galois group of X is the affine group scheme given by Tannakian duality. The classical Hodge conjecture states that the motivic Galois group and the Hodge group are isomorphic.

In this chapter, we carry these relations over to the parallel world of function fields and prove the analog of the Hodge conjecture when $Q = \mathbb{F}_q(t)$. We shall first introduce the function field analogs of abelian varieties, pure uniformizable Anderson A-modules. As in the classical situation, we define isogenies between Anderson A-modules and the category $\mathscr{PUM}^I$ of pure uniformizable Anderson A-modules up to isogeny. We then assign a pure rigid analytically trivial dual Anderson A-motive $\mathcal{M}^*(\underline{E})$ to a pure uniformizable Anderson A-module $\underline{E}$. Through the Betti homology realization of a pure uniformizable Anderson A-module $\underline{E}$, we further associate a pure Q-Hodge-Pink structure $\mathcal{H}(\underline{E})$ with $\underline{E}$. We show that the functors $\omega' \circ \mathcal{P}' \circ \mathcal{M}^*$ and $\varpi \circ \mathcal{H}$ are isomorphic, so that we obtain the following "commutative" diagram:

$$\begin{tikzcd}
\mathscr{PRDA}^I_+ \arrow[r, "\mathcal{P}'"] & \mathscr{PT}' \arrow[r, "\mathcal{R}"] \arrow[d, "\mathcal{T}"] \arrow[dr, "\omega'"] & \mathscr{PT} \arrow[d, "\omega"] \\
\mathscr{PUM}^I \arrow[u, "\mathcal{M}^*"] \arrow[r, "\mathcal{H}"] & \mathscr{H}odge_Q \arrow[r, "\varpi"] & \mathscr{V}ec_Q .
\end{tikzcd}$$

We consider a pure uniformizable Anderson A-module over $k \subset \mathbb{C}_\infty$ and the Tannakian categories over Q generated by $(\mathcal{P}' \circ \mathcal{M}^*)(\underline{E})$ and $\mathcal{H}(\underline{E})$. We call the linear algebraic group given by Tannakian duality the Galois group $\Gamma_{\underline{E}}$ and the Hodge-Pink group $G_{\underline{E}}$ of $\underline{E}$, respectively. The first section of this chapter ends with the definition of a functor $\mathcal{T} : \mathscr{PT}' \to \mathscr{H}odge_Q$ that induces a Q-group scheme homomorphism $\mu : G_{\underline{E}} \to \Gamma_{\underline{E}}$.

In the second section we prove the Hodge conjecture for function fields; that is, μ is an isomorphism. The proof uses the equivalent conditions given in Proposition 1.2.15. In order to show that μ is faithfully flat, we want in particular to find a corresponding pure dual sub-t-motive in the Tannakian category generated by $\mathcal{P}(\mathcal{M}^*(\underline{E}))$ to a pure sub-Q-Hodge-Pink structure in the Tannakian category generated by $\mathcal{H}(\underline{E})$. This is done through F-modules that are roughly spoken rigid sheaves on rigid analytic spaces. We may assign F-modules to pure rigid analytically trivial Anderson A-motive over $\mathbb{C}_\infty$ and sub-Q-Hodge-Pink structures. Then the rigid analytic GAGA principle allows a return to algebraic sheaves, which give rise to pure rigid analytically dual Anderson A-motives over $\mathbb{C}_\infty$ and pure dual t-motives.

4.1 The map μ from the Hodge-Pink group to the Galois group

We first introduce Anderson A-modules over k, which form a category that we denote by $\mathscr{M}$. The next subsection deals with the relations between Anderson A-modules over k and dual Anderson A-motives. We give a functor $\mathcal{M}^*$ from $\mathscr{M}$ to the category $\mathscr{DA}_+$ of dual Anderson A-motives of positive rank and dimension that is in fact an equivalence of categories. We call an Anderson A-module pure if the corresponding dual Anderson A-motive is pure. Isogenies and uniformizability of Anderson A-modules are discussed next. The upshot is that the category $\mathscr{PUM}^I$ of pure uniformizable Anderson A-modules over k up to isogeny is equivalent to the category $\mathscr{PRDA}_+^I$ of pure rigid analytically trivial dual Anderson A-motives of positive rank and dimension up to isogeny. We define a functor $\mathcal{E}$ such that $\mathcal{E}$ and $\mathcal{M}^*$ are "inverses up to isomorphism".

When $A = \mathbb{F}_q[t]$, we may associate a pure Q-Hodge-Pink structure $\mathcal{H}(\underline{E})$ with a pure uniformizable Anderson A-module $\underline{E}$ over $\mathbb{C}_\infty$. We then give an alternative functor $\mathcal{D} : \mathscr{PRDA}_+^I \to \mathscr{Hodge}_Q$ that is isomorphic to $\mathcal{H} \circ \mathcal{E}$. Further, we define a functor $\mathcal{T} : \mathscr{PT} \to \mathscr{Hodge}_Q$ such that $\mathcal{T} \circ \mathcal{P}' \cong \mathcal{D}$. Finally, we show that the functors $\varpi \circ \mathcal{T}$ and ω' are isomorphic so that we have a "commutative" diagram:

$$\begin{tikzcd}
\mathscr{PRDA}_+^I \arrow[r, "\mathcal{P}'"] \arrow[d, "\mathcal{E}"'] \arrow[dr, "\mathcal{D}"] & \mathscr{PT}' \arrow[r, "\mathcal{R}"] \arrow[d, "\mathcal{T}"] \arrow[dr, "\omega'"] & \mathscr{PT} \arrow[d, "\omega"] \\
\mathscr{PUM}^I \arrow[r, "\mathcal{H}"] & \mathscr{Hodge}_Q \arrow[r, "\varpi"] & \mathscr{Vec}_Q.
\end{tikzcd}$$

By Lemma 1.2.14 there is a Q-group scheme homomorphism μ from the Hodge-Pink group of $\mathcal{T}(\underline{P})$ to the Galois group of $\underline{P}$, where $\underline{P}$ is a pure dual t-motive over $\mathbb{C}_\infty$.

4.1.1 Anderson A-modules

Just as abelian varieties are higher dimensional generalizations of elliptic curves, t-modules were developed by Anderson as a higher dimensional generalization of Drinfeld A-modules when $A = \mathbb{F}_q[t]$ [And86]. We slightly generalize their definition such that A may be the ring of integers of an arbitrary function field. We call the objects thus obtained Anderson A-modules, indicating the fact that they are A-module schemes.

Definition 4.1.1. Let $(k, \gamma : A \to k)$ be an A-field and r, d positive integers.

(i) An *(abelian) Anderson A-module of rank r, dimension d and characteristic γ over k* is a pair $\underline{E} = (E, \varphi)$, where $E \cong \mathbb{G}_{a,k}^d$ is an additive algebraic group scheme and

$$\begin{aligned}
\varphi : A &\to \operatorname{End}_{k,\mathbb{F}_q}(E) \cong \operatorname{Mat}_{d\times d}(k[\tau]), \\
a &\mapsto \varphi_a := \varphi(a),
\end{aligned}$$

is a ring homomorphism such that

$$(\mathrm{T}_0\varphi_a - \gamma(a))^d = 0 \text{ on } \mathrm{T}_0 E \cong \operatorname{Mat}_{d\times 1}(k)$$

and $\mathcal{M}^*(E) := \operatorname{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, E)$ is a locally free A_k-module of rank r under:

$$\begin{aligned}
A \ni a :& \quad \mathrm{m} \mapsto \varphi_a \circ \mathrm{m}, \\
k \ni \beta :& \quad \mathrm{m} \mapsto \mathrm{m} \circ \beta.
\end{aligned}$$

(ii) A *morphism* $f : (E, \varphi) \to (E', \psi)$ of Anderson A-modules is a homomorphism of A-module schemes; that is, $f \in \mathrm{Hom}_k(E, E')$ such that $\psi_a \circ f = f \circ \varphi_a$ for all $a \in A$.

Anderson A-modules over k form a category $\mathscr{M}$, and we denote the set of all morphisms $\underline{E} \to \underline{E}'$ of Anderson A-modules by $\mathrm{Hom}_{\mathscr{M}}(\underline{E}, \underline{E}')$.

Remark 4.1.2. The condition that $\mathcal{M}^*(E)$ is a locally free A_k-module of rank r is equivalent to the usual condition in the definition of Anderson A-modules that $\mathcal{M}_*(E) := \mathrm{Hom}_{k,\mathbb{F}_q}(E, \mathbb{G}_{a,k})$ is a locally free A_k-module of rank r under:

$$\begin{aligned} A \ni a: &\quad \mathrm{m} \mapsto \mathrm{m} \circ \varphi_a, \\ k \ni \beta: &\quad \mathrm{m} \mapsto \beta \circ \mathrm{m}. \end{aligned}$$

The reason for changing this is that $\mathcal{M}^*(E)$ gives rise to a functor $\mathcal{M}^* : \mathscr{M} \to \mathscr{D}\mathscr{A}_+$, whereas $\mathcal{M}_*(E)$ provides a functor $\mathcal{M}^* : \mathscr{M} \to \mathscr{A}_+$ (see Section 4.1.2).

Example 4.1.3. A *Drinfeld A-module* (E, φ) *of rank r over k* is an Anderson A-module (E, φ) of rank r and dimension 1 over k. Thus

$$\mathrm{T}_0 \circ \varphi = \gamma \quad \text{and} \quad \exists\, a \in A : \ \varphi_a \neq \gamma(a)\tau^0.$$

In order to show in Proposition 4.1.26 that the period lattice of a uniformizable Anderson A-module $\underline{E}$ of rank r also has rank r, we need to introduce the notion of a torsion submodule of $\underline{E}$.

Definition 4.1.4 ([Har08, Def. 2.2.1])**.** Let $\underline{E} = (E, \varphi)$ be an Anderson A-module over k and $\mathfrak{a} = (a_1, \dots a_n)$ be a non-zero ideal in A. The $\mathfrak{a}$*-torsion submodule of* $\underline{E}$ is

$$\underline{E}[\mathfrak{a}] := E[\mathfrak{a}] := \varphi[\mathfrak{a}] := \ker(E \overset{(\varphi_{a_1}, \dots, \varphi_{a_n})}{\longrightarrow} E^n = E \times_{\mathrm{Spec}\, k} \dots \times_{\mathrm{Spec}\, k} E).$$

4.1.2 *From dual Anderson A-motives to Anderson A-modules*

First we define a functor $\mathcal{M}^*$ from the category $\mathscr{M}$ of Anderson A-modules of rank r and dimension d over k to the category $\mathscr{D}\mathscr{A}_+$ of dual Anderson A-motives of rank r and dimension d over k that is an equivalence of categories. We then call an Anderson A-module $\underline{E}$ pure if the dual Anderson A-motive $\mathcal{M}^*(\underline{E})$ is pure. In the next section, we define isogenies of Anderson A-modules and prove that two Anderson A-modules are isogenous if and only if the associated dual Anderson A-motives are isogenous. Finally, we prove that an Anderson A-motive is uniformizable if and only if the corresponding dual Anderson A-motive is rigid analytically trivial. Thus we can construct the desired equivalence of categories $\mathcal{E}$ from the category $\mathscr{P}\mathscr{R}\mathscr{D}\mathscr{A}_+^I$ of pure rigid analytically trivial dual Anderson A-motives of positive rank and dimension up to isogeny to the category $\mathscr{P}\mathscr{U}\mathscr{M}^I$ of pure uniformizable Anderson A-modules up to isogeny.

The equivalence of the categories $\mathscr{M}$ and $\mathscr{D}\mathscr{A}_+$

We want to construct a functor $\mathcal{M}^*$ from the category $\mathscr{M}$ of Anderson A-modules over k to the category $\mathscr{D}\mathscr{A}_+$ of dual Anderson A-motives of positive rank and dimension over k. Then we show that $\mathcal{M}^*$ is an equivalence of categories, preserving ranks and dimensions.

Recall that $\mathcal{M}_*(E) = \mathrm{Hom}_{k,\mathbb{F}_q}(E, \mathbb{G}_{a,k})$ is a locally free A_k-module of rank r under:

$$\begin{aligned} A \ni a: &\quad \mathrm{m} \mapsto \quad \mathrm{m} \circ \varphi_a, \\ k \ni \beta: &\quad \mathrm{m} \mapsto \quad \beta \circ \mathrm{m}. \end{aligned} \tag{4.1}$$

Then $\mathcal{M}_*(E)$ is equipped with a τ-action

$$\tau : \quad \mathrm{m} \mapsto \quad \tau \circ \mathrm{m} = \mathrm{Frob}_{q,\mathbb{G}_{a,k}} \circ \mathrm{m},$$

which induces an A_k-homomorphism $\tau_{\mathcal{M}_*(E)} : F^*\mathcal{M}_*(E) \to \mathcal{M}_*(E)$ (cf. Lemma 1.1.3). This defines a functor $\mathcal{M}_* : \mathscr{M} \to \mathscr{A}_+$, $(E, \varphi) \mapsto (\mathcal{M}_*(E), \tau_{\mathcal{M}_*(E)})$, that is known to be an anti-equivalence of categories.

Similarly, we want to associate a dual Anderson A-motive $\mathcal{M}^*(\underline{E}) = (\mathcal{M}^*(E), \sigma_{\mathcal{M}^*(E)})$ with an Anderson A-module $\underline{E} = (E, \varphi)$. By definition, $\mathcal{M}^*(E) = \mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, E)$ is a locally free A_k-module of rank r under

$$\begin{aligned} A \ni a : \quad & \mathrm{m} \mapsto \quad \varphi_a \circ \mathrm{m}, \\ k \ni \beta : \quad & \mathrm{m} \mapsto \quad \mathrm{m} \circ \beta, \end{aligned}$$

and we add a τ-action

$$\tau : \quad \mathrm{m} \mapsto \quad \mathrm{m} \circ \tau = \mathrm{m} \circ \mathrm{Frob}_{q,\mathbb{G}_{a,k}} .$$

This means, that $a \in A$ acts by multiplication on the left and $\beta \in k$ and τ by multiplication on the right on $\mathrm{Mat}_{d\times 1}(k[\tau]) \cong \mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, E)$. We recall the identification (1.2) induced by the dagger operation

$$\mathrm{Hom}_{k,\mathbb{F}_q}(E, E') \cong \mathrm{Mat}_{e\times d}(k[\tau]) \xrightarrow{\dagger} \mathrm{Mat}_{d\times e}(k[\sigma])$$

and thus obtain corresponding actions on $\mathrm{Mat}_{1\times d}(k[\sigma])$

$$\begin{aligned} A \ni a : \quad & \mathrm{m} \mapsto \quad \mathrm{m} \cdot \varphi_a^\dagger, \\ k \ni \beta : \quad & \mathrm{m} \mapsto \quad \beta \cdot \mathrm{m}, \\ \sigma : \quad & \mathrm{m} \mapsto \quad \sigma \cdot \mathrm{m}. \end{aligned}$$

The actions of $a \in A$ and $\beta \in k$ on $\mathrm{Mat}_{1\times d}(k[\sigma])$ obviously commute, but σ is a ς^*-linear map since

$$\begin{aligned} \sigma((\sum_i a_i \otimes \beta_i) \cdot \mathrm{m}) &= \sigma(\sum_i \beta_i \cdot \mathrm{m} \cdot \varphi_{a_i}^\dagger) = \sum_i \beta_i^{(-1)} \cdot \sigma \cdot \mathrm{m} \cdot \varphi_{a_i}^\dagger \\ &= (\sum_i a_i \otimes \beta_i^{(-1)}) \sigma(\mathrm{m}) \end{aligned}$$

for all $\sum_i a_i \otimes \beta_i \in A_k$ and $\mathrm{m} \in \mathrm{Mat}_{1\times d}(k[\sigma])$. Thus $\mathcal{M}^*(E)$ is an $A_k[\sigma]$-module as desired. Further, we define $\sigma_{\mathcal{M}^*(E)} : \varsigma^*\mathcal{M}^*(E) \to \mathcal{M}^*(E)$ to be the A_k-homomorphism induced by σ (see Lemma 1.1.3).

Recall that there is an isomorphism

$$\operatorname{coker} \tau_{\mathcal{M}_*(E)} = \mathcal{M}_*(E)/\tau_{\mathcal{M}_*(E)}(F^*\mathcal{M}_*(E)) \cong (\mathrm{T}_0 E)^\vee$$

induced by the map $\mathcal{M}_*(E) \to (\mathrm{T}_0 E)^\vee$, $\mathrm{m} \mapsto \mathrm{T}_0(\mathrm{m})$ [Har08, Lem. 2.1.9]. This isomorphism is an essential part of the proof of the anti-equivalence of the category $\mathscr{A}_+$ of non-dual Anderson A-motives of positive rank and dimension over k and the category $\mathscr{M}$ of Anderson A-modules over k. We apply Lemma 1.1.22 to see that the Zariski-tangent space to E at the identity and $\operatorname{coker} \sigma_{\mathcal{M}^*(E)}$ are similarly related:

Lemma 4.1.5. *Let (E_1,φ) and (E_2,ψ) be Anderson A-modules of respective dimension d_1 and d_2 and $(\mathcal{M}^*(E_1),\sigma_{\mathcal{M}^*(E_1)})$ and $(\mathcal{M}^*(E_2),\sigma_{\mathcal{M}^*(E_2)})$ be defined as above. Suppose $f \in \mathrm{Hom}_{A_k}(\mathcal{M}^*(E_1),\mathcal{M}^*(E_2))$ such that $\varsigma^* f\circ\sigma_{\mathcal{M}^*(E_1)} = \sigma_{\mathcal{M}^*(E_2)}\circ f$. Then the following diagram commutes and has exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & \varsigma^*\mathcal{M}^*(E_1) \arrow[r, "\sigma_{\mathcal{M}^*(E_1)}"] \arrow[d, "\varsigma^* f"'] & \mathcal{M}^*(E_1) \arrow[r, "\mathrm{T}_0"] \arrow[d, "f"'] & \mathrm{T}_0E_1 \arrow[r] \arrow[d, "\mathrm{T}_0 f"'] & 0 \\
0 \arrow[r] & \varsigma^*\mathcal{M}^*(E_2) \arrow[r, "\sigma_{\mathcal{M}^*(E_2)}"] & \mathcal{M}^*(E_2) \arrow[r, "\mathrm{T}_0"] & \mathrm{T}_0E_2 \arrow[r] & 0.
\end{tikzcd}$$

In particular, if (E,φ) is an Anderson A-module over k then

$$\operatorname{coker}\sigma_{\mathcal{M}^*(E)} = \mathcal{M}^*(E)/\sigma_{\mathcal{M}^*(E)}(\varsigma^*\mathcal{M}^*(E)) \cong \mathrm{T}_0E.$$

In order to show that $(E,\varphi)\mapsto(\mathcal{M}^*(E),\sigma_{\mathcal{M}^*(E)})$ defines an essentially surjective functor $\mathcal{M}^* : \mathscr{M} \to \mathscr{DA}_+$, we construct an "inverse" functor $\mathcal{E} : \mathscr{DA}_+ \to \mathscr{M}$ such that $(\mathcal{M}^*(\mathcal{E}(\mathrm{M})),\sigma_{\mathcal{M}^*(\mathcal{E}(\mathrm{M}))}) \cong \underline{\mathrm{M}}$ for any dual Anderson A-motives M holds.

Let $\underline{\mathrm{M}} \in \mathrm{Ob}(\mathscr{DA}_+)$ be a dual Anderson A-motive over k of dimension d. By Lemma 2.1.5 we have $\mathrm{M} \cong \mathrm{Mat}_{1\times d}(k[\sigma])$ and the action of $a \in A$ on $\mathrm{Mat}_{1\times d}(k[\sigma])$ is given by right multiplication by a matrix $\varphi_a \in \mathrm{Mat}_{d\times d}(k[\sigma])$. Therefore $\varphi_a^\dagger \in \mathrm{Mat}_{d\times d}(k[\tau]) \cong \mathrm{End}_{k,\mathbb{F}_q}(\mathbb{G}^d_{a,k})$ and we can define a ring homomorphism

$$\varphi_{\mathcal{E}(\underline{\mathrm{M}})} : A \to \mathrm{End}_{k,\mathbb{F}_q}(\mathbb{G}^d_{a,k}), \quad a \mapsto \varphi_a^\dagger.$$

Set $\mathcal{E}(\mathrm{M}) := \mathbb{G}^d_{a,k}$ and $\mathcal{E}(\underline{\mathrm{M}}) := (\mathcal{E}(\mathrm{M}),\varphi_{\mathcal{E}(\underline{\mathrm{M}})})$.

Proposition 4.1.6. *Let $\underline{\mathrm{M}} = (\mathrm{M},\sigma_{\mathrm{M}}) \in \mathrm{Ob}(\mathscr{DA}_+)$ be a dual Anderson A-motive of rank r and dimension d over k. Then $\mathcal{E}(\underline{\mathrm{M}})$ is an Anderson A-module of rank r and dimension d over k. Furthermore, the functor $\mathcal{E} : \mathscr{DA}_+ \to \mathscr{M}$, $\underline{\mathrm{M}} \mapsto \mathcal{E}(\underline{\mathrm{M}})$, is well-defined, preserves ranks and dimensions and satisfies $(\mathcal{M}^*(\mathcal{E}(\mathrm{M})),\sigma_{\mathcal{M}^*(\mathcal{E}(\mathrm{M}))}) \cong \underline{\mathrm{M}}$ for all dual Anderson A-motives $\underline{\mathrm{M}}$.*

Proof. To see that $\mathcal{M}^*(\underline{E})$ is an Anderson A-module of rank r and dimension d over k, it remains to show that the two conditions $(\mathrm{T}_0\varphi_a - \gamma(a))^d = 0$ on $\mathrm{T}_0\mathcal{E}(\mathrm{M}) = \mathrm{T}_0\mathbb{G}^d_{a,k}$ and $\mathcal{M}^*(\mathcal{E}(\mathrm{M}))$ is a locally free A_k-module of rank r are satisfied.

The latter is clear since $\mathrm{M} \cong \mathrm{Mat}_{1\times d}(k[\sigma]) \cong \mathcal{M}^*(\mathcal{E}(\mathrm{M}))$ is locally free of rank r by assumption. Furthermore, we know from Lemma 4.1.5:

$$\begin{aligned}
& (a\otimes 1 - 1\otimes\gamma(a))^d = 0 && \text{on } \operatorname{coker}\sigma_{\mathrm{M}} \\
\Leftrightarrow\quad & (\overline{\mathrm{m}}\cdot\varphi_a^\dagger - \gamma(a)\cdot\overline{\mathrm{m}})^d = 0 && \forall\, \overline{\mathrm{m}} \in \operatorname{coker}\sigma_{\mathrm{M}} \subseteq \mathrm{Mat}_{1\times d}(k[\sigma]) \\
\Leftrightarrow\quad & (\mathrm{T}_0\varphi_a - \gamma(a))^d = 0 && \text{on } \mathrm{T}_0\mathcal{E}(\mathrm{M}).
\end{aligned}$$

□

Now we have collected the necessary ingredients to finally prove:

Theorem 4.1.7. *The functor*

$$\mathcal{M}^* : \mathscr{M} \to \mathscr{DA}_+, \quad \underline{E} = (E,\varphi) \mapsto \mathcal{M}^*(\underline{E}) = (\mathcal{M}^*(E),\sigma_{\mathcal{M}^*(E)})$$

with $\mathcal{M}^*(E) = \mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k},E)$ and $\sigma_{\mathcal{M}^*(E)} : \varsigma^*\mathrm{M} \to \mathrm{M}, \mathrm{m}\otimes 1 \mapsto \sigma\mathrm{m}$, is a covariant equivalence of categories, preserving ranks and dimensions.

Proof. For ease of notation, set $(\mathrm{M}, \sigma_{\mathrm{M}}) := \mathcal{M}^*(\underline{E}) = (\mathrm{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, E), \sigma_{\mathcal{M}^*(E)})$. By what has been said before, it suffices to show that $\dim \operatorname{coker} \sigma_{\mathrm{M}} = d$ and the A_k-homomorphism σ_{M} is injective to see that $\mathcal{M}^*(\underline{E})$ is a dual Anderson A-motive. The former is clear by Lemma 4.1.5. The induced ς^*-linear map $\sigma : \mathrm{M} \to \mathrm{M}$ is injective since $\mathrm{M} \cong \mathrm{Mat}_{1\times d}(k[\sigma])$ has no zero-divisors and injectivity of σ_{M} follows.

$\mathcal{M}^*$ is then an equivalence of categories if it is essentially surjective and fully faithful, so by applying Lemma 4.1.6, it remains to show that $\mathcal{M}^*$ is fully faithful.

Let $\underline{E} = (E, \varphi)$ and $\underline{E}' = (E', \psi)$ be two Anderson A-modules of dimension d and d', respectively, and write $\mathrm{M} := \mathcal{M}^*(E)$ and $\mathrm{M}' := \mathcal{M}^*(E')$. Let $f \in \mathrm{Hom}_{\mathscr{M}}(\underline{E}, \underline{E}')$; that is, $f : \underline{E} \to \underline{E}'$ is a homomorphism of k-group schemes such that $\psi_a \circ f = f \circ \varphi_a$. Then

$$\mathcal{M}^*(f) = (\mathrm{m} \mapsto f \circ \mathrm{m}) : \mathcal{M}^*(E) \to \mathcal{M}^*(E')$$

is an A_k-homomorphism that satisfies $\sigma_{\mathrm{M}} \circ \varsigma^* f = f \circ \sigma_{\mathrm{M}'}$ by Lemma 4.1.5.

Consider now $f \in \mathrm{Hom}_k(\mathcal{M}^*(\underline{E}), \mathcal{M}^*(\underline{E}'))$. This means, $f : \mathcal{M}^*(E) \to \mathcal{M}^*(E')$ is an A_k-homomorphism such that $\sigma_{\mathcal{M}^*(E)} \circ \varsigma^* f = f \circ \sigma_{\mathcal{M}^*(E')}$. Moreover, $\psi_a \circ f = f \circ \varphi_a$ holds for all $a \in A$ since

$$(f \circ \varphi_a)(\mathrm{m}) = f \circ (\varphi_a \circ \mathrm{m}) = f(a \cdot \mathrm{m}) = a(f \circ \mathrm{m}) = (\psi_a \circ f)(\mathrm{m})$$

for all $\mathrm{m} \in \mathcal{M}^*(E)$. □

Let us apply Lemma 1.1.21 to emphasize the close relation between Anderson A-modules and dual Anderson A-motives. The following isomorphism is unique to the dual setting.

Corollary 4.1.8. *Let (E, φ) be an Anderson A-module over k, $\mathcal{M}^* = (\mathcal{M}^*(E), \sigma_{\mathcal{M}^*(E)})$ its associated dual Anderson A-motive and σ denote the ς^*-liner map induced by $\sigma_{\mathcal{M}^*(E)}$. We then obtain*

$$\operatorname{coker}(\sigma - \mathrm{id}_{\mathcal{M}^*(E)}) = \mathcal{M}^*(E)/(\sigma - \mathrm{id}_{\mathcal{M}^*(E)})\mathcal{M}^*(E) \cong E(k).$$

Because Drinfeld A-modules of rank r over k correspond to (dual) Anderson A-motives of rank r and dimension 1 over k, we call such a (dual) Anderson A-motive a *(dual) Drinfeld A-motive of rank r over k.*

We see that a Drinfeld $\mathbb{F}_q[t]$-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r over k as defined in Example 2.1.6 corresponds to a Drinfeld $\mathbb{F}_q[t]$-module (E, φ) of dimension r over k since setting $\mathbf{m} := (1, \sigma, \ldots, \sigma^{r-1})^{\mathrm{tr}}$ and

$$t = \varphi_t^\dagger = \theta + \alpha_1^{(-1)}\sigma + \ldots + \alpha_r^{(-r)}\sigma^r \in k[\sigma]$$

implies

$$\begin{array}{rcl} 1 & \mapsto & \sigma, \\ & \vdots & \\ \sigma^{r-2} & \mapsto & \sigma^{r-1}, \\ \sigma^{r-1} & \mapsto & \sigma^r = \left((t-\theta) - \alpha_1^{(-1)}\sigma - \ldots - \alpha_{r-1}^{(-(r-1))}\sigma^{r-1}\right)/\alpha_r^{(-r)}. \end{array}$$

Hence,

$$\Phi_{\mathbf{m}} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\alpha_r^{(-r)} & -\alpha_1^{(-1)}/\alpha_r^{(-r)} & \cdots & -\alpha_{r-1}^{(-(r-1))}/\alpha_r^{(-r)} \end{pmatrix}$$

represents σ_{M} with respect to the $k[t]$-basis $\mathbf{m}$ for M, as desired. If we are given a dual Drinfeld $\mathbb{F}_q[t]$-motive of rank r over k, we get back to a Drinfeld $\mathbb{F}_q[t]$-module (E, φ) of rank r over k by putting $E := \mathbb{G}_{a,k}$ and

$$\begin{aligned} \varphi_t &:= t^\dagger = (\theta + \alpha_1^{(-1)}\sigma + \ldots + \alpha_r^{(-r)}\sigma^r)^\dagger \\ &= \theta + \alpha_1\tau + \ldots + \alpha_r\tau^r \in k[\tau]. \end{aligned}$$

Similarly, we get from a Drinfeld $\mathbb{F}_q[t]$-motive to a Drinfeld $\mathbb{F}_q[t]$-module of the same rank and the other way around.

Definition 4.1.9. An Anderson A-module $\underline{E}$ is said to be *pure of weight* $\frac{d}{r}$ if $\mathcal{M}^*(\underline{E})$ is pure of weight $\frac{l}{n}$.

Moreover, the functors $\mathcal{E}$ and $\mathcal{M}^*$ are inverse up to isomorphism by Proposition 4.1.6. The rest of this subsection deals with showing that $\mathcal{E}$ induces a functor from the category $\mathscr{PRDA}_+^I$ to the category of pure uniformizable Anderson A-modules up to isogeny, which is well-defined and an equivalence of categories.

Isogenies

An isogeny between abelian varieties X and Y is a homomorphism $X \to Y$ that is surjective with finite kernel. Furthermore, the relation of isogeny is an equivalence relation for abelian varieties. We similarly define isogenies of Anderson A-modules. We show that a morphism f of Anderson A-modules over k is an isogeny if and only if $\mathcal{M}^*(f)$ is an isogeny of dual Anderson A-motives over k. From Corollary 2.2.6 it follows that the relation of isogeny is an equivalence relation for Anderson A-modules so that we may define the category of pure Anderson A-modules up to isogeny.

Definition 4.1.10. (i) We call a morphism f of Anderson A-modules (E, φ) and (E', ψ) an *isogeny* if $f : E \to E'$ is finite and surjective as a k-scheme morphism. An isogeny f is said to be *separable* if $\ker f$ is geometrically reduced and *inseparable*, otherwise.

(ii) We say that two Anderson A-modules $\underline{E}$ and $\underline{E}'$ are *isogenous* if there is an isogeny $f \in \mathrm{Hom}_{\mathscr{M}}(\underline{E}, \underline{E}')$.

Recall that a morphism f of affine k-schemes is called *finite* if the induced ring homomorphism f^* is finite.

Lemma 4.1.11 ([Har08, Lem. 2.3.2])**.** *Let E and E' be additive group schemes of dimension d and d', respectively, and $f : E \to E'$ a morphism of group schemes with induced ring homomorphism $f^* : k[x_1, \ldots, k_d] \to k[x_1, \ldots, k_{d'}]$. Then the following conditions are equivalent:*

(i) f is finite and surjective,

(ii) f^* is finite and $d = d'$,

(iii) f^* is finite and injective.

Our goal is to restrict the functor $\mathcal{E}$ to a functor from the category of dual Anderson A-motives up to isogeny to the category of Anderson A-modules up to isogeny. In order to do this we state the following:

Theorem 4.1.12. *Let $\underline{E} = (E, \varphi)$ and $\underline{E}' = (E', \psi)$ be Anderson A-modules over k. Then an $f \in \operatorname{Hom}_{\mathscr{M}}(\underline{E}, \underline{E}')$ is an isogeny if and only if $\mathcal{M}^*(f) \in \operatorname{Hom}_k(\mathcal{M}^*(\underline{E}), \mathcal{M}^*(\underline{E}'))$ is an isogeny.*

Proof. Let $\underline{\mathrm{M}} := (\mathrm{M}, \sigma_{\mathrm{M}}) := \mathcal{M}^*(\underline{E})$ and $\underline{\mathrm{N}} := (\mathrm{N}, \sigma_{\mathrm{N}}) := \mathcal{M}^*(\underline{E}')$ be the corresponding dual Anderson A-motives. By abuse of notation, we write

$$\sigma : \mathrm{M} \to \mathrm{M} \quad \text{and} \quad \sigma : \mathrm{N} \to \mathrm{N}$$

for the ς^*-linear maps induced by σ_{M} and σ_{N} respectively. We suppose first that $\mathcal{M}^*(f) : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ is an isogeny. It suffices to show that f is finite by Lemma 4.1.11 and Proposition 2.2.3. The following diagram is commutative with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker(\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{M}})) \arrow[d] & \\
0 \arrow[r] & \mathrm{M} \arrow[r] \arrow[d, "\mathcal{M}^*(f)"] & \mathrm{M} \arrow[r] \arrow[d, "\mathcal{M}^*(f)"] & \mathrm{M}/(\sigma - \mathrm{id}_{\mathrm{M}})\mathrm{M} \arrow[r] \arrow[d, "\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{M}})"] & 0 \\
0 \arrow[r] & \mathrm{N} \arrow[r] \arrow[d] & \mathrm{N} \arrow[r] \arrow[d] & \mathrm{N}/(\sigma - \mathrm{id}_{\mathrm{N}})\mathrm{N} \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} \mathcal{M}^*(f) \arrow[r] \arrow[d] & \operatorname{coker} \mathcal{M}^*(f) \arrow[r] \arrow[d] & \operatorname{coker}(\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{N}})) \arrow[d] & \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

From the Snake Lemma we get an injection $\ker(\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{M}})) \hookrightarrow \operatorname{coker} \mathcal{M}^*(f)$ and thus

$$\dim_k \ker(\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{M}})) \leq= \dim_k \operatorname{coker} \mathcal{M}^*(f) < \infty.$$

Lemma 4.1.5 provides $\ker f \cong \ker(\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{M}}))$ so that

$$\dim_k \Gamma(\ker f, \mathcal{O}_{\ker f}) = \dim_k \ker(\mathcal{M}^*(f) \bmod (\sigma - \mathrm{id}_{\mathrm{M}})) < \infty.$$

Whence f is finite and an isogeny.

For the converse direction, suppose that f is an isogeny. By Corollary 4.1.13, E and E' are of the same rank and dimension so that the same holds for the associated dual Anderson A-motives M and N. It remains to show that $\mathcal{M}^*(f)$ is injective by Lemma 2.2.2. Suppose $\mathrm{m} \in \mathrm{M} = \operatorname{Hom}_{k,\mathbb{F}_q}(\mathbb{G}_{a,k}, E)$ such that $\mathcal{M}^*(f)(\mathrm{m}) = f \circ \mathrm{m} = 0$. This implies that m factors through $\ker f$; that is, $\mathrm{m} : \mathbb{G}_{a,k} \to \ker f \hookrightarrow E$. Since $\mathbb{G}_{a,k}$ is reduced and connected, m factors through the reduced subscheme of the connected component of $\ker f$, which contains 0. Since $\ker f$ is finite, this subscheme must be trivial. Hence, m is the zero map and $\mathcal{M}^*(f)$ an isogeny. □

From Proposition 2.4.9 and Corollary 2.2.6 we directly deduce the following:

Corollary 4.1.13. *(i) Let $f : \underline{E} \to \underline{E}'$ be an isogeny between Anderson A-modules $\underline{E}$ and $\underline{E}'$ over k. Then $\underline{E}$ and $\underline{E}'$ are Anderson A-modules of the same rank and dimension over k and $\underline{E}$ is pure if and only if $\underline{E}'$ is pure.*

(ii) The relation of isogeny is an equivalence relation for Anderson A-modules over k.

We can thus define the category of Anderson A-modules up to isogeny. The morphisms in the category of Anderson A-modules up to isogeny will be the quasi-morphisms whose definition reads as follows:

Definition 4.1.14. We let $\underline{E}$ and $\underline{E}'$ be Anderson A-modules over k and put

$\mathrm{QHom}_{\mathscr{M}}(\underline{E}, \underline{E}')$	$:= \mathrm{Hom}_k(\underline{E}, \underline{E}') \otimes_A Q$	the Q-vector space of *quasi-morphisms* and
$\mathrm{QEnd}_{\mathscr{M}}(\underline{E})$	$:= \mathrm{End}_k(\underline{E}) \otimes_A Q$	the Q-algebra of *quasi-endomorphisms.*

(i) We define the *category $\mathscr{M}^I$ of Anderson A-modules up to isogeny* as follows:

- Objects of $\mathscr{M}^I$: Anderson A-modules;
- Morphisms of $\mathscr{M}^I$: The quasi-morphisms in $\mathrm{QHom}_k(\underline{E}, \underline{E}')$.

(ii) We define the full *subcategory $\mathscr{P}\mathscr{M}^I$ of pure Anderson A-modules up to isogeny* by restriction.

Thus $\mathcal{E}$ induces a well-defined functor $\mathscr{P}\mathscr{D}\mathscr{A}_+^I \to \mathscr{P}\mathscr{M}^I$ that is an equivalence of categories.

Uniformizable Anderson A-modules

We assume for the rest of this chapter, that $k \subseteq \mathbb{C}_\infty$ is a complete field that contains Q_∞ as in Section 2.5. We define the exponential function $\exp_{\underline{E}} : \mathrm{T}_0 E \to E(k)$ of an Anderson A-module $\underline{E} = (E, \varphi)$ over k and study when $\underline{E}$ is uniformizable; that is, there is a short exact sequence

$$0 \longrightarrow \Lambda_{\underline{E}} \longrightarrow \mathrm{T}_0 E \longrightarrow E(k) \longrightarrow 0.$$

Note that then $\mathrm{T}_0 E/\Lambda_{\underline{E}} \cong E(k)$, which is analogous to the uniformizability of an abelian variety over $K \subset \mathbb{C}$. In order to define $\exp_{\underline{E}}$, we put $||x|| := \sup_{i,j} |x_{i,j}|$ for any matrix x with entries in k.

Definition 4.1.15 ([BH07a, Cor. 8.8])**.** Let $\underline{E} = (E, \varphi)$ be an Anderson A-module of dimension d over k. The unique *exponential function attached to $\underline{E}$*, $\exp_{\underline{E}} : \mathrm{T}_0 E \to E(k)$, satisfies the following conditions:

(a) For all $a \in A$: $\exp_{\underline{E}} \circ \mathrm{T}_0 \varphi_a = \varphi_a \circ \exp_{\underline{E}}$,

(b) Let ρ be an arbitrary coordinate system for E. Then there exists a unique sequence $(e_{(i)})$ in $\mathrm{Mat}_{d\times d}(k)$ such that

$$e_{(0)} = \mathrm{id} \quad \text{and} \quad \lim_{i\to\infty} q^{-i} \log ||e_{(i)}|| = \lim_{i\to\infty} \log ||e_{(i)}^{(-i)}|| = -\infty,$$

thereby defining an entire rigid analytic function

$$\mathrm{Exp}_{\underline{E}} : x \mapsto \sum_{j=0}^{\infty} e_{(j)} x^{(j)} : \mathrm{Mat}_{d\times 1}(k) \to \mathrm{Mat}_{d\times 1}(k),$$

which satisfies $\mathrm{Exp}_{\underline{E}} \circ \mathrm{T}_0 \rho = \rho \circ \exp_{\underline{E}}$.

For later purposes, we also want to have a "local" inverse of the exponential function of an Anderson A-module.

Definition 4.1.16 (Cf. [Böc02, Lem. 9.14])**.** Let $\underline{E} = (E, \varphi)$ be an Anderson A-module of dimension d over k. The *analytic logarithm attached to* $\underline{E}$ is defined to be a the unique map $\log_{\underline{E}} : V \to \mathrm{T}_0 E$, where V is a sufficiently small neighborhood of $0 \in E(k)$, satisfying the following conditions:

(a) $\exp_{\underline{E}} \circ \log_{\underline{E}} = \mathrm{id} = \log_{\underline{E}} \circ \exp_{\underline{E}}$ on V,

(b) Let ρ be an arbitrary coordinate system for E. Then there exists a unique sequence $(l_{(i)})$ in $\mathrm{Mat}_{d\times d}(k)$ such that

$$\lim_{i\to\infty} q^{-i} \log ||l_{(i)}|| = \lim_{i\to\infty} \log ||l_{(i)}^{(-i)}|| = 0$$

thereby defining a rigid analytic function

$$\mathrm{Log}_{\underline{E}} : x \mapsto \sum_{i=0}^{\infty} l_{(i)} x^{(i)} : (\epsilon\text{-ball in } \mathrm{Mat}_{d\times 1}(k)) \to \mathrm{Mat}_{d\times 1}(k)$$

which satisfies $\mathrm{Log}_{\underline{E}} \circ \mathrm{T}_0\rho = \rho \circ \log_{\underline{E}}$.

Definition 4.1.17. Let $\underline{E}$ be an Anderson A-module over k.

(i) We call the A-module $\Lambda_{\underline{E}} := \ker \exp_{\underline{E}}$ the *period lattice of* $\underline{E}$ and elements in $\Lambda_{\underline{E}}$ *periods of* $\underline{E}$. We define the *rank of* $\Lambda_{\underline{E}}$ to be $\mathrm{rank}_A \Lambda_{\underline{E}} := \dim_Q \Lambda_{\underline{E}} \otimes_A Q$.

(ii) We say that $\underline{E}$ is *uniformizable* if $\exp_{\underline{E}}$ is surjective.

Anderson shows in the case $A = \mathbb{F}_q[t]$ that the period lattice $\Lambda_{\underline{E}}$ of an Anderson A-module $\underline{E}$ of rank r is discrete in $\mathrm{T}_0 E$ and finitely generated over A. The rank of $\Lambda_{\underline{E}}$ is at most r with equality if and only if $\underline{E}$ is uniformizable, whence the name (see [And86, Lem. 2.4.1 and Thm. 4]).

If $\underline{E}$ is a uniformizable Anderson A-module over k, we have $\mathrm{T}_0 E/\Lambda_{\underline{E}} \cong E(k)$. In analogy with uniformizability of abelian varieties, we define its first Betti homology to be

$$H_{\mathrm{B}}(\underline{E}, A) := \Lambda_{\underline{E}}, \quad \text{and further} \quad H_{\mathrm{B}}(\underline{E}, B) := \Lambda_{\underline{E}} \otimes_A B$$

for any A-algebra B. Its Betti cohomology realization is

$$H^1_{\mathrm{B}}(\underline{E}, B) := \mathrm{Hom}_A(\Lambda_{\underline{E}}, B)$$

for any A-algebra B.

As the following theorem states, any Drinfeld A-module over k is an example for a uniformizable Anderson A-module over k.

Theorem 4.1.18 ([Gos96, Thm. 4.6.9])**.** *Let $\underline{E}$ be a Drinfeld A-module of rank r over k and $\Lambda_{\underline{E}}$ its period lattice. Then $\Lambda_{\underline{E}}$ has rank r and furthermore the functor $\underline{E} \mapsto \Lambda_{\underline{E}}$ from the category of Drinfeld modules of rank r over k to the category of finitely generated A-modules of rank r that are discrete in $\mathrm{T}_0 E \cong k$ is an equivalence of categories.*

If the dimension of an Anderson A-module $\underline{E}$ is greater than one, $\exp_{\underline{E}}$ need not be surjective (for an example, see [Gos96, Exmp. 5.5.9]). In the following subsection, we prove that an Anderson A-module over k is uniformizable if and only if the corresponding dual Anderson A-motive is rigid analytically trivial.

Uniformizability

Throughout this section, we fix a uniformizable Anderson A-module $\underline{E} = (E, \varphi)$ of rank r and dimension d over k. Our goal is to show that $\underline{E}$ is uniformizable if and only if the associated dual Anderson A-motive $\mathcal{M}^*(\underline{E})$ is rigid analytically trivial. Recall that the map

$$\delta : \mathrm{Mat}_{1\times d}(k[\sigma]) \to \mathrm{Mat}_{d\times 1}(k), \quad \sum_{i=0}^{\infty} \alpha_{(i)}\sigma^i \mapsto \sum_{i=0}^{\infty} \left(\alpha_{(i)}^{(i)}\right)^{\mathrm{tr}},$$

defined in Lemma 1.1.21, provides an isomorphism

$$\operatorname{coker}(\sigma - 1) = \mathcal{M}^*(E)/(\sigma - 1)\mathcal{M}^*(E) \cong E(k), \tag{4.2}$$

where σ denotes the ς^*-liner map induced by $\sigma_{\mathcal{M}^*(E)}$ (Corollary 4.1.8). Using the ring homomorphism $i^* : \mathbb{F}_q[t] \to A$, $t \mapsto a$, we will see that we may pass from elements in $E(k)$ to elements of the a-adic completion $\mathcal{M}^*(E)_a = \varprojlim \mathcal{M}^*(E)/a^n\mathcal{M}^*(E)$ of $\mathcal{M}^*(E)$ via the "switcheroo" (Lemma 4.1.22). In order to prove that $\Lambda_{\underline{E}} \cong \mathcal{M}^*(E)(1)^{\sigma}$, we want to relate periods of $\underline{E}$ and $\mathcal{M}^*(E)$-cycles; that is, convergent and σ-invariant elements in $\mathcal{M}^*(E)_a$. At first, we put elements in T_0E in bijective canonical correspondence with k-valued points of E. This is done through the "a-division towers".

Definition 4.1.19. Fix an $a \in A \setminus \mathbb{F}_q$ and an $x \in E(k)$.

(i) A sequence $x_{(0)}, x_{(1)}, x_{(2)}, \ldots \in E(k)$ is an *a-division tower above x* if

$$\varphi_a(x_{(n)}) = \begin{cases} x_{(n-1)} & \text{if } n > 0 \\ x & \text{if } n = 0 \end{cases}$$

(ii) An a-division tower $(x_{(n)})_{n=0}^{\infty}$ is said to be *convergent* if $\lim_{n\to\infty} \|\rho(x_{(n)})\| = 0$ for all coordinate systems ρ.

The latter condition allows us to apply the analytic logarithm attached to an Anderson A-module to an element $x_{(n)}$ of a convergent a-division tower $(x_{(n)})_{n=0}^{\infty}$ for n sufficiently large. Let us give an example of an a-division tower that shows how to get from a period of $\underline{E}$ to an a-division tower above 0.

Example 4.1.20. Let $\xi \in \mathrm{T}_0E$ be a solution of the equation $\exp_{\underline{E}}(\xi) = x$. The sequence $x_{(n)} := \exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right)$, $n \geq 0$, defines an a-division tower $(x_{(n)})_{n=0}^{\infty}$ above x since

$$\varphi_a(x_{(n)}) = \varphi_a\left(\exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right)\right) = \exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-((n-1)+1)}\xi\right) = x_{(n-1)}$$

for $n > 0$. Moreover, $(x_{(n)})_{n=0}^{\infty}$ is convergent because we have for all coordinate systems ρ

$$\begin{aligned}
\lim_{n\to\infty} \|\rho\left(\exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right)\right)\| &= \lim_{n\to\infty} \| \mathrm{Exp}_{\underline{E}}\left(\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right)\right) \| \\
&= \lim_{n\to\infty} \| \sum_{i=0}^{\infty} e_{(i)} \left(\mathrm{T}_0\rho((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi)\right)^{(i)} \| \\
&\leq \lim_{n\to\infty} \|\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\right) \sum_{i=0}^{\infty} e_{(i)} \left(\mathrm{T}_0\rho(\xi)\right)^{(i)} \| \\
&\leq \lim_{n\to\infty} \|\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\right) \mathrm{Exp}_{\underline{E}}\left(\mathrm{T}_0\rho(\xi)\right) \| \\
&= 0.
\end{aligned}$$

In fact, we can also recover the “logarithm” $\xi \in \ker(\mathrm{Exp}_{\underline{E}})$ from a convergent a-division tower above 0.

Proposition 4.1.21 ([ABP, §1.9.3])**.** *The convergent a-division towers above $x \in E(k)$ are in bijective correspondence with elements $\xi \in \mathrm{T}_0E$. If $\xi \in \mathrm{T}_0E$ corresponds to the convergent a-division tower $(x_{(n)})_{n=0}^{\infty}$ we call ξ the* logarithm of $(x_{(n)})_{n=0}^{\infty}$.

Proof. We have seen in the previous example that a $\xi \in \mathrm{T}_0E$ defines a convergent a-division tower $(x_{(n)})_{n=0}^{\infty}$ above x with $x_{(n)} = \exp_{\underline{E}}((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi)$, $n \geq 0$.

Conversely, if $(x_{(n)})_{n=0}^{\infty}$ is a convergent a-division tower we find an $\xi \in \mathrm{T}_0E$ by following the idea shown in the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\mathrm{T}_0\varphi_a"] & \overset{\log(x_{(N)})\in}{\mathrm{T}_0E} \arrow[r, "\exp_{\underline{E}}"] \arrow[d, "\mathrm{T}_0\varphi_a"] & \overset{x_{(N)}\in}{E(k)} \arrow[r] \arrow[d, "\varphi_a"] & 0\,. \\
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\mathrm{T}_0\varphi_a"] & \mathrm{T}_0E \arrow[r, "\exp_{\underline{E}}"] \arrow[d, "\mathrm{T}_0\varphi_a"] & E(k) \arrow[r] \arrow[d, "\varphi_a"] & 0 \\
& \vdots \arrow[d, "\mathrm{T}_0\varphi_a"] & \vdots \arrow[d, "\mathrm{T}_0\varphi_a"] & \vdots \arrow[d, "\varphi_a"] & \\
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\mathrm{T}_0\varphi_a"] & \mathrm{T}_0E \arrow[r, "\exp_{\underline{E}}"] \arrow[d, "\mathrm{T}_0\varphi_a"] & E(k) \arrow[r] \arrow[d, "\varphi_a"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & \underset{\ni\xi:=(\mathrm{T}_0\varphi_a)^{N+1}\log(x_{(N)})}{\mathrm{T}_0E} \arrow[r, "\exp_{\underline{E}}"] & \underset{\ni x}{E(k)} \arrow[r] & 0
\end{tikzcd}$$

More formally, there exists a unique $\xi \in \mathrm{T}_0E$ with the following properties:

- ξ is the common value $((\mathrm{T}_0\varphi_a)^{n+1}\log_{\underline{E}}(x_{(n)}))$ for all $n \gg 0$.

  Since $(x_{(n)})_{n=0}^{\infty}$ is convergent there is an $N > 0$ so that $\|\rho(x_{(n)})\| < \epsilon$ for all coordinate systems ρ and $n \geq N$. Hence, $\log_{\underline{E}}(x_{(n)})$ is defined for all $n \geq N$. Let $\xi := (\mathrm{T}_0\varphi_a)^{N+1}\log_{\underline{E}}(x_{(N)})$. Because

  $$\begin{aligned}(\mathrm{T}_0\varphi_a)^{(n-1)+1}\log_{\underline{E}}(x_{(n-1)}) &= (\mathrm{T}_0\varphi_a)^{(n-1)+1}\left(\log_{\underline{E}}\left(\varphi_a(x_{(n)})\right)\right) \\ &= (\mathrm{T}_0\varphi_a)^{n+1}\left(\log_{\underline{E}}(x_{(n)})\right) \quad \text{for all } n \geq N,\end{aligned}$$

  we find that $\xi = (\mathrm{T}_0\varphi_a)^{n+1}\log_{\underline{E}}(x_{(n)})$ for all $n \geq N$.

- $x_{(n)} = \exp_{\underline{E}}((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi)$ for all n; since

  $$\begin{aligned}\exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right) &= \exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}(\mathrm{T}_0\varphi_a)^{n+1}\left(\log_{\underline{E}}(x_{(n)})\right)\right) \\ &= x_{(n)} \quad \text{for all } n \geq N, \\ \exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right) &= \exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\left((\mathrm{T}_0\varphi_a)^{N+1}\left(\log_{\underline{E}}(x_{(N)})\right)\right)\right) \\ &= \varphi_a^{N-n}\left(\exp_{\underline{E}}\left(\log_{\underline{E}}(x_{(N)})\right)\right) \\ &= x_{(n)} \quad \text{for all } n < N.\end{aligned}$$

- $\exp_{\underline{E}}(\xi) = x$; since

$$\begin{aligned}\exp_{\underline{E}}(\xi) &= \exp_{\underline{E}}\left((\mathrm{T}_0\varphi_a)^{N+1}\left(\log_{\underline{E}}(x_{(N)})\right)\right)\\ &= \varphi_a^{N+1}\left(\exp_{\underline{E}}\left(\log_{\underline{E}}(x_{(N)})\right)\right)\\ &= x.\end{aligned}$$

- "Vieta formula": $\lim_{n\to\infty} ||\mathrm{T}_0\rho(\xi - (\mathrm{T}_0\varphi_a)^{n+1}x_{(n)})|| = 0$ for all coordinate systems ρ.

  Since $(x_{(n)})_{n=0}^{\infty}$ is convergent there is an $N > 0$ so that for all $n \geq N$ we have $\|\rho(x_{(n)})\| < \epsilon$ and $y_{(n)} := \log_{\underline{E}}(x_{(n)})$ is defined. As above, we have $\xi = (\mathrm{T}_0\varphi_a)^{n+1}y_{(n)}$ for $n \gg 0$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} ||\mathrm{T}_0\rho\left(\xi - (\mathrm{T}_0\varphi_a)^{n+1}x_{(n)}\right)|| &= \lim_{n\to\infty} ||\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{n+1}\left(y_{(n)} - \exp_{\underline{E}}(y_{(n)})\right)\right)||\\ &= \lim_{n\to\infty} ||\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{n+1}\right)\sum_{i\geq 1} e_{(i)}\mathrm{T}_0\rho(y_{(n)})^{(i)}||\\ &= \lim_{n\to\infty} ||\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{n+1}\right)\\ &\qquad \cdot \sum_{i\geq 1} e_{(i)}\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{-(n+1)}\xi\right)^{(i)}||\\ &\leq \lim_{n\to\infty} ||\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{(1-q)(n+1)}\right)\sum_{i\geq 1} e_{(i)}\mathrm{T}_0\rho(\xi)^{(i)}||\\ &\leq \lim_{n\to\infty} ||\mathrm{T}_0\rho\left((\mathrm{T}_0\varphi_a)^{(1-q)(n+1)}\right)\mathrm{Exp}_{\underline{E}}\left(\mathrm{T}_0\rho(\xi)\right)||\\ &\leq \lim_{n\to\infty} ||(\mathrm{T}_0\varphi_a)^{(1-q)(n+1)}\rho(x)||\\ &= 0.\end{aligned}$$ □

As already mentioned, we want to pass from k-valued points of E to elements of $\mathcal{M}^*(E)_a$ via the "switcheroo", which makes use of the isomorphism (4.2).

Lemma 4.1.22 (The switcheroo [ABP, §1.7.3])**.** *Let* $(\mathrm{M}, \sigma_{\mathrm{M}})$ *be a dual Anderson A-motive and a a non-constant element in A. Further,* σ_{M} *induces a* ς^**-linear map* $\sigma : \mathrm{M} \to \mathrm{M}$ *and a* $\varsigma_{A/a}^*$*-linear map* $\sigma : \mathrm{M}_a \to \mathrm{M}_a$*. We then define the groups:*

$$\begin{aligned}G_1 := G_1(a, \mathrm{M}) &:= \{(x, y, z) \in \mathrm{M}\times\mathrm{M}\times\mathrm{M} \mid x = ay + (\sigma - 1)z\}\,,\\ G_2 := G_2(a, \mathrm{M}) &:= \{(x, y) \in \mathrm{M}\times(\mathrm{M}/(\sigma-1)\mathrm{M}) \mid x \equiv ay \bmod (\sigma-1)\mathrm{M}\}\,,\\ G_3 := G_3(a, \mathrm{M}) &:= \{(x, z) \in \mathrm{M}\times(\mathrm{M}/a\mathrm{M}) \mid x \equiv (\sigma-1)z \bmod a\mathrm{M}\}\,,\end{aligned}$$

The two sequences

$$0 \longrightarrow \mathrm{M} \xrightarrow{\mathrm{m}\mapsto(0,(\sigma-1)\mathrm{m},-a\mathrm{m})} G_1 \xrightarrow{(x,y,z)\mapsto(x,y \bmod (\sigma-1))} G_2 \longrightarrow 0$$

$$0 \longrightarrow \mathrm{M} \xrightarrow{\mathrm{m}\mapsto(0,(\sigma-1)\mathrm{m},-a\mathrm{m})} G_1 \xrightarrow{(x,y,z)\mapsto(x,z \bmod a)} G_3 \longrightarrow 0$$

are exact since M *is free over* A_k *and* $k[\sigma]$ *and a is central in* $A_k[\sigma]$*. Therefore,* G_2 *and* G_3 *are isomorphic to the same quotient of* G_1*, and hence canonically isomorphic to each other.*

In order to find the solution corresponding to an a-division tower $(x_{(n)})_{n=0}^{\infty}$ above $x \in E(k) \cong \mathrm{Mat}_{d\times 1}(k)$, we define the *Anderson generating function*

$$f := \sum_{n=0}^{\infty} x_{(n)}^{\mathrm{tr}} a^n \in \mathrm{Mat}_{1\times d}(A \otimes_{\mathbb{F}_q[t]} k[\![t]\!]).$$

The concept of rigid analytic trivializations corresponds to the theory of scattering matrices for (non-zero) pure uniformizable Anderson A-modules of rank r over k (with an additional "σ-structure") that was studied by Anderson when $A = \mathbb{F}_q[t]$ in [And86, §3]. That is, if $\lambda_1, \ldots, \lambda_r$ is a basis of the period lattice of $\underline{E}$ and $\tilde{\Phi}$ represents $\tau_{\mathcal{M}_*(E)}$ with respect to a $k[t]$-basis $\{\mathrm{m}_1, \ldots, \mathrm{m}_r\}$ of $\mathcal{M}_*(E)$, the scattering matrix $\tilde{\Psi}$ is defined by

$$\tilde{\Psi}_{ij} := -\sum_{k=0}^{\infty} \exp_{\underline{E}}\left(\frac{\lambda_j}{t^{k+1}}\right)^{(i-1)} t^k, \quad 1 \leq i, j \leq r, \tag{4.3}$$

such that then

$$\tilde{\Psi}^{(1)} = \tilde{\Phi}\tilde{\Psi}$$

holds. Furthermore, [And86, Thm. 5] states that giving such an Anderson A-module is equivalent to defining a scattering matrix $\tilde{\Psi}$.

The next proof shows that Anderson generating functions play a similar role for defining a rigid analytic trivialization in the dual setting. We will see this in more detail in Example 4.1.25.

Theorem 4.1.23. *Let* $(\mathrm{M}, \sigma_{\mathrm{M}})$ *be the dual Anderson A-motive associated with* (E, φ) *and* $\sigma : \mathrm{M}_a \to \mathrm{M}_a$ *the* $\varsigma_{A(a)}^*$*-linear map induced by* σ_{M}. *Choose a coordinate system* ρ *for* E *and an* $x = x_{(-1)} \in \mathrm{Mat}_{1\times d}(k) \cong E(k)$. *Further, let* $\mathbf{n} \in \mathrm{Mat}_{d\times 1}(k[\sigma])$ *be a vector whose entries form a* $k[\sigma]$*-basis for* M *such that* $\mathbf{n}$ *is compatible with* ρ *and* $\mathrm{Mat}_{1\times d}(k[\sigma]) \xrightarrow{\cdot\mathbf{n}} \mathrm{M}$.

(i) The a-division towers $(x_{(n)})_{n=0}^{\infty}$ *above* $x \in \mathrm{Mat}_{1\times d}(k) \cong E(k)$ *and solutions* $\gamma \in \mathrm{M}_a$ *of the* $(\sigma - 1)$-division equation $x^{\mathrm{tr}}\mathbf{n} = (\sigma - 1)\gamma$ *are canonically in bijection via the "switcheroo" of the preceding lemma.*

(ii) If $\gamma \in \mathrm{M}_a$ *satisfies* $x^{\mathrm{tr}}\mathbf{n} = (\sigma - 1)\gamma$ *and* $(x_{(n)})_{n=0}^{\infty}$ *is the a-division tower canonically corresponding to* γ*, then the following are equivalent:*

  (a) γ *is convergent,*

  (b) $f^{(\nu)} := \sum_{n=0}^{\infty} (x_{(n)}^{\mathrm{tr}})^{(\nu)} a^n \in \mathrm{Mat}_{1\times d}(A \otimes_{\mathbb{F}_q[t]} k\langle \frac{t}{\theta} \rangle) \subseteq \mathrm{Mat}_{1\times d}(A(1))$ *for* $\nu = 1, 2, \ldots,$

  (c) $(x_{(n)})_{n=0}^{\infty}$ *is convergent.*

Proof. Using the ring homomorphism $i^* : \mathbb{F}_q[t] \to A$, $t \mapsto a$, and the coordinate system ρ, we may assume, without loss of generality, that $A = \mathbb{F}_q[t]$ and $E = \mathbb{G}_{a,k}$. We write

$$\varphi_t = \sum_{i=0}^{\infty} \alpha_{(i)} \tau^i,$$

where $\alpha_{(i)} \in \mathrm{Mat}_{d\times d}(k), \alpha_{(i)} = 0$ for $i \gg 0$, and fix a basis $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(\mathrm{M})$ so that the map $\mathrm{Mat}_{1\times r}(k[t]) \xrightarrow{\cdot\mathbf{m}} \mathrm{M}$ is bijective and σ_{M} in M is represented by

$$\Phi_{\mathbf{m}} = \sum_{j=0}^{\infty} \beta_{(j)} t^j$$

with $\beta_{(j)} \in \mathrm{Mat}_{r\times r}(k), \beta_{(j)} = 0$ for $j \gg 0$. Let us first prove (i).
"$\Rightarrow$" A t-division tower $(x_{(n)})_{n=0}^{\infty}$ above x satisfies $(x_{(n-1)}^{\mathrm{tr}}\mathbf{n}, x_{(n)}^{\mathrm{tr}}\mathbf{n}) \in G_2(t, \mathrm{M})$ since

$$\delta(x_{(n-1)}^{\mathrm{tr}}) = x_{(n-1)} = \varphi_t(x_{(n)}) = \varphi_t(\delta(x_{(n)}^{\mathrm{tr}})) \overset{\text{Lem. 1.1.21}}{=} \delta(x_{(n)}^{\mathrm{tr}}\varphi_t^{\dagger})$$

in $E(k) \cong \mathrm{M}/(\sigma-1)\mathrm{M}$ corresponds to

$$x_{(n-1)}^{\mathrm{tr}}\mathbf{n} \equiv t x_{(n)}^{\mathrm{tr}}\mathbf{n} \bmod (\sigma-1)\mathrm{M}.$$

We will at first solve for all $n \geq 0$ the equation

$$x_{(n-1)}^{\mathrm{tr}}\mathbf{n} = t x_{(n)}^{\mathrm{tr}}\mathbf{n} + (\sigma-1)\phi_{(n)}\mathbf{m} \tag{4.4}$$

for $\phi_{(n)} \in \mathrm{Mat}_{1\times r}(k[t])$ so that we obtain the corresponding element $(x_{(n-1)}^{\mathrm{tr}}\mathbf{n}, \phi_{(n)}\mathbf{m}) \in G_3(t, \mathrm{M})$. Secondly, multiplying (4.4) with t^n and summing up provides

$$x^{\mathrm{tr}}\mathbf{n} = x_{(-1)}^{\mathrm{tr}}\mathbf{n} = \sum_{n=0}^{\infty}(t^n x_{(n-1)} - t^{n+1}x_{(n)})^{\mathrm{tr}}\mathbf{n} = (\sigma-1)\sum_{n=0}^{\infty} t^n\phi_{(n)}\mathbf{m}. \tag{4.5}$$

As desired, we have then found the canonically corresponding solution

$$\gamma := \sum_{n=0}^{\infty} t^n \phi_{(n)}\mathbf{m} \in \mathrm{M}_t.$$

In order to solve (4.4), we define $\tilde{\alpha}_{(\nu)} \in \mathrm{Mat}_{d\times r}(k[t]),\ \nu = 1, 2, \ldots,\ \tilde{\alpha}_{(\nu)} = 0$ for $\nu \gg 0$ by requiring that

$$\tilde{\alpha}_{(\nu)}\mathbf{m} = -\sum_{i=0}^{\infty}(\alpha_{(\nu+i)}^{(-i)})^{\mathrm{tr}}\sigma^i\mathbf{n}$$

holds. Consider for all $n \geq 0$ the equations

$$\begin{aligned}
\varphi_t x_{(n)} - x_{(n-1)} &= \varphi_t x_{(n)} - \sum_{j=0}^{\infty}\alpha_{(j)}x_{(n)}^{(j)} \\
&= \sum_{j=1}^{\infty}\alpha_{(j)}x_{(n)}^{(j)}(\tau^j - 1) \\
&= \sum_{j=1}^{\infty}\alpha_{(j)}x_{(n)}^{(j)} \cdot \sum_{l=0}^{j-1}\tau^l(\tau-1) \\
&= \left(\sum_{\nu=1}^{\infty}\sum_{i=0}^{\infty}\alpha_{(\nu+i)}x_{(n)}^{(\nu+i)}\right)\tau^i(\tau-1) \\
&= \left(\sum_{\nu=1}^{\infty}\sum_{i=0}^{\infty}\tau^i\alpha_{(\nu+i)}^{(-i)}x_{(n)}^{(\nu)}\right)(\tau-1) && \mid\cdot\mathbf{n}\ \ \mid^{\dagger} \\
\Leftrightarrow \quad (tx_{(n)} - x_{(n-1)})^{\mathrm{tr}}\mathbf{n} &= (1-\sigma)\sum_{\nu=1}^{\infty}(x_{(n)}^{\mathrm{tr}})^{(\nu)}\tilde{\alpha}_{(\nu)}\mathbf{m} && \mid\cdot t^n \\
\Leftrightarrow \quad (t^{n+1}x_{(n)} - t^n x_{(n-1)})^{\mathrm{tr}}\mathbf{n} &= (1-\sigma)\sum_{\nu=1}^{\infty}(x_{(n)}^{\mathrm{tr}})^{(\nu)}t^n\tilde{\alpha}_{(\nu)}\mathbf{m}
\end{aligned} \tag{4.6}$$

Since multiplication by $t^{n+1} = (\varphi_t^\dagger)^{n+1}$ corresponds to $\mathrm{T}_0\varphi_t$ on the tangent space $\mathrm{T}_0 E \cong \mathrm{M}/\sigma\mathrm{M}$ at the identity, we have

$$\left((\mathrm{T}_0\varphi_t)^{n+1}x_{(n)} - (\mathrm{T}_0\varphi_t)^n x_{(n-1)}\right)^{\mathrm{tr}}\mathbf{n} \equiv (1-\sigma)\sum_{\nu=1}^{\infty}(x_{(n)}^{\mathrm{tr}})^{(\nu)}t^n\tilde{\alpha}_{(\nu)}\mathbf{m} \bmod \sigma\mathrm{M}. \tag{4.7}$$

To get to (4.5) we sum up in (4.6) and obtain

$$x^{\mathrm{tr}}\mathbf{n} = \sum_{n=0}^{\infty}(t^{n+1}x_{(n)} - t^n x_{(n-1)})^{\mathrm{tr}}\mathbf{m} = (1-\sigma)\sum_{\nu=1}^{\infty}f^{(\nu)}\tilde{\alpha}_{(\nu)}\mathbf{m}$$

with $f := \sum_{n=0}^{\infty}(x_{(n)}^{\mathrm{tr}})t^n$. Hence, $\gamma = \sum_{\nu=1}^{\infty}f^{(\nu)}\tilde{\alpha}_{(\nu)}\mathbf{m}$ is the solution of the $(\sigma-1)$-division equation with lefthand side $x^{\mathrm{tr}}\mathbf{n}$. By summing up in equation (4.7), we find that the following congruences hold:

$$\begin{aligned}\left((\mathrm{T}_0\varphi_t)^{n+1}x_{(n)} - x\right)^{\mathrm{tr}}\mathbf{m} &= \sum_{i=0}^{n}\left((\mathrm{T}_0\varphi_t)^{i+1}x_{(i)} - (\mathrm{T}_0\varphi_t)^i x_{(i-1)}\right)^{\mathrm{tr}}\mathbf{m}\\ &\equiv (1-\sigma)\sum_{\nu=1}^{\infty}f_{\le n}^{(\nu)}\tilde{\alpha}_{(\nu)}\mathbf{m} \quad \bmod \sigma\mathrm{M},\end{aligned}$$

with $f_{\le n} := \sum_{i=0}^{n}(x_{(i)}^{\mathrm{tr}})t^i$.

"$\Leftarrow$" Given a solution $\gamma =: y\mathbf{m} = \left(\sum_{n=0}^{\infty}y_{(n)}t^n\right)\mathbf{m} \in \mathrm{M}_t,\ y_{(n)} \in \mathrm{Mat}_{1\times r}(k)$, of the $(\sigma-1)$-division equation $x^{\mathrm{tr}}\mathbf{n} = (\sigma-1)y\mathbf{m}$, we set

$$y_{\le n} := \sum_{i=0}^{n}y_{(i)}t^i$$

so that $(x^{\mathrm{tr}}\mathbf{n}, y_{\le n}\mathbf{m}) \in G_3(t^{n+1},\mathrm{M})$. By solving

$$x^{\mathrm{tr}}\mathbf{n} = t^{n+1}\psi_{(n)}\mathbf{m} + (\sigma-1)y_{\le n}\mathbf{m} \tag{4.8}$$

for $\psi_{(n)} \in \mathrm{Mat}_{1\times d}(k[\sigma])$ we find an element $(x^{\mathrm{tr}}\mathbf{n}, \psi_{(n)}\mathbf{m}) \in G_2(t^{n+1},\mathrm{M})$ for each $n \ge 0$. Set $x_{(n)} := \delta(\psi_{(n)})$. Then $(x_{(n)})_{n=0}^{\infty}$ is a t-division tower above x since $\forall n \ge 1$:

$$\begin{aligned} && x^{\mathrm{tr}}\mathbf{n} &\equiv t^n\psi_{(n-1)}\mathbf{m} \bmod (\sigma-1)\mathrm{M}\\ && &\equiv t^{n+1}\psi_{(n)}\mathbf{m} \bmod (\sigma-1)\mathrm{M}\\ \Leftrightarrow && \psi_{(n-1)}\mathbf{m} &= t\psi_{(n)}\mathbf{m} = \psi_{(n)}\varphi_t^\dagger\mathbf{m} \bmod (\sigma-1)\mathrm{M}\\ \Leftrightarrow && x_{(n-1)} &= \delta(\psi_{(n-1)}) = \varphi_t(\delta(\psi_{(n)})) = \varphi_t(x_{(n)}).\end{aligned}$$

in $E(k)$. It is thus the canonically corresponding t-division tower above x to the given solution $\gamma = y\mathbf{m}$.

In order to solve (4.8) for each $\psi_{(n)} =: \sum_{i\in\mathbb{Z}}c_{(i)}t^i$, $c_{(i)} = 0$ for $i < 0$ and $i \gg 0$, we define coefficients $\tilde{\beta}_{(\nu)} \in \mathrm{Mat}_{d\times r}(k[\tau]),\ \nu = 1,2,\ldots,\ \tilde{\beta}_{(\nu)} = 0$ for $\nu \gg 0$ by requiring that

$$\tilde{\beta}_{(\nu)}\mathbf{m} = -\left(\sum_{i=1}^{\infty}\beta_{(\nu+i)}t^{i-1}\right)\mathbf{m}$$

is satisfied. Further, let $U \in \mathrm{Mat}_{d\times r}(k[t])$ be the unique solution of the equation $\mathbf{n} = U\mathbf{m}$ and n so large it exceeds the degree of t in each entry of U.

$$\begin{aligned}
&t^{-(n+1)}(x^{\mathrm{tr}}\mathbf{m} + (1-\sigma)y_{\leq n}\mathbf{m}) \\
&= t^{-(n+1)}(x^{\mathrm{tr}}U + y_{\leq n} - y_{\leq n}\Phi)\mathbf{m} \\
&= t^{-(n+1)}(x^{\mathrm{tr}}U + y_{\leq n} - \sum_{\nu=0}^{n}\sum_{j=0}^{\infty} y_{(n)}^{(-1)}\beta_{(j)}t^{\nu+j})\mathbf{m} \\
&= t^{-(n+1)}(x^{\mathrm{tr}}U + \sum_{n=0}^{\infty} y_{(n)}t^n - \sum_{\nu=0}^{n}\sum_{j=0}^{\infty} y_{(n-\nu)}^{(-1)}\beta_{(j)}t^{n-\nu+j})\mathbf{m} \\
&= \underbrace{t^{-(n+1)}(x^{\mathrm{tr}}U + \sum_{n=0}^{\infty} y_{(n)}t^n - \sum_{\nu=0}^{n}\sum_{n-\nu+j\leq n} y_{(n-\nu)}^{(-1)}\beta_{(j)}t^{n-\nu+j})\mathbf{m}}_{=0 \text{ since } c_{(i)}=0 \text{ for } i<0} \\
&\quad -\sum_{\nu=0}^{n}\sum_{n-\nu+j>n} y_{(n-\nu)}^{(-1)}\beta_{(j)}t^{j-\nu-1}\mathbf{m} \\
&= -\left(\sum_{\nu=0}^{n}\sum_{j>\nu} y_{(n-\nu)}^{(-1)}\beta_{(j)}t^{j-\nu-1}\right)\mathbf{m} \\
&= -\left(\sum_{\nu=0}^{n} y_{(n-\nu)}^{(-1)}(\sum_{i=1}^{\infty}\beta_{(\nu+i)}t^{i-1})\right)\mathbf{m} \\
&= \left(\sum_{\nu=0}^{n} y_{(n-\nu)}^{(-1)}\tilde{\beta}^{\dagger}_{(\nu)}\right)\mathbf{m}
\end{aligned}$$

Thus we may set $\psi_{(n)} := \sum_{\nu=0}^{n} y_{(n-\nu)}^{(-1)}\tilde{\beta}^{\dagger}_{(\nu)}$ and $x_{(n)} := \delta(\psi_{(n)})$ so that the t-division tower $(x_{(n)})_{n=0}^{\infty}$ above x satisfies the relations

$$x_{(n)} = \delta\left(\sum_{\nu=0}^{n} y_{(n-\nu)}^{(-1)}\tilde{\beta}^{\dagger}_{(\nu)}\right) = \sum_{\nu=0}^{n}\tilde{\beta}_{(\nu)}\left((y_{(n-\nu)}^{(-1)})^{\mathrm{tr}}\right)$$

for all $n \gg 0$. This bijective correspondence is clearly independent of the choice of $\mathbf{m}$.

The proof of (ii) proceeds by showing (a)⇒(c)⇒(b)⇒(a) with the help of (i). Suppose that $\gamma \in \mathrm{M}_t$ is convergent and let $y = \sum_{n=0}^{\infty} y_{(n)}t^n \in \mathrm{Mat}_{1\times r}(k[\![t]\!])$ with $y_{(n)} \in \mathrm{Mat}_{1\times r}(k)$, be the unique solution of $\gamma = y\mathbf{m}$. Then $\lim_{n\to\infty}||y_{(n)}|| = 0$ and by (i) there is an $N \in \mathbb{N}$ such that the corresponding t-division tower $(x_{(n)})_{n=0}^{\infty}$ is given by

$$x_{(n)} = \sum_{\nu=0}^{n}\tilde{\beta}_{(\nu)}\left((y_{(n-\nu)}^{(-1)})^{\mathrm{tr}}\right)$$

for all $n > N$ and satisfies

$$\lim_{n\to\infty}||x_{(n)}|| = 0.$$

This means $(x_{(n)})_{n=0}^{\infty}$ is convergent as desired.

Because $(\mathrm{T}_0\varphi_t - \theta)^d = 0$ holds on T_0E, we can write $\mathrm{T}_0\varphi_t = \theta\cdot\mathbf{1}_d + N$ for some nilpotent matrix $N \in \mathrm{Mat}_{d\times d}(k)$, where $\mathbf{1}_d \in \mathrm{Mat}_{d\times d}(k)$ denotes the identity matrix. By the "Vieta formula" we have

$$\lim_{n\to\infty} ||\xi - (\mathrm{T}_0\varphi_t)^{n+1}x_{(n)}|| = \lim_{n\to\infty} ||\xi - \theta^{n+1}x_{(n)}|| = 0,$$

and thus $\sup_{n=0}^{\infty} |\theta|^{n+1}||x_{(n)}|| < \infty$. For $\nu > 0$ we see that also $\sup_{n=0}^{\infty} |\theta|^{q(n+1)}||x_{(n)}^{(\nu)}|| < \infty$ and further

$$\lim_{n\to\infty} |\theta|^n||x_{(n)}^{(\nu)}|| = \lim_{n\to\infty} |\theta|^{(1-q)n-1} \cdot \left(|\theta|^{q(n+1)}||x_{(n)}^{(\nu)}||\right) = 0.$$

We thus conclude that the $(q^\nu)^{\mathrm{th}}$-power of the Anderson generating function

$$f^{(\nu)} = \sum_{n=0}^{\infty}(x_{(n)}^{\mathrm{tr}})^{(\nu)}t^n \in \mathrm{Mat}_{1\times d}(k[\![t]\!])$$

assigned to $(x_{(n)})_{n=0}^{\infty}$ lies in $\mathrm{Mat}_{1\times d}(k\langle \frac{t}{\theta}\rangle)$. By (i) we find that the corresponding solution is $\gamma = y\mathbf{m}$ with

$$y = \sum_{\nu=1}^{\infty} f^{(\nu)}\tilde{\alpha}_{(\nu)} \in \mathrm{Mat}_{1\times r}(k\langle \frac{t}{\theta}\rangle) \subseteq \mathrm{Mat}_{1\times r}(k\langle t\rangle),$$

which finishes the proof. □

Corollary 4.1.24. *The A-module* $\mathrm{M}(1)^\sigma$ *is isomorphic to the period lattice* $\Lambda_{\underline{E}} = \ker \exp_{\underline{E}}$.

Proof. As before, we write $\sigma : \mathrm{M}_a \to \mathrm{M}_a$ for the $\varsigma_{A/(a)}^*$-linear map induced by σ_{M}. The convergent a-division towers above 0 are in bijective correspondence with periods of $\underline{E}$ by the logarithm construction of Proposition 4.1.21. Moreover according to Theorem 4.1.23, the M-cycles, that is, the convergent solutions of the $(\sigma - 1)$-division equation, correspond bijectively to the convergent a-division towers above 0. □

Recall any Drinfeld A-module is uniformizable (Theorem 4.1.18). We want to take a closer look at the above correspondences and investigate the periods of a Drinfeld $\mathbb{F}_q[t]$-module (E,φ) of rank r over k and the σ-invariants of its associated Drinfeld $\mathbb{F}_q[t]$-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$. Through Anderson generating functions, we find a rigid analytic trivialization Ψ of the matrix Φ that represents σ_{M} with respect to a fixed $k[t]$-basis for M. In Section 5.1 we further investigate the linear independence of the entries of $\Psi^{-1}(\theta)$ for Drinfeld $\mathbb{F}_q[t]$-motives of rank 2 over $\mathbb{C}_\infty$.

Example 4.1.25 (Cf. [Pel08])**.** Let $A = \mathbb{F}_q[t]$, (E,φ) be a Drinfeld $\mathbb{F}_q[t]$-module of rank r over k, $(\mathcal{M}^*(E), \sigma_{\mathcal{M}^*(E)})$ the corresponding dual Drinfeld $\mathbb{F}_q[t]$-motive of rank r over k and $(\mathcal{M}_*(E), \tau_{\mathcal{M}_*(E)})$ the associated Drinfeld $\mathbb{F}_q[t]$-motive of rank r over k. We thus have

$$\varphi_t = \theta + \alpha_1\tau + \ldots + \alpha_r\tau^r \in k[\tau] \cong \mathrm{End}_{k,\mathbb{F}_q}(E),$$

by choosing a coordinate system ρ. Then

$$\Phi_{\mathbf{m}^*} := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\alpha_r^{(-r)} & -\alpha_1^{(-1)}/\alpha_r^{(-r)} & \cdots & -\alpha_{r-1}^{(r-1)}/\alpha_r^{(-r)} \end{pmatrix}$$

represents $\sigma_{\mathcal{M}^*(E)}$ with respect to the $k[t]$-basis $\mathbf{m}^* = (1, \sigma, \ldots, \sigma^{r-1})^{\mathrm{tr}}$ for $\mathcal{M}^*(E)$, and

$$\tilde{\Phi}_{\mathbf{m}_*} := \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\theta)/\alpha_r & -\alpha_1/\alpha_r & \cdots & -\alpha_{r-1}/\alpha_r \end{pmatrix}$$

represents $\tau_{\mathcal{M}_*(E)}$ with respect to the $k[t]$-basis $\mathbf{m}_* = (1, \tau, \ldots, \tau^{r-1})^{\mathrm{tr}}$ for $\mathcal{M}_*(E)$. For a period $\lambda \in \Lambda_{\underline{E}}$ the corresponding convergent t-division tower is $\left(\exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right)\right)_{n=0}^{\infty}$, which defines the corresponding Anderson generating function

$$f_\lambda(t) := \sum_{n=0}^{\infty} \exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right) t^n \in \mathbb{T} := k\langle t\rangle.$$

Since

$$\begin{aligned} \exp_{\underline{E}}\left(\frac{\lambda}{\theta^n}\right) &= \varphi_t\left(\exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right)\right) \\ &= \theta \cdot \exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right) + \alpha_1 \cdot \exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right)^{(1)} + \ldots + \alpha_r \cdot \exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right)^{(r)}, \end{aligned}$$

multiplying by t^n and summing up we get

$$\begin{aligned} \theta f_\lambda(t) + \alpha_1 f_\lambda^{(1)}(t) + \ldots + \alpha_r f_\lambda^{(r)}(t) &= \sum_{n=0}^{\infty} \exp_{\underline{E}}\left(\frac{\lambda}{\theta^n}\right) t^n = \exp_{\underline{E}}(\lambda) + t f_\lambda(t) \\ &= t f_\lambda(t). \end{aligned} \tag{4.9}$$

We put $\mathbf{n} := (1)$ so that $\mathcal{M}^*(E) \overset{\cdot \mathbf{n}}{\cong} k[\sigma]$. The corresponding solution $\gamma \in \mathcal{M}^*(E)_t$ of $(\sigma - 1)\gamma = 0$ is then

$$\gamma = f_\lambda^{(1)} \tilde{\alpha}_{(1)} \mathbf{m}^* + \ldots + f_\lambda^{(r)} \tilde{\alpha}_{(r)} \mathbf{m}^*$$

with

$$\begin{aligned} \tilde{\alpha}_{(1)}\mathbf{m}^* &= -(\alpha_1 + \alpha_2^{(-1)}\sigma + \ldots + \alpha_r^{(-(r-1))}\sigma^{r-1})\mathbf{n}, \\ \tilde{\alpha}_{(2)}\mathbf{m}^* &= -(\alpha_2 + \alpha_3^{(-1)}\sigma + \ldots + \alpha_r^{(-(r-2))}\sigma^{r-2})\mathbf{n}, \\ &\vdots \\ \tilde{\alpha}_{(r)}\mathbf{m}^* &= -\alpha_r \mathbf{n}. \end{aligned}$$

We have

$$\begin{aligned} \gamma &= -(\alpha_1 f_\lambda^{(1)} + \ldots + \alpha_r f_\lambda^{(r)} \quad \alpha_2^{(-1)} f_\lambda^{(1)} + \ldots + \alpha_{r-1}^{(-1)} f_\lambda^{(r-1)} \quad \cdots \quad \alpha_r^{(-(r-1))} f_\lambda^{(1)})\mathbf{m}^* \\ &= -(f_\lambda^{(1)} \quad \cdots \quad f_\lambda^{(r)}) \cdot A^{(1)} \cdot \mathbf{m}^* \end{aligned}$$

where

$$A := \begin{pmatrix} \alpha_1^{(-1)} & \alpha_2^{(-2)} & \cdots & \alpha_r^{(-r)} \\ \vdots & \vdots & & 0 \\ \alpha_{r-1}^{(-1)} & \alpha_r^{(-2)} & & \vdots \\ \alpha_r^{(-1)} & 0 & \cdots & 0. \end{pmatrix} \in \mathrm{GL}_r(k)$$

Let $\lambda_1, \ldots, \lambda_r$ be a $\mathbb{F}_q[t]$-basis for $\Lambda_{\underline{E}}$. We write $f_i := f_{\lambda_i}$, $i = 1, \ldots, r$ and γ_i, $i = 1, \ldots, r$, for the corresponding solutions. From the linear independence of the set $\{\lambda_1, \ldots, \lambda_r\}$ it follows that $\{f_1, \ldots, f_r\}$ is linearly independent over $\mathbb{F}_q$ and that

$$\tilde{\Psi} := -\begin{pmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(r-1)} \\ \vdots & \vdots & & \vdots \\ f_r & f_r^{(1)} & \cdots & f_r^{(r-1)} \end{pmatrix} \in \mathrm{Mat}_{r\times r}(\mathbb{T})$$

is invertible by [Gos96, Lem. 1.3.3]. We set $\Theta := \tilde{\Psi}^{(1)} \cdot A^{(1)} \in \mathrm{GL}_r(\mathbb{T})$ so that the vector

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_r \end{pmatrix} = \Theta \cdot \mathbf{m}^*$$

comprises an $\mathbb{F}_q[t]$-basis for $\mathcal{M}^*(E)(1)^\sigma$. In comparison with (4.3), we find that $\tilde{\Psi}^{\mathrm{tr}}$ is the scattering matrix associated with $\underline{E}$. Indeed,

$$\tilde{\Psi} \cdot \tilde{\Phi}_{\mathsf{m}_*}^{\mathrm{tr}} = -\begin{pmatrix} f_1^{(1)} & f_1^{(2)} & \cdots & \left((t-\theta)f_1 - \alpha_1 f_1^{(1)} - \ldots - \alpha_{r-1} f_1^{(r-1)}\right)/\alpha_r \\ \vdots & \vdots & & \vdots \\ f_r^{(1)} & f_r^{(2)} & \cdots & \left((t-\theta)f_r - \alpha_1 f_r^{(1)} - \ldots - \alpha_{r-1} f_r^{(r-1)}\right)/\alpha_r \end{pmatrix} \overset{(4.9)}{=} \tilde{\Psi}^{(1)} \tag{4.10}$$

implies that $\tilde{\Phi}_{\mathsf{m}_*} \cdot \tilde{\Psi}^{\mathrm{tr}} = (\tilde{\Psi}^{\mathrm{tr}})^{(1)}$ holds. Moreover,

$$\tilde{\Phi}_{\mathsf{m}_*}^{\mathrm{tr}} \cdot A^{(1)} = \begin{pmatrix} (t-\theta) & 0 & \cdots & 0 \\ 0 & \alpha_2^{(-1)} & \cdots & \alpha_r^{(-(r-1))} \\ \vdots & \vdots & & \\ 0 & \alpha_r^{(-1)} & & 0 \end{pmatrix} = A \cdot \Phi_{\mathbf{m}^*} \tag{4.11}$$

so that

$$\Theta = \tilde{\Psi}^{(1)} \cdot A^{(1)} \overset{(4.10)}{=} \tilde{\Psi} \cdot \tilde{\Phi}_{\mathsf{m}_*}^{\mathrm{tr}} \cdot A^{(1)} \overset{(4.11)}{=} \tilde{\Psi} \cdot A \cdot \Phi_{\mathbf{m}^*} = \Theta^{(-1)} \cdot \Phi_{\mathbf{m}^*}$$

and $\Psi := \Theta^{-1} \in \mathrm{GL}_r(\mathbb{T})$ is hence a rigid analytic trivialization of $\Phi_{\mathbf{m}^*}$, meaning

$$\Psi^{(-1)} = \Phi_{\mathbf{m}^*} \Psi.$$

From [ABP04, Prop. 3.1.3] we may further deduce that $\Psi \in \mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_{r\times r}(\mathbb{E})$.

Proposition 4.1.26 (Cf. [And86, Thm. 4])**.** *Let $A = \mathbb{F}_q[t]$, $\underline{E} = (E, \varphi)$ be an Anderson A-module of rank r and dimension d over $\mathbb{C}_\infty$ and $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ the corresponding dual Anderson A-motive.*

(i) $\mathrm{rank}_A \mathrm{M}(1)^\sigma = \mathrm{rank}_A \Lambda_{\underline{E}} = r$,

(ii) $\mathcal{M}^*(\underline{E})$ is rigid analytically trivial,

(iii) $\underline{E}$ is uniformizable.

Proof. For the same reasons as above, we may assume without loss of generality that $E = \mathbb{G}_{a,k}^d$ and $A = \mathbb{F}_q[t]$. We first prove (i)$\Leftrightarrow$(ii) as in Lemma 2.6.1 and next (ii)$\Rightarrow$(iii)$\Rightarrow$(i).

We show that if $\mu_1, \dots, \mu_m$ are linearly independent over A, then they are linearly independent over $A(1)$ in $\mathrm{M}(1)$. This implies that the natural map $\phi : \mathrm{M}(1)^\sigma \otimes_Q Q(1) \to \mathrm{M}(1)$ is injective, which shows $\mathrm{rank}_A \mathrm{M}(1)^\sigma \leq \mathrm{rank}_{A(1)} \mathrm{M}(1) = \mathrm{rank}_{A_k} \mathrm{M} = r$. Clearly equality holds if and only if ϕ is also surjective; that is, if and only if $\underline{\mathrm{M}}$ is rigid analytically trivial. For the sake of contradiction, we assume that $m \geq 2$ is minimal such that $\mu_1, \dots, \mu_m$ are linearly independent over A, but $\mu_1, \dots, \mu_m$ are not linearly independent over $A(1)$ in $\mathrm{M}(1)$. This means, there are $\alpha_i \in A(1)$ such that

$$\sum_{i=1}^{m} \alpha_i \mu_i = 0.$$

Without loss of generality, we suppose $\alpha_1 = 1$. Since $\sigma_{\mathrm{M}}(\varsigma^* \mathrm{M})$ is a cotorsion A_k-submodule of the locally free A_k-module M, we also have

$$\sum_{i=1}^{m} \alpha_i^{(-1)} \mu_i = 0$$

and thus $\sum_{i=1}^m (\alpha_i - \alpha_i^{(-1)}) \mu_i = \sum_{i=2}^m (\alpha_i - \alpha_i^{(-1)}) \mu_i = 0$. Hence, $\alpha_i \in A$, which contradicts the assumption and proves the injectivity of ϕ, as desired.

To show that (ii) implies (iii), suppose $x \in \mathrm{Mat}_{d\times 1}(\mathbb{C}_\infty) \cong E(\mathbb{C}_\infty)$. By assumption, there is a rigid analytic trivialization $\Psi \in \mathrm{GL}_r(\mathbb{C}_\infty\langle t\rangle)$ such that $\Psi^{(-1)} = \Phi_{\mathbf{m}} \Psi$. Define $a_{(n)} \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty)$ by the rule

$$\sum_{n=0}^{\infty} a_{(n)} t^n = x^{\mathrm{tr}} U \Psi.$$

Necessarily we have $\lim_{n\to\infty} ||a_{(n)}|| = 0$ so that there is an $N \in \mathbb{N}$ such that

$$||a_{(n)}|| < \frac{1}{2} \quad \text{for all } n > N.$$

Since $\mathbb{C}_\infty$ is algebraically closed, we can find an $b_{(n)}$ which satisfies

$$a_{(n)} = b_{(n)}^{(-1)} - b_{(n)} \tag{4.12}$$

for $n \leq N$. For $n > N$ we define $b_{(n)} := \sum_{j=1}^\infty a_{(n)}^{(j)}$ which then also satisfies (4.12). Since then

$$\lim_{n\to\infty} ||b_{(n)}|| = 0$$

we can define $Z := \sum_{n=0}^\infty b_{(n)} t^n \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty\langle t\rangle)$ so that

$$x^{\mathrm{tr}} U \Psi = Z^{(-1)} - Z.$$

We write $\Theta := \Psi^{-1}$ as before. Hence $\Theta = \Theta^{(-1)} \Phi$ and

$$x^{\mathrm{tr}} U = (Z\Theta)^{(-1)} \Phi - (Z\Theta).$$

Thus $\gamma := (Z\Theta)\mathbf{m} \in \mathrm{M}_t$ is a convergent solution of the $(\sigma - 1)$-division equation

$$x^{\mathrm{tr}} \mathbf{n} = (\varsigma - 1)\gamma.$$

By Theorem 4.1.23 this corresponds to a convergent division tower $(x_{(n)})_{n=0}^{\infty}$ above x whose logarithm $\xi \in \mathrm{T}_0E$ satisfies $\exp(\xi) = x$.

Suppose now that $\exp_{\underline{E}}$ is surjective and a is a non-constant element of A. From [BH07a, Thm. 8.6], we see that the $A/\mathfrak{a}$-module $\underline{E}[\mathfrak{a}](\mathbb{C}_\infty)$ is of dimension r where $\mathfrak{a} := (a) \subseteq A$. Moreover, $a^{-1}\Lambda_{\underline{E}}/\Lambda_{\underline{E}} \cong \exp_{\underline{E}}(a^{-1}\Lambda_{\underline{E}}) = \underline{E}[\mathfrak{a}](\mathbb{C}_\infty)$. We conclude that

$$\operatorname{rank}_A \mathrm{M}(1)^\sigma \overset{\text{Cor. 4.1.24}}{=} \operatorname{rank}_A \Lambda_{\underline{E}} = \dim_{A/\mathfrak{a}} a^{-1}\Lambda_{\underline{E}}/\Lambda_{\underline{E}} = r,$$

whence the assertion. □

We may now state the following:

Definition 4.1.27. (i) We define the strictly full *subcategory* $\mathscr{PUM}^I \subset \mathscr{M}^I$ *of pure uniformizable Anderson A-modules up to isogeny* by restriction.

(ii) Let $\underline{E}$ be a pure uniformizable Anderson A-module over $\mathbb{C}_\infty$ and $\underline{\mathrm{M}}$ be the corresponding pure rigid analytically trivial dual Anderson A-motive over $\mathbb{C}_\infty$. The *Galois group* $\Gamma_{\underline{E}}$ *of* $\underline{E}$ is the Galois group associated with $\underline{\mathrm{M}}$ by Definition 2.8.2.

Therefore, the functor $\mathcal{E} : \mathscr{DA}_+ \to \mathscr{M}$ induces an equivalence $\mathscr{PRDA}_+^I \to \mathscr{PUM}^I$ of categories, which by abuse of notation we also denote by $\mathcal{E}$. Similarly, we write $\mathcal{M}^*$ for the equivalence $\mathscr{PUM}^I \to \mathscr{PRDA}_+^I$ of categories defined by $\mathcal{M}^* : \mathscr{M} \to \mathscr{DA}_+$.

4.1.3 From Anderson A-modules to Q-Hodge-Pink structures

Recall that the first Betti homology group of an abelian variety over $\mathbb{C}$ carries a rational Hodge structure. Similarly, we will now associate a pure Q-Hodge-Pink structure $\mathcal{H}(\underline{E})$ of weight $-\frac{d}{r}$ with a pure uniformizable Anderson A-module $\underline{E} = (E, \varphi)$ of rank r, dimension d and weight $\frac{d}{r}$ over $k \subset \mathbb{C}_\infty$. In analogy with the classical case, we define the Q-vector space underlying $\mathcal{H}(\underline{E})$ to be $H := H_{\mathrm{B}}(\underline{E}, Q) = \Lambda_{\underline{E}} \otimes_A Q$, where $\Lambda_{\underline{E}}$ is the period lattice of $\underline{E}$. Following Schindler in [Sch09, §5.1], we denote the ideal $(a \otimes 1 - 1 \otimes \gamma(a) | a \in A)$ by J. Let $z = \frac{a}{b}$ be a uniformizing parameter of Q at ∞ and put $\zeta := \gamma(z) = \gamma(a)\gamma(b)^{-1}$. By [Sch09, Lem. 5.1.1], $z - \zeta$ is a uniformizing parameter of $\mathcal{O}_{C_{\mathbb{C}_\infty} V(J)}$ and further

$$\widehat{\mathcal{O}_{C_{\mathbb{C}_\infty} V(J)}} = \mathbb{C}_\infty[\![z - \zeta]\!].$$

We then define a map

$$\begin{aligned} f : H \otimes_Q \mathbb{C}_\infty[\![z - \zeta]\!] \quad &\to \quad \mathrm{T}_0E \\ (\lambda \otimes \beta) \otimes \sum_{i=0}^{\infty} \alpha_i (z - \zeta)^i \quad &\mapsto \quad \sum_{i=0}^{\infty} \beta\alpha_i (\mathrm{T}_0\varphi_a \mathrm{T}_0\varphi_b^{-1} - \gamma(a)\gamma(b)^{-1})^i \lambda \end{aligned}$$

Note f is well-defined because $\underline{E}$ is of dimension d. Therefore $(\mathrm{T}_0\varphi_a - \gamma(a))^d = 0$ holds on T_0E for all $a \in A$. By [And86, Cor. 3.3.6], we find that f is surjective so we have a short exact sequence

$$0 \longrightarrow \mathfrak{q}_H \longrightarrow H \otimes_Q \mathbb{C}_\infty[\![z - \zeta]\!] \xrightarrow{f} \mathrm{T}_0E \longrightarrow 0.$$

with $\mathfrak{q}_H := \ker f$. By construction, $\mathfrak{q}_H$ is a $\mathbb{C}_\infty[\![z-\zeta]\!]$-lattice in $H \otimes_Q \mathbb{C}_\infty(\!(z-\zeta)\!)$. This defines a pure Q-pre-Hodge-Pink structure $\underline{H} = (H, W, \mathfrak{q}_H)$ of weight $-\frac{d}{r}$. Let $\underline{H}_\infty = (H_\infty, W_\infty, \mathfrak{q}_{H_\infty})$ be its associated pure Q_∞-pre-Hodge-Pink structure. Since $\Lambda_{\underline{E}}$ is discrete in T_0E, we have

$$H_\infty \cap \mathfrak{q}_{H_\infty} = (H \otimes_Q Q_\infty) \cap \mathfrak{q}_H = \{0\}.$$

This condition implies that $\underline{H}_\infty$ is semistable and thus $\underline{H}$ a pure Q-Hodge-Pink $\underline{H}$ structure.

Definition 4.1.28. (i) We define $\mathcal{H} : \mathscr{PUM}^I \to \mathscr{Hodge}_Q$ to be the fully faithful functor that sends a pure uniformizable Anderson A-module $\underline{E}$ of rank r, dimension d and weight $\frac{d}{r}$ to the pure Q-Hodge-Pink structure $\underline{H} = (H, W, \mathfrak{q}_H)$ of rank r and weight $-\frac{d}{r}$ as defined above.

(ii) Let $\underline{E}$ be a pure uniformizable Anderson A-module over $k \subset \mathbb{C}_\infty$. We call the Hodge-Pink group of $\mathcal{H}(\underline{E})$ the *Hodge-Pink group of* $\underline{E}$ and denote it by $G_{\underline{E}}$.

4.1.4 From dual Anderson A-motives to Q-Hodge-Pink structures

The fully faithful functor $\mathcal{H} \circ \mathcal{E}$ associates a pure Q-Hodge-Pink structure of weight $-\frac{d}{r}$ with a pure rigid analytically trivial dual Anderson A-motive of rank r, dimension d and weight $\frac{l}{n}$ over $\mathbb{C}_\infty$ with $d, r > 0$. Following unpublished ideas of Pink in the non-dual setting, we construct a "direct" functor $\mathcal{D} : \mathscr{PRDA}_+^I \to \mathscr{Hodge}_Q$, which is isomorphic to $\mathcal{H} \circ \mathcal{E} : \mathscr{PRDA}_+^I \to \mathscr{Hodge}_Q$. This will later make it possible to assign a dual Anderson sub-A-motive $\underline{\mathrm{M}}'$ over $\mathbb{C}_\infty$ to a strict sub-Q-Hodge-Pink structure $\underline{H}'$ of $\mathcal{D}(\underline{\mathrm{M}})$ such that $\mathcal{D}(\underline{\mathrm{M}}') = \underline{H}'$.

Fix such a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r, dimension d and weight $\frac{l}{n}$ over $\mathbb{C}_\infty$ in $\mathscr{PRDA}_+^I$. We want to study its relations to the pure Q-Hodge Pink structure $(H, W, \mathfrak{q}_H) := \mathcal{H}(\mathcal{E}(\underline{\mathrm{M}}))$ of rank r and weight $-\frac{d}{r}$, which we have associated in the previous section with the pure uniformizable Anderson A-module $\underline{E} := \mathcal{E}(\underline{\mathrm{M}})$ of weight $\frac{d}{r}$. By definition of the functor $\mathcal{H}$ and Corollary 4.1.24, we have

$$H = \Lambda_{\underline{E}} \otimes_A Q \cong \mathrm{M}(1)^\sigma \otimes_A Q.$$

We want to define the $\mathbb{C}_\infty[\![z-\zeta]\!]$-lattices $\mathfrak{q}_H$ in terms of $(\mathrm{M}, \sigma_{\mathrm{M}})$. Consider the two short exact sequences

$$0 \longrightarrow \mathfrak{q}_H \longrightarrow \mathfrak{p}_H \longrightarrow \mathrm{T}_0 E \longrightarrow 0$$

(see Subsection 4.1.3), and

$$0 \to \sigma_{\mathrm{M}}(\varsigma^*\mathrm{M}) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \to \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \to \operatorname{coker} \sigma_{\mathrm{M}} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \to 0.$$

In order to prove that the two sequences are isomorphic, we state two lemmas on the relations between $(\mathrm{M}(\infty), \sigma_{\mathrm{M}(\infty)})$ and $(\mathrm{M}(1)^\sigma \otimes_A A(\infty), \sigma_{\mathrm{M}(1)^\sigma \otimes_A A(\infty)})$ at $t = \theta^{q^i}$ for all $i \in \mathbb{Z}$, where $\sigma_{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}$ is the natural isomorphism

$$\varsigma^*_{A(\infty)}(\mathrm{M}(1)^\sigma \otimes_A A(\infty)) \xrightarrow{\sim} \mathrm{M}(1)^\sigma \otimes_A A(\infty).$$

We note first that $\varsigma^*_{A(\infty)} : A(\infty) \to A(\infty)$ induces ring homomorphisms

$$\varsigma^*_{\mathbb{C}_\infty[\![t-\theta]\!]} : \mathbb{C}_\infty[\![t-\theta]\!] \to \mathbb{C}_\infty[\![t-\theta^{q^{-1}}]\!] \quad \text{and} \quad \varsigma^*_{\mathbb{C}_\infty[\![t-\theta^q]\!]} : \mathbb{C}_\infty[\![t-\theta^q]\!] \to \mathbb{C}_\infty[\![t-\theta]\!].$$

Lemma 4.1.29 (Cf. [BH07a, Prop. 3.4])**.** *There is a well-defined map*

$$\psi : \begin{pmatrix} \mathrm{M}(\infty) = \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} A(\infty) \\ \sigma_{\mathrm{M}(\infty)} = \sigma_{\mathrm{M}} \otimes \mathrm{id}_{A(\infty)} \end{pmatrix} \to \begin{pmatrix} \mathrm{M}(1)^\sigma \otimes_A A(\infty) \\ \sigma_{\mathrm{M}(1)^\sigma \otimes_A A(\infty)} \end{pmatrix}.$$

Proof. Suppose that $\mathbf{m} := (\mathrm{m}_1, \ldots, \mathrm{m}_r)^{\mathrm{tr}}$ is a $\mathbb{C}_\infty[t]$-basis for M and $\Phi = \sum_{i=0}^{\infty} \phi_{(i)} t^i \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\{t\})$, with $\phi_{(i)} \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty)$, represents $\sigma_{\mathrm{M}(\infty)}$ on $\mathrm{Mat}_{1\times r}(\mathbb{C}_\infty\{t\}) \cong \mathrm{M}(\infty)$. By Proposition 2.5.8 there is a rigid analytic trivialization $\Psi \in \mathrm{GL}_r(\mathbb{C}_\infty\langle t\rangle)$ of Φ such that $\Psi^{(-1)} = \Phi\Psi$ and the vector $\Psi^{-1}\mathbf{m}$ comprises an $\mathbb{F}_q[t]$-basis for $\mathrm{M}(1)^\sigma$.

We write $\Psi = \sum_{n=0}^{\infty} \psi_{(n)} t^n \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\langle t\rangle)$ with $\psi_{(n)} \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty)$. Because $\Psi = \Phi^{(1)}\Psi^{(1)}$, for $n \geq 0$ the n^{th} coefficient is

$$\psi_{(n)} = \phi_{(0)}^{(1)}\psi_{(n)}^{(1)} + g_{(n)} \quad \text{with} \quad g_{(n)} := \sum_{i=1}^{n} \phi_{(i)}^{(1)}\psi_{(n-i)}^{(1)}.$$

Pick $\alpha \in \mathbb{C}_\infty$ such that $|\alpha| > 1$ and consider the affinoid covering $\{\mathrm{Sp}\,\mathbb{C}_\infty\langle \frac{t}{\alpha^j}\rangle\}\}$, $j \in \mathbb{N}$, which is an admissible covering of $\mathbb{A}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}$. Hence, it suffices to show that $\Psi \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\langle \frac{t}{\alpha^j}\rangle)$ to see that $\Psi \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\{t\})$.

Put $\beta := \alpha^{j+1}$ for an arbitrary $j \in \mathbb{N}$. We want to show that $|\psi_{(n)}||\beta|^n$ is bounded since then $\lim_{n\to\infty} |\psi_{(n)}||\alpha^j|^n = 0$, as desired.

Since $\Phi^{(1)} \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\{t\})$ and $\Psi \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\langle t\rangle)$, we have zero sequences

$$\lim_{i\to\infty} |\phi_{(i)}^{(1)}||\beta|^i = 0 \quad \text{and} \quad \lim_{n\to\infty} |\psi_{(n)}| = 0.$$

Hence, there is an $N \in \mathbb{N}$ such that for all $n > N$ and $i \in \{0, \ldots, n\}$

$$|\phi_{(i)}^{(1)}||\beta|^i|\psi_{(n-i)}|^{q-1} \leq \frac{1}{2}.$$

For $i = 0$ we have

$$|\psi_{(n)} - g_{(n)}| = |\phi_{(0)}^{(1)}\psi_{(n)}^{(1)}| \leq \frac{1}{2}|\psi_{(n)}| < |\psi_{(n)}|$$

and thus $|\psi_{(n)}| = |g_{(n)}| \leq \max_{i=1,\ldots,n}\{|\phi_{(i)}^{(1)}\psi_{(n-i)}^{(1)}|\}$. This gives us

$$\begin{aligned} |\psi_{(n)}||\beta|^n &\leq |\beta|^n \max_{i=1,\ldots,n}\{|\phi_{(i)}^{(1)}\psi_{(n-i)}^{(1)}|\} \leq \max_{i=1,\ldots,n}\{|\phi_{(i)}^{(1)}||\beta|^i|\psi_{(n-i)}|^{q-1}|\beta|^{n-i}|\psi_{(n-i)}|\} \\ &\leq \frac{1}{2}\max_{i=1,\ldots,n}\{|\beta|^{n-i}|\psi_{(n-i)}|\}. \end{aligned}$$

Therefore, $\Psi \in \mathrm{GL}_r(\mathbb{C}_\infty\langle t\rangle) \cap \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty\{t\})$ and we may define

$$\psi : \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} A(\infty) \to \mathrm{M}(1)^\sigma \otimes_A A(\infty), \quad \mathrm{m}_i \otimes a \mapsto \Psi^{-1}\mathrm{m}_i \otimes a.$$ □

Lemma 4.1.30. *The cokernel of ψ, defined as above, is supported at $t = \theta^{q^i}$, $i > 0$. In particular,*

$$\mathrm{M}(1)^\sigma \otimes_A \mathbb{C}_\infty\langle \frac{t}{\theta}\rangle \cong \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty\langle \frac{t}{\theta}\rangle \quad \textit{and} \quad \mathrm{M}(1)^\sigma \otimes_A \mathbb{C}_\infty[\![t-\theta]\!] \cong \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!].$$

Proof. Assume to the contrary, that $\mathrm{supp}(\mathrm{coker}(\psi))$ is not equal to $\{\theta^{q^i} \mid i > 0\}$. Note that $\sigma_{\mathrm{M}(1)^\sigma\otimes_A A(\infty)} : \varsigma^*_{A(\infty)}(\mathrm{M}(1)^\sigma \otimes_A A(\infty)) \xrightarrow{\sim} \mathrm{M}(1)^\sigma \otimes_A A(\infty)$ induces a map

$$\varsigma^*_{\mathbb{C}_\infty[\![t-t_0]\!]}\left(\frac{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}{\mathrm{M}(\infty)} \otimes_{A(\infty)} \mathbb{C}_\infty[\![t-t_0]\!]\right) \to \frac{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}{\mathrm{M}(\infty)} \otimes_{A(\infty)} \mathbb{C}_\infty[\![t-t_0]\!],$$

which by abuse of notation we also denote by $\sigma_{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}$. Since the cokernel of $\sigma_{\mathrm{M}(\infty)}$ is supported at $t = \theta$, we have

$$\sigma_{\mathrm{M}(\infty)}\left(\varsigma^*_{A(\infty)}\mathrm{M}(\infty)\right) \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - t_0^{q^{-1}}]\!] \cong \mathrm{M}(\infty) \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - t_0^{q^{-1}}]\!]$$

if $t_0^{q^{-1}} \neq \theta$. This provides

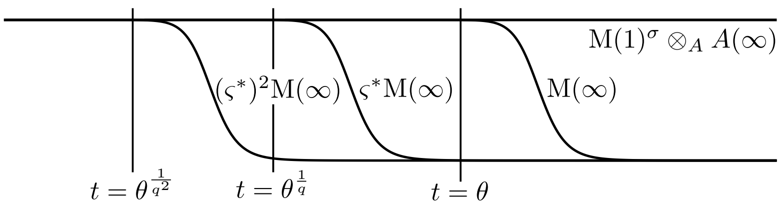
$$\begin{aligned} &\sigma_{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}\left(\varsigma^*_{\mathbb{C}_\infty[\![t-t_0]\!]}\left(\frac{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}{\mathrm{M}(\infty)} \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - t_0]\!]\right)\right) \\ &\quad = \frac{\sigma_{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}\left(\varsigma^*_{A(\infty)}\left(\mathrm{M}(1)^\sigma \otimes_A A(\infty)\right)\right)}{\sigma_{\mathrm{M}(\infty)}\left(\varsigma^*_{A(\infty)}\mathrm{M}(\infty)\right)} \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - t_0^{q^{-1}}]\!] \\ &\quad \cong \frac{\mathrm{M}(1)^\sigma \otimes_A A(\infty)}{\mathrm{M}(\infty)} \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - t_0^{q^{-1}}]\!]. \end{aligned}$$

That is, if $t_0 \in \mathrm{supp}(\mathrm{coker}(\psi))$ and $t_0^{q^{-1}} \neq \theta$, then $t_0^{q^{-1}}$ is also contained in the support of $\mathrm{coker}(\psi))$. We assume that $t_0 \neq \theta^{q^i}$ for $i > 0$ so that we may iterate this argument. This provides $\{t_0^{q^{-i}} \mid i > 0\} \subseteq \mathrm{supp}(\mathrm{coker}(\psi))$. Because M is rigid analytically trivial we have $\mathrm{M}(1)^\sigma \otimes_A A(1) \cong \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} A(1)$ and thus $|t_0| > 1$. Hence, $|t_0|^{q^{-i}} \to 1$ for $i \to \infty$ and 1 is a limit point of $\{t_0^{q^{-i}} \mid i > 0\} \subseteq \mathrm{supp}(\mathrm{coker}(\psi))$. This contradicts that $\mathrm{supp}(\mathrm{coker}(\psi))$ is discrete and proves that $\mathrm{coker}(\psi)$ is supported at $t = \theta^{q^i}$, $i > 0$. $\square$

Moreover, we have

$$\begin{aligned} \varsigma^*_{\mathbb{C}_\infty[\![t-\theta^q]\!]}\left(\mathrm{M}(\infty) \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - \theta^q]\!]\right) &= \varsigma^*_{A(\infty)}\mathrm{M}(\infty) \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - \theta]\!] \\ &\xhookrightarrow{\sigma_{\mathrm{M}(\infty)} \otimes \mathrm{id}} \mathrm{M}(\infty) \otimes_{A(\infty)} \mathbb{C}_\infty[\![t - \theta]\!]. \end{aligned}$$

Recall that $(\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}}, \sigma_{\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}}})$ is the rigid σ-sheaf over $A(\infty)$ on $\mathrm{Sp}\, k$ associated with $\underline{\mathrm{M}}$ such that $\mathrm{M}(\infty) = \Gamma(\mathfrak{A}(\infty), \mathcal{F}^{\mathrm{rig}}_{\mathrm{M}})$ and $\sigma_{\mathrm{M}(\infty)}$ is the induced $A(\infty)$-homomorphism on global sections. For $j \geq 1$ we denote the global sections of the pullback $(\varsigma^*_{A(\infty)})^j \mathcal{F}^{\mathrm{rig}}_{\mathrm{M}}$ by $(\varsigma^*_{A(\infty)})^j \mathrm{M}(\infty)$ and the induced $A(\infty)$-homomorphisms by $\sigma_{(\varsigma^*_{A(\infty)})^j \mathrm{M}(\infty)}$. Then we can picture the relations of the corresponding rigid σ-sheaves over $A(\infty)$ on $\mathrm{Sp}\, \mathbb{C}_\infty$ at $t = \theta^{q^i}$, $i \in \mathbb{Z}$, as follows:



.

We are now able to show that the two short exact sequences

$$0 \longrightarrow \mathfrak{q}_H \longrightarrow \mathfrak{p}_H \longrightarrow \mathrm{T}_0 E \longrightarrow 0$$

and

$$0 \to \sigma_{\mathrm{M}}(\varsigma^*\mathrm{M}) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t - \theta]\!] \to \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t - \theta]\!] \to \mathrm{coker}\, \sigma_{\mathrm{M}} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t - \theta]\!] \to 0$$

are isomorphic. From Lemma 4.1.30 and Corollary 4.1.24, we know that

$$\mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \cong \mathrm{M}(1)^\sigma \otimes_A \mathbb{C}_\infty[\![t-\theta]\!] \cong (\Lambda_{\underline{E}} \otimes_A Q) \otimes_Q \mathbb{C}_\infty[\![z-\zeta]\!] = \mathfrak{p}_H$$

holds. Further,

$$\begin{aligned} \operatorname{coker} \sigma_{\mathrm{M}} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] &= \operatorname{coker} \sigma_{\mathrm{M}} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!]/(t-\theta)^d \\ &= \operatorname{coker} \sigma_{\mathrm{M}} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[t]/(t-\theta)^d \\ &= \operatorname{coker} \sigma_{\mathrm{M}} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[t] \\ &= \operatorname{coker} \sigma_{\mathrm{M}} \\ &\cong \mathrm{T}_0 E \end{aligned}$$

by the condition $(t-\theta)^d \operatorname{coker} \sigma_{\mathrm{M}} = 0$ and Lemma 4.1.5. The two sequences are thus isomorphic and hence

$$\mathfrak{q}_H \cong \sigma_{\mathrm{M}}(\varsigma^*\mathrm{M}) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \subseteq \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \cong \mathfrak{p}_H.$$

Observe that we have $(z-\zeta)^d \mathfrak{p}_H \subseteq \mathfrak{q}_H$.

Definition 4.1.31. (i) We define $\mathcal{D} : \mathscr{PRDA}_+^I \to \mathscr{Hodge}_Q$ to be the fully faithful functor that sends a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ of rank r, dimension d and weight $\frac{l}{n}$ over $\mathbb{C}_\infty$ to the Q-Hodge-Pink structure of rank r with underlying Q-vector space $H := \mathrm{M}(1)^\sigma \otimes_A Q$ and lattice $\mathfrak{q}_H := \sigma_{\mathrm{M}}(\varsigma^*\mathrm{M}) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!]$ that is pure of weight $-\frac{d}{r}$.

(ii) Let $\underline{\mathrm{M}}$ be a pure rigid analytically trivial dual Anderson A-motive of positive rank and dimension over $k \subset \mathbb{C}_\infty$. We call the Hodge-Pink group of $\mathcal{D}(\underline{\mathrm{M}})$ the *Hodge-Pink group of* $\underline{\mathrm{M}}$ and denote it by $G_{\underline{\mathrm{M}}}$.

We observe that the functors $\mathcal{D}$ and $\mathcal{H} \circ \mathcal{E}$ are isomorphic by construction.

Remark 4.1.32 (Cf. [Gos94, §2.6]). Let $\underline{H} := (H, W, \mathfrak{q}_H) := \mathcal{D}(\underline{\mathrm{M}})$ be the pure Q-Hodge-Pink structure associated with a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ over $\mathbb{C}_\infty$. Recall that $H_{\mathbb{C}_\infty} = \mathfrak{p}_H/(z-\zeta)\mathfrak{p}_H \cong \mathrm{M}/(t-\theta)\mathrm{M}$ and the Hodge-Pink filtration $F = (F^i H_{\mathbb{C}_\infty})_{i\in\mathbb{Z}}$ of $H_{\mathbb{C}_\infty}$ is given by

$$F^i H_{\mathbb{C}_\infty} := \text{image of } \mathfrak{p}_H \cap (z-\zeta)^i \mathfrak{q}_H \text{ in } H_{\mathbb{C}_\infty} \quad \text{for all } i \in \mathbb{Z}.$$

By the definition of the functor $\mathcal{D}$, we have $H_{\mathbb{C}_\infty} \cong \mathrm{M}/(t-\theta)\mathrm{M}$ and

$$F^i H_{\mathbb{C}_\infty} \cong \text{image of } \mathrm{M} \cap (t-\theta)^i \sigma_{\mathrm{M}}(\varsigma^*\mathrm{M}) \text{ in } \mathrm{M}/(t-\theta)\mathrm{M} \quad \text{for all } i \in \mathbb{Z}.$$

As we will see in Remark 5.2.3, $\operatorname{Hom}_{\mathbb{C}_\infty}(\mathrm{M}/(t-\theta)\mathrm{M}, \mathbb{C}_\infty\, \mathrm{d}t)$ admits interpretation as the first de Rham cohomology group of $\underline{\mathrm{M}}$.

Motivated by this, we put

$$H^1_{\mathrm{DR}}(\underline{\mathrm{M}}, \mathbb{C}_\infty) := \operatorname{Hom}_{\mathbb{C}_\infty}(\mathrm{M}/(t-\theta)\mathrm{M}, \mathbb{C}_\infty\, \mathrm{d}t).$$

Consider the decreasing filtration of $\mathrm{M}/(t-\theta)\mathrm{M}$

$$F^i \mathrm{M}/(t-\theta)\mathrm{M} := \text{image of } \mathrm{M} \cap (t-\theta)^i \sigma_{\mathrm{M}}(\varsigma^*\mathrm{M}) \text{ in } \mathrm{M}/(t-\theta)\mathrm{M} \quad \text{for all } i \in \mathbb{Z}.$$

Similarly to the induced filtration of the dual space (Definition 3.1.2), the *de Rham filtration of* $\underline{\mathrm{M}}$ is defined to be

$$F^i H^1_{\mathrm{DR}}(\underline{\mathrm{M}}, \mathbb{C}_\infty) := \{\mathrm{m} \in \mathrm{M}/(t-\theta)\mathrm{M} \to \mathbb{C}_\infty\, \mathrm{d}t \mid \forall j < i : \mathrm{m}|_{F^{-j}\mathrm{M}/(t-\theta)\mathrm{M}} = 0\}.$$

Observe that we have $F^n H^1_{\mathrm{DR}}(\underline{\mathrm{M}}, \mathbb{C}_\infty) = 0$ for $n > d$ since

$$F^{-d}\mathrm{M}/(t-\theta)\mathrm{M} = \mathrm{M}/(t-\theta)\mathrm{M} \quad \text{and} \quad F^1\mathrm{M}/(t-\theta)\mathrm{M} = 0.$$

4.1.5 *Construction of μ*

Suppose $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ be a pure rigid analytically trivial dual Anderson A-motive of rank r, dimension d and weight $\frac{l}{n}$ over $\mathbb{C}_\infty$ in $\mathscr{PRDA}_+^I$. Let $\underline{H} = (H, W, \mathfrak{q}_H)$ be the associated pure Q-Hodge-Pink structure and $\underline{P} = (\underline{\mathrm{M}}, 0)$ be the associated pure dual t-motive. Our goal is to construct a homomorphism μ of Q-group schemes from the Hodge-Pink group $G_{\underline{\mathrm{M}}}$ to the Galois group $\Gamma_{\underline{\mathrm{M}}}$. In order to do this, we define a functor

$$\mathcal{T} : \mathscr{PT} \to \mathscr{Hodge}_Q, \quad \underline{\mathrm{M}}(i) \mapsto \mathcal{D}(\underline{\mathrm{M}}) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-i}$$

that satisfies $\mathcal{D}(\underline{\mathrm{N}}) \cong \mathcal{T}(\mathcal{P}(\underline{\mathrm{N}}))$ for any pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{N}}$ over $\mathbb{C}_\infty$.

We want to show that the functors $\varpi \circ \mathcal{T}$ and ω' are isomorphic so that the following diagram "commutes":

$$\begin{tikzcd}
\mathscr{PRDA}_+^I \arrow[r, "\mathcal{P}'"] \arrow[dr, "\mathcal{D}"'] & \mathscr{PT}' \arrow[r, "\mathcal{R}"] \arrow[d, "\mathcal{T}"] \arrow[dr, "\omega'"] & \mathscr{PT} \arrow[d, "\omega"] \\
& \mathscr{Hodge}_Q \arrow[r, "\varpi"] & \mathscr{Vec}_Q.
\end{tikzcd}$$

Lemma 4.1.33. *There is a functorial isomorphism η between the functors $\varpi \circ \mathcal{T} : \mathscr{PT}' \to \mathscr{Vec}_Q$ and $\omega' = \omega \circ \mathcal{R} : \mathscr{PT}' \to \mathscr{Vec}_Q$.*

Proof. For each pure dual t-motives $\underline{\mathrm{M}}(i) \in \mathrm{Ob}(\mathscr{PT}')$ we want to define an isomorphism $\eta_{\underline{\mathrm{M}}(i)} : (\varpi \circ \mathcal{T})(\underline{\mathrm{M}}(i)) \xrightarrow{\sim} (\omega')(\underline{\mathrm{M}}(i))$ so that there is a commutative diagram

$$\begin{tikzcd}
\omega'(\underline{\mathrm{M}}(i)) \arrow[r, "\eta_{\underline{\mathrm{M}}(i)}"] \arrow[d, "\omega'(f)"'] & (\varpi \circ \mathcal{T})(\underline{\mathrm{M}}(i)) \arrow[d, "(\varpi\circ\mathcal{T})(f)"] \\
\omega'(\underline{\mathrm{N}}(j)) \arrow[r, "\eta_{\underline{\mathrm{N}}(j)}"] & (\varpi \circ \mathcal{T})(\underline{\mathrm{N}}(j))
\end{tikzcd}$$

for all homomorphisms $f : \underline{\mathrm{M}}(i) \to \underline{\mathrm{N}}(j)$ of pure dual t-motives $\underline{\mathrm{M}}(i), \underline{\mathrm{N}}(j) \in \mathrm{Ob}(\mathscr{PT}')$. For an arbitrary pure dual t-motive $\underline{\mathrm{M}}(i)$, we have

$$\omega'(\underline{\mathrm{M}}(i)) = \omega(\mathcal{P}(\underline{\mathrm{M}}) \otimes \omega(\mathcal{P}(\underline{\mathrm{C}})^i) \quad \text{and} \quad \varpi(\mathcal{T}(\underline{\mathrm{M}}(i))) = \varpi(\mathcal{D}(\underline{\mathrm{M}})) \otimes \varpi(\mathcal{D}(\underline{\mathrm{C}})^i)$$

Thus it suffices to prove that the functors $\omega \circ \mathcal{P} \cong \omega \circ \mathcal{R} \circ \mathcal{P}'$ and $\varpi \circ \mathcal{D}$ are isomorphic.

To see this, let $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ be an arbitrary pure rigid analytically trivial dual Anderson A-motive of rank r over $\mathbb{C}_\infty$ and $\underline{P} = (P, \sigma_P)$ its associated pure dual t-motive in $\mathscr{PT}$. Suppose that $\Phi_{\mathbf{m}}$ represents σ_{M} with respect to a $\mathbb{C}_\infty[t]$-basis $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(\mathrm{M})$ for M. We tensor the entries of $\mathbf{m}$ with Q so that we obtain a $\mathbb{C}_\infty(t)$-basis $\mathbf{p} \in \mathrm{Mat}_{r\times 1}(P)$ for P and $\Phi_{\mathbf{p}} := \Phi_{\mathbf{m}}$ represents σ_P with respect to the basis $\mathbf{p}$. Furthermore, a rigid trivialization $\Psi_{\mathbf{m}}$ of $\Phi_{\mathbf{m}}$ also provides a rigid trivialization of $\Phi_{\mathbf{p}}$. By Proposition 2.5.8 we find that $(\omega\circ\mathcal{P})(\underline{\mathrm{M}}) = \omega(\underline{P}) = P(1)^\sigma \cong \mathrm{M}(1)^\sigma \otimes_A Q$ holds. Further we have $(\varpi\circ\mathcal{D})(\underline{\mathrm{M}}) = \mathrm{M}(1)^\sigma \otimes_A Q$ by definition. Then there is a canonical isomorphism $\eta_{\underline{\mathrm{M}}} : P(1)^\sigma \to \mathrm{M}(1)^\sigma \otimes_A Q$ such that for any homomorphism $f : \underline{\mathrm{M}} \to \underline{\mathrm{N}}$ of dual Anderson A-motives commutativity of the following diagram follows automatically:

$$\begin{tikzcd}
(\omega \circ \mathcal{P})(\underline{\mathrm{M}}) \arrow[r, "\eta_{\underline{\mathrm{M}}}"] \arrow[d, "(\omega\circ\mathcal{P})(f)"'] & (\varpi \circ \mathcal{D})(\underline{\mathrm{M}}) \arrow[d, "(\varpi\circ\mathcal{D})(f)"] \\
(\omega \circ \mathcal{P})(\underline{\mathrm{N}}) \arrow[r, "\eta_{\underline{\mathrm{N}}}"] & (\varpi \circ \mathcal{D})(\underline{\mathrm{N}}).
\end{tikzcd}$$

Hence, the functors $\varpi \circ \mathcal{T}$ and $\omega' = \omega \circ \mathcal{R}$ are isomorphic. $\square$

Consider now an arbitrary pure dual t-motive $\underline{P}$ over $\mathbb{C}_\infty$. We denote the restriction of $\mathcal{T}$ to $\langle\langle \underline{P} \rangle\rangle$ by $\mathcal{T}_{\underline{P}} : \langle\langle \underline{P} \rangle\rangle \to \langle\langle \mathcal{T}(\underline{P}) \rangle\rangle$. This is well-defined since $\mathcal{T}$ is an exact tensor functor.

Corollary 4.1.34. *Let $\underline{P}$ be a pure dual t-motive over $\mathbb{C}_\infty$. Then there is a functorial isomorphism η' between the functors $\varpi_{\mathcal{T}(\underline{P})} \circ \mathcal{T}_{\underline{P}} : \langle\langle \underline{P} \rangle\rangle \to \mathscr{V}ec_Q$ and $\omega'_{\underline{P}} = \omega_{\underline{P}} \circ \mathcal{R}_{\underline{P}} : \langle\langle \underline{P} \rangle\rangle \to \mathscr{V}ec_Q$.*

From Lemma 1.2.14 follows the existence of the desired map between the Galois group and Hodge-Pink group of a pure dual t-motive over $\mathbb{C}_\infty$.

Corollary 4.1.35. *Let $\underline{P}$ be a pure dual t-motive over $\mathbb{C}_\infty$. There is a unique Q-group scheme homomorphism $\mu : G_{\mathcal{T}(\underline{P})} \to \Gamma_{\underline{P}}$ such that $\mathcal{T}_{\underline{P}} \cong \omega^\mu : \mathscr{R}ep_Q(\Gamma_{\underline{P}}) \to \mathscr{R}ep_Q(G_{\mathcal{T}(\underline{P})})$.*

4.2 Equality of the Hodge-Pink group and the Galois group

We consider a pure dual t-motive $\underline{P}$ over $\mathbb{C}_\infty$ together with its associated pure Q-Hodge-Pink structure $\underline{H} := \mathcal{T}(\underline{P})$. We then prove the Hodge conjecture for function fields, that is, the Q-group scheme homomorphism $\mu : G_{\underline{H}} \to \Gamma_{\underline{P}}$ defined in the previous section is an isomorphism. By Tannakian duality, the Tannakian categories $\langle\langle \underline{P} \rangle\rangle$ and $\langle\langle \underline{H} \rangle\rangle$ generated, respectively, by the pure dual t-motive $\underline{P}$ and the pure Q-Hodge-Pink structure $\underline{H}$, are equivalent.

We proceed by showing that μ is faithfully flat and a closed immersion. The latter is easily seen (Proposition 4.2.17), but the proof of the former assertion requires some preparatory work. In the first subsection, we define F-modules following Hartl in [Har10]. We may then associate an F-module $\underline{\dot{M}}$ over $\dot{D}^\infty_{\mathbb{C}_\infty}$ with a pure dual Anderson A-motive over $\mathbb{C}_\infty$. Next we assign a pair of F-modules $(\underline{\dot{M}}', \underline{\dot{N}}')$ over $\dot{D}^\infty_{\mathbb{C}_\infty}$ to a pure strict sub-Q-Hodge-Pink structure $\underline{H}'$ in $\langle\langle \underline{H} \rangle\rangle$. Through the classification of corresponding σ-bundles studied in [HP04], we show that $\underline{\dot{M}}'$ is a sub-F-module of $\underline{\dot{M}}$. With the help of the rigid analytic GAGA principle, we associate a pure rigid analytically trivial dual Anderson sub-A-motive $\underline{\mathrm{M}}'$ over $\mathbb{C}_\infty$ with $\underline{H}'$ such that $\mathcal{D}(\underline{\mathrm{M}}') = \underline{H}'$.

Combining this and Proposition 1.2.15 (i), we prove that μ is faithfully flat. This yields the Hodge conjecture for function fields.

4.2.1 F-Modules

F-modules were studied by Hartl in [Har10] where they were called σ-modules whose underlying map σ, in their notation, corresponds to our map F. Of importance to us is that F-modules over specific "coefficient rings" admit a classification. Through this we put the F-modules, which we associate with dual Anderson A-motives and sub-Q-Hodge-Pink structures, in relation. We first fix notation and afterwards state the definitions and results we need later. We refer the reader for more details on F-modules and their classification to [Har10] and [HP04], respectively.

Consider the ring of formal power series $\mathbb{F}_q[\![\zeta]\!]$ over $\mathbb{F}_q$ in the indeterminant ζ and its field of fractions, the field of formal Laurent series $\mathbb{F}_q(\!(\zeta)\!)$ over $\mathbb{F}_q$ in ζ. Let R be a valuation ring of Krull dimension one that contains $\mathbb{F}_q[\![\zeta]\!]$ and is complete and separated with respect to the ζ-adic topology. Let L be the fraction field of R and $\overline{L}$ be the completion of an algebraic closure of L, which is automatically algebraically closed by [Gos96, Prop. 2.1].[1] Further, let $L[\![z]\!]$ denote the ring of formal power series in z and $L[\![z]\!][z^{-1}]$ be the ring of Laurent series in z with finite principal part. We let n and n' be rational numbers such that $n' \geq n > 0$ and define the *coefficient ring of an F-module* $\mathcal{R}$ to be one of the following L-algebras introduced in Section 1.3:

$$\begin{aligned}
D(\zeta^n)_L^\infty &:= L\langle \frac{z}{\zeta^n} \rangle &&= \{\sum_{i=0}^{\infty} \alpha_i z^i \in L[\![z]\!] : \lim_{i\to\infty} |\alpha_i \zeta^{ni}| = 0\}, \\
& L\langle \frac{z}{\zeta^n} \rangle [z^{-1}] &&:= \{\sum_{i \gg -\infty}^{\infty} \alpha_i z^i \in L[\![z]\!][z^{-1}] : \lim_{i\to\infty} |\alpha_i \zeta^{ni}| = 0\}, \\
A(\zeta^n, \zeta^{n'})_L^\infty &:= L\langle \frac{z}{\zeta^n}, \frac{\zeta^{n'}}{z} \rangle &&= \{\sum_{i\in\mathbb{Z}} \alpha_i z^i : \lim_{i\to\infty} |\alpha_i \zeta^{ni}| = 0,\ \lim_{i\to-\infty} |\alpha_i \zeta^{n'i}| = 0\}, \\
\dot{D}(\zeta^n)_L^\infty &:= L\langle \frac{z}{\zeta^n}, z^{-1}\} &&= \{\sum_{i\in\mathbb{Z}} \alpha_i z^i : \lim_{i\to\pm\infty} |\alpha_i \zeta^{n'i}| = 0 \text{ for all } n' \geq n\}, \\
\dot{D}_L^\infty &:= L\{z, z^{-1}\} &&= \{\sum_{i\in\mathbb{Z}} \alpha_i z^i : \lim_{i\to\pm\infty} |\alpha_i \zeta^{ni}| = 0 \text{ for all } n > 0\}.
\end{aligned}$$

The corresponding rigid L-spaces to $D(\zeta^n)_L^\infty$, $A(\zeta^n,\zeta^{n'})_L^\infty$, $\dot{D}(\zeta^n)_L^\infty$, and $\dot{D}_L^\infty$ are

$$\begin{aligned}
\mathfrak{D}(\zeta^n)_L^\infty &:= \operatorname{Sp} D(\zeta^n)_L^\infty = \{x \in \overline{L} : |x| \leq |\zeta|^n\} \\
&= \text{the disk centered at } z=0 \text{ with radius } |\zeta|^n, \\
\mathfrak{A}(\zeta^n,\zeta^{n'})_L^\infty &= \operatorname{Sp} A(\zeta^n,\zeta^{n'})_L^\infty = \{x \in \overline{L} : |\zeta|^{n'} \leq |x| \leq |\zeta|^n\} \\
&= \text{the annulus centered at } z=0 \text{ with inner radius } |\zeta|^{n'} \text{ and outer radius } |\zeta|^n, \\
\dot{\mathfrak{D}}(\zeta^n)_L^\infty &:= \operatorname{Sp} \dot{D}(\zeta^n)_L^\infty = \{x \in \overline{L} : 0 < |x| \leq |\zeta|^n\} \\
&= \text{the } \textit{punctured disk} \text{ centered at } z=0 \text{ with radius } |\zeta|^n, \\
\dot{\mathfrak{D}}_L^\infty &:= \operatorname{Sp} \dot{D}_L^\infty = \{x \in \overline{L} : 0 < |x| < 1\} \\
&= \text{the } \textit{punctured unit disk} \text{ centered at } z=0,
\end{aligned}$$

respectively.

Set $F := \mathrm{Frob}_{q,\operatorname{Sp} L}$ and extend the induced map $F^* = \mathrm{Frob}_{q,L} : L \to L$, $\alpha \mapsto \alpha^q$, to a homomorphism

$$F^*\left(\sum_{i=-\infty}^{\infty} \alpha_i z^i\right) := \sum_{i=-\infty}^{\infty} \alpha_i^q z^i,$$

by mapping z to itself. Let $\mathcal{R}$ be one of the coefficient rings and denote its image under F^* by $\mathcal{R}^{F^*}$. Note, we then have $\left(D(\zeta^n)_L^\infty\right)^{F^*} = F^*(D(\zeta^n)_L^\infty) = D(\zeta^{qn})_L^\infty$ and similarly for $L\langle \frac{z}{\zeta^n}\rangle[z^{-1}]$ and $\dot{D}(\zeta^n)_L^\infty$. We further set

$$\left(A(\zeta^n,\zeta^{n'})_L^\infty\right)^{F^*} := A(\zeta^n,\zeta^{qn'})_L^\infty \supset A(\zeta^{qn},\zeta^{qn'})_L^\infty = F^*(A(\zeta^n,\zeta^{n'})_L^\infty)$$

if $n' \geq qn$ and $\left(\dot{D}_L^\infty\right)^{F^*} := \dot{D}_L^\infty$.

Let $\mathcal{R}$ be one of the coefficient rings. Then there is a natural inclusion $\iota : \mathcal{R} \to \mathcal{R}^{F^*}$, which sends an $f \in \mathcal{R} \subseteq \mathcal{R}^{F^*}$ to itself.

[1] In the next subsections, when $A = \mathbb{F}_q[t]$ we let $z = \frac{1}{t}$ be a local parameter at ∞ of C and set $\zeta = \frac{1}{\theta}$. Then (L,γ) with $\gamma : A \hookrightarrow Q_\infty \to L, t \mapsto \theta$, is an A-field and we may relate F-modules and pure dual Anderson A-motives over $\overline{L} = \mathbb{C}_\infty$.

Definition 4.2.1 ([Har10, Def. 1.2.1]). Let $\mathcal{R}$ be either $D(\zeta^n)_L^\infty$, $L\langle\frac{z}{\zeta^n}\rangle[z^{-1}]$ or $A(\zeta^n,\zeta^{n'})_L^\infty$, and r an integer. If M is an $\mathcal{R}$-module, we denote $F^*M := M \otimes_{\mathcal{R},F^*} \mathcal{R}^{F^*}$ and $\iota^*M := M \otimes_{\mathcal{R},\iota} \mathcal{R}^{F^*}$.

(i) An *F-module $\underline{M}$ of rank r over $\mathcal{R}$* is a pair (M,τ_M), where M is a locally free coherent $\mathcal{R}$-module M of rank r and $\tau_M : F^*M \to \iota^*M$ is an isomorphism of $\mathcal{R}^{F^*}$-modules.

(ii) A homomorphism $f : M \to N$ of $\mathcal{R}$-modules is a *morphism between two F-modules $\underline{M} = (M,\tau_M)$ and $\underline{N} = (N,\tau_N)$ over $\mathcal{R}$* if it satisfies $\tau_N \circ F^*f = \iota^*f \circ \tau_M$.

(iii) We define the *tensor product* $\underline{M} \otimes \underline{N}$ of two F-modules $\underline{M} = (M,\tau_M)$ and $\underline{N} = (N,\tau_N)$ over $\mathcal{R}$ to be the pair $M \otimes_{\mathcal{R}} N$ together with the isomorphism $\tau_{\underline{M}\otimes\underline{N}} := \tau_M \otimes \tau_N$ and similarly for $n \geq 1$ the *symmetric power* $\operatorname{Sym}^n \underline{M}$ and the *alternating power* $\bigwedge^n \underline{M}$ of an F-module $\underline{M}$.

(iv) The *inner hom* $\mathcal{H}om(\underline{M},\underline{N})$ of two F-modules $\underline{M}$ and $\underline{N}$ is

$$\mathcal{H}om(\underline{M},\underline{N}) := (\mathcal{H}om(M,N), \tau_{\mathcal{H}om(M,N)}) \text{ with } \tau_{\mathcal{H}om(M,N)}(F^*f) := \tau_N \circ F^*f \circ \tau_M^{-1}.$$

The F-module $\underline{\mathbb{1}}_{\mathcal{R}} := (\mathbb{1}_{\mathcal{R}}, \tau_{\mathbb{1}_{\mathcal{R}}})$ over $\mathcal{R}$ consisting of $\mathcal{R}$ itself and $\mathrm{id}_{\mathcal{R}^{F^*}} : \mathcal{R}^{F^*} \xrightarrow{\sim} \mathcal{R}^{F^*}$ is a unit object with respect to the tensor product, so that the *dual* $\underline{M}^\vee$ of an F-module $\underline{M}$ over $\mathcal{R}$ is given by $\mathcal{H}om(\underline{M},\underline{\mathbb{1}}_{\mathcal{R}})$.

We denote the additive rigid tensor category of F-modules over $\mathcal{R}$ by $\mathscr{F}_{\mathcal{R}}$ and the abelian group of all morphisms of F-modules from $\underline{M}$ to $\underline{N}$ by $\operatorname{Hom}_{\mathscr{F}_{\mathcal{R}}}(\underline{M},\underline{N})$, for $\underline{M}$, $\underline{N} \in \operatorname{Ob}(\mathscr{F}_{\mathcal{R}})$.

For the definition of F-modules over $\dot{D}(\zeta^n)_L^\infty$ and $\dot{D}_L^\infty$, we work with sheaves on the rigid spaces $\dot{\mathfrak{D}}(\zeta^n)_L^\infty$ and $\dot{\mathfrak{D}}_L^\infty$, respectively. One could define F-modules over $D(\zeta^n)_L^\infty$ and $A(\zeta^n,\zeta^{n'})_L^\infty$ in the same way.

Definition 4.2.2 ([Har10, Def. 1.2.2]). Let $\mathcal{R}$ be either $\dot{D}(\zeta^n)_L^\infty$ or $\dot{D}_L^\infty$, and r an integer. By abuse of notation, we denote the map $\operatorname{Sp}\mathcal{R} \to \operatorname{Sp}\mathcal{R}$ that sends an $x \in \operatorname{Sp}\mathcal{R} \subseteq \overline{L}$ to $x^{q^{-1}}$ by F also.

(i) An *F-module $\underline{M}$ of rank r over $\mathcal{R}$* is a pair (M,τ_M) where M is a locally free coherent sheaf M of $\mathcal{O}_{\operatorname{Sp}\mathcal{R}}$-modules of rank r on $\operatorname{Sp}\mathcal{R}$ and $\tau_M : F^*M \xrightarrow{\sim} \iota^*M$ is an $\mathcal{O}_{\operatorname{Sp}\mathcal{R}^{F^*}}$-module homomorphism.

(ii) An $\mathcal{O}_{\operatorname{Sp}\mathcal{R}}$-module homomorphism $f : M \to M$ is a *morphism between two F-modules $\underline{M} = (M,\tau_M)$ and $\underline{N} = (N,\tau_N)$ over $\mathcal{R}$* if it satisfies $\tau_N \circ F^*f = \iota^*f \circ \tau_M$.

(iii) We define the *tensor product* $\underline{M} \otimes \underline{N}$ of two F-modules $\underline{M} = (M,\tau_M)$ and $\underline{N} = (N,\tau_N)$ over $\mathcal{R}$ to be the pair $M \otimes_{\mathcal{O}_{\operatorname{Sp}\mathcal{R}}} N$ together with the isomorphism $\tau_{\underline{M}\otimes\underline{N}} := \tau_M \otimes \tau_N$ and similarly for $n \geq 1$ the *symmetric power* $\operatorname{Sym}^n \underline{M}$ and the *alternating power* $\bigwedge^n \underline{M}$ of an F-module $\underline{M}$.

(iv) The *inner hom* $\mathcal{H}om(\underline{M},\underline{N})$ of two F-modules $\underline{M}$ and $\underline{N}$ is the inner hom $\mathcal{H}om(M,N)$ of the locally free sheaves M and N of $\mathcal{O}_{\operatorname{Sp}\mathcal{R}}$-modules together with the $\mathcal{O}_{\operatorname{Sp}\mathcal{R}^{F^*}}$-module homomorphism

$$\tau_{\mathcal{H}om(M,N)} : F^*\,\mathcal{H}om(M,N) \xrightarrow{\sim} \iota^*\,\mathcal{H}om(M,N), \quad F^*f \mapsto \tau_N \circ F^*f \circ \tau_M^{-1}.$$

The F-module $\underline{\mathbb{1}}_{\mathcal{R}} := (\mathbb{1}_{\mathcal{R}}, \tau_{\mathbb{1}_{\mathcal{R}}})$ over $\mathcal{R}$ consisting of the structure sheaf $\mathcal{O}_{\operatorname{Sp}\mathcal{R}}$ and $\mathrm{id}_{\mathcal{O}_{\operatorname{Sp}\mathcal{R}^{F^*}}} : \mathcal{O}_{\operatorname{Sp}\mathcal{R}^{F^*}} \xrightarrow{\sim} \mathcal{O}_{\operatorname{Sp}\mathcal{R}^{F^*}}$ is a unit object with respect to the tensor product, so that the *dual* $\underline{M}^\vee$ of an F-module $\underline{M}$ over $\mathcal{R}$ is given by setting $\underline{M}^\vee := \mathcal{H}om(\underline{M},\underline{\mathbb{1}}_{\mathcal{R}})$.

We denote the additive rigid tensor category of F-modules over $\mathcal{R}$ by $\mathscr{F}_{\mathcal{R}}$ and the abelian group of all morphisms of F-modules from $\underline{M}$ to $\underline{N}$ by $\mathrm{Hom}_{\mathscr{F}_{\mathcal{R}}}(\underline{M}, \underline{N})$ for $\underline{M}$, $\underline{N} \in \mathrm{Ob}(\mathscr{F}_{\mathcal{R}})$.

By abuse of notation we will denote the isomorphism $\tau_M : F^*M \to \iota^*M$ underlying an F-module (M, τ_M) by $\tau_M : F^*M \to M$.

Note that one can also define F-modules over $\dot{D}(\zeta^n)^{\infty}_{\overline{L}}$, $\dot{D}^{\infty}_{\overline{L}}$, $D(\zeta^n)^{\infty}_{\overline{L}}$ and $A(\zeta^n, \zeta^{n'})^{\infty}_{\overline{L}}$ as smooth locally free rigid analytic σ-sheaves since the underlying homomorphism is an isomorphism. We preferred to be consistent with the terminology of [Har10] for clarity and include the definition of F-modules over $L\langle \frac{z}{\zeta^n} \rangle [z^{-1}]$.

Example 4.2.3. We assume that z is invertible and let d be an arbitrary integer.

(i) If $\mathcal{R}$ is either $L\langle \frac{z}{\zeta^n} \rangle [z^{-1}]$ or $A(\zeta^n, \zeta^{n'})^{\infty}_{\overline{L}}$, we define the F-module $\underline{\mathcal{O}(d)}$ of rank 1 over $\mathcal{R}$ to be
$$\underline{\mathcal{O}(d)} := (\mathcal{O}(d), \tau_{\mathcal{O}(d)}) := (\mathcal{R}, z^{-d}F^*).$$

(ii) Correspondingly, if $\mathcal{R}$ is either $\dot{D}(\zeta^n)^{\infty}_{\overline{L}}$ or $\dot{D}^{\infty}_{\overline{L}}$, we define the F-module $\underline{\mathcal{O}(d)}$ of rank 1 over $\mathcal{R}$ to be
$$\underline{\mathcal{O}(d)} := (\mathcal{O}(d), \tau_{\mathcal{O}(d)}) := (\mathcal{O}_{\mathrm{Sp}\,\mathcal{R}}, z^{-d}F).$$

We are particularly interested in F-modules over $\dot{D}^{\infty}_{\overline{L}}$, which we associate with pure rigid analytically trivial dual Anderson A-motives over $\overline{L}$ and sub-Q-Hodge-Pink-structures. Hartl and Pink introduced the corresponding σ-bundles in [HP04]. From [Har10, Prop. 1.4.1] it follows that the categories $\mathscr{F}_{\dot{D}(\zeta^n)^{\infty}_{\overline{L}}}$ and $\mathscr{F}_{\dot{D}^{\infty}_{\overline{L}}}$ are equivalent, so that we also may carry definitions and results stated in [HP04] over to F-modules over $\dot{D}(\zeta^n)^{\infty}_{\overline{L}}$, as done by Hartl in [Har10, §1]. In the following, we let $\mathcal{R}$ be either $\dot{D}(\zeta^n)^{\infty}_{\overline{L}}$ or $\dot{D}^{\infty}_{\overline{L}}$. We deduce now the properties of F-modules over $\mathcal{R}$, which we need later.

Corollary 4.2.4 ([HP04, Cor. 5.4])**.** *Every F-module of rank 1 over $\mathcal{R}$ is isomorphic to $\mathcal{O}(d)$ for a unique integer d.*

Let $\underline{M}$ be an F-module of rank r over $\mathcal{R}$. Then there is by the previous corollary a unique integer d such that $\bigwedge^r \underline{M} \cong \mathcal{O}(d)$. We then say that $\underline{M}$ is of *degree* d and, if $r \in \mathbb{N}^{>0}$, of *weight* $\mathrm{wt}(\underline{M}) := \frac{d}{r}$.

Further we say that a non-zero F-module $\underline{M}$ of positive rank r over $\mathcal{R}$ is *semi-stable* if $\mathrm{wt}(\underline{N}) \leq \mathrm{wt}(\underline{M})$ for all non-zero sub-F-modules $\underline{N}$ over $\mathcal{R}$ and *stable* if $\mathrm{wt}(\underline{N}) < \mathrm{wt}(\underline{M})$ for all proper non-zero sub-F-modules $\underline{N}$ over $\mathcal{R}$.

In order to obtain a similar assertion as in the last corollary for an F-module of positive rank r and degree d over $\mathcal{R}$, we have to define the pullback of an F-module over $\mathcal{R}$.

Definition 4.2.5 ([HP04, §7])**.** Let r be a positive integer and consider the morphism $[r] : \mathrm{Sp}\,\mathcal{R} \to \mathrm{Sp}\,\mathcal{R}$, $x \mapsto x^r$, that induces
$$[r]^* : \mathcal{R} \to \mathcal{R},\ \sum_i \alpha_i z^i \mapsto \sum_i \alpha_j z^{ri}.$$

(i) We define the *pullback* $[r]^*\underline{M}$ of an F-module $\underline{M} = (M, \tau_M)$ over $\mathcal{R}$ to be the pullback $[r]^*M$ together with the induced isomorphism
$$F^*([r]^*M) = [r]^*(F^*M) \xrightarrow{[r]^*\tau_M} [r]^*M.$$

(ii) The *pushforward* $[r]_*\underline{M}$ of an F-module $\underline{M} = (M, \tau_M)$ over $\mathcal{R}$ is the pushforward $[r]_*M$ together with the induced isomorphism

$$F^*([r]_*M) = [r]_*(F^*M) \xrightarrow{[r]_*\tau_M} [r]_*M.$$

We are now ready to give the basic example of F-modules over $\mathcal{R}$. These are also called *building blocks* since the classification of an arbitrary F-module of positive rank r and degree d over $\mathcal{R}$ provides a decomposition into a direct sum of such building blocks by [HP04, Thm. 11.1].

Example 4.2.6 ([HP04, §8]). For every pair (d, r) of relatively prime integers r, d with $r > 0$, we define the F-module $\dot{\underline{M}}_{d,r} := (\dot{M}_{d,r}, \tau_{\dot{M}_{d,r}})$ over $\mathcal{R}$ to be the pullback $[r]^*\underline{\mathcal{O}(d)}$ of $\underline{\mathcal{O}(d)}$ that is a semi-stable F-module of rank r and weight $\frac{d}{r}$ over $\mathcal{R}$ by [HP04, Prop. 7.6]. We can choose a basis such that $\dot{M}_{d,r} \cong \mathcal{O}_{\mathrm{Sp}\,\mathcal{R}}^{\oplus r}$, so that we can represent $\tau_{\dot{M}_{d,r}} : \dot{M}_{d,r} \to \dot{M}_{d,r}$ with respect to this basis as the isomorphism $F^*\mathcal{O}_{\mathrm{Sp}\,\mathcal{R}}^{\oplus r} \xrightarrow{\sim} \mathcal{O}_{\mathrm{Sp}\,\mathcal{R}}^{\oplus r}$ followed by multiplication by $A_{d,r}$, with

$$A_{d,r} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ z^{-d} & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_r(\mathcal{O}_{\mathrm{Sp}\,\mathcal{R}}).$$

We then get the desired classification of F-modules over $\mathcal{R}$ as follows:

Theorem 4.2.7 ([HP04, Thm. 11.1]). *Every F-module over $\mathcal{R}$ is isomorphic to one of the form $\oplus_{i=1}^k \dot{\underline{M}}_{d_i,r_i}$, where d_i, r_i are integers such that $r_i > 0$ and $\gcd(d_i, r_i) = 1$ for $i = 1, \ldots, k$.*

4.2.2 From dual Anderson A-motives to F-modules

Let us now define a functor $\dot{\mathcal{F}}$ from the category $\mathscr{PDA}^I$ of pure dual Anderson A-motives up to isogeny over $\mathbb{C}_\infty$ to the category $\mathscr{F}_{\dot{D}^\infty_{\mathbb{C}_\infty}}$ of F-modules over $\dot{D}^\infty_{\mathbb{C}_\infty}$ by assigning such an F-module over $\dot{D}^\infty_{\mathbb{C}_\infty}$ with a pure dual Anderson A-motive $\underline{\mathrm{M}}$ over $\mathbb{C}_\infty$.

Suppose $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}})$ is a pure dual Anderson A-motive of rank r, dimension d and weight $\frac{l}{n}$. By definition of purity there is a locally free sheaf $\mathcal{M}$ of rank r on $\mathbb{P}^1_{\mathbb{C}_\infty}$ such that $\Gamma(\mathbb{A}^1_{\mathbb{C}_\infty}, \mathcal{M}|_{\mathbb{A}^1_{\mathbb{C}_\infty}}) = \mathrm{M}$ and

$$\sigma^n_{\mathrm{M}} := \sigma_{\mathrm{M}} \circ \varsigma^*(\sigma_{\mathrm{M}}) \circ \ldots \circ (\varsigma^*)^{n-1}\sigma_{\mathrm{M}} : (\varsigma^*)^n\mathrm{M} \to \mathrm{M}$$

induces an isomorphism

$$(\varsigma^*)^n\mathcal{M}_{\infty_{\mathbb{C}_\infty}} \to \mathcal{M}(l \cdot \infty_{\mathbb{C}_\infty})_{\infty_{\mathbb{C}_\infty}}$$

of the stalks of $\mathcal{M}$ at $\infty_{\mathbb{C}_\infty}$. By definition, we have $\mathrm{supp}(\mathrm{coker}\,\sigma_{\mathrm{M}}) = \theta$, so that

$$\mathrm{supp}(\mathrm{coker}\,\sigma^n_{\mathrm{M}}) = \{t = \theta^{q^{-i}} \mid i = 0, \ldots n-1\},$$

and $\sigma^n_{\mathcal{M}} : (\varsigma^*)^n\mathcal{M} \to \mathcal{M}(l \cdot \infty_{\mathbb{C}_\infty})$ is an isomorphism on $\mathbb{P}^1_{\mathbb{C}_\infty} \setminus \{t = \theta^{q^{-i}} \mid i = 0, \ldots n-1\}$. Hence, by allowing a pole at $z = 0$, σ_{M} induces an isomorphism

$$\varsigma^*\mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty\langle \frac{z}{\zeta^{q^n}} \rangle[z^{-1}] \xrightarrow{\sim} \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty\langle \frac{z}{\zeta^{q^n}} \rangle[z^{-1}],$$

where $z := \frac{1}{t}$ is a local parameter at $\infty_{\mathbb{C}_\infty}$. We apply $(\varsigma^*)^{-1} = F^*$ to its inverse and obtain an isomorphism $\tau_M : F^*M \xrightarrow{\sim} M$ where we define M to be the locally free coherent sheaf of $\mathcal{O}_{\mathrm{Sp}\,\mathbb{C}_\infty\langle\frac{z}{\zeta^{q^n}}\rangle[z^{-1}]}$-modules with global sections

$$\mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty\langle\frac{z}{\zeta^{q^n}}\rangle[z^{-1}].$$

We call (M, τ_M) the *F-module over $\mathbb{C}_\infty\langle\frac{z}{\zeta^{q^n}}\rangle[z^{-1}]$ associated with* $\underline{\mathrm{M}}$. Further, let $\underline{\dot{M}}$ be the corresponding F-module over $\dot{D}^\infty_{\mathbb{C}_\infty}$ by [Har10, Prop. 1.4.1].

Definition 4.2.8. We let $\dot{\mathcal{F}} : \mathscr{PDA}^I \to \mathscr{F}_{\dot{D}^\infty_{\mathbb{C}_\infty}}$ be the functor that sends a pure dual Anderson A-motive $\underline{\mathrm{M}}$ over $\mathbb{C}_\infty$ to the F-module $\underline{\dot{M}}$ over $\dot{D}^\infty_{\mathbb{C}_\infty}$ as defined above.

The purity condition gives us additional information at $\infty_{\mathbb{C}_\infty}$ that amounts to the following assertion on the classification of $\underline{\dot{M}}$.

Proposition 4.2.9. *Let $\underline{\mathrm{M}}$ be a pure dual Anderson A-motive of rank r, dimension d and weight $\frac{l}{n}$ over $\mathbb{C}_\infty$ with $\gcd(l,n) = 1$. Then $\underline{\dot{M}} := \dot{\mathcal{F}}(\underline{\mathrm{M}})$ is isomorphic to $\underline{\dot{M}}_{-l,n}^{\oplus r/n}$.*

Proof. Suppose $\underline{\dot{M}} = (\dot{M}, \tau_{\dot{M}})$. By Theorem 4.2.7 there is a decomposition into a direct sum $\oplus_{i=1}^m \underline{\dot{M}}_{d_i,r_i}$ with integers d_i, r_i such that $r_i > 0$ and $\gcd(d_i, r_i) = 1$ for $i = 1, \ldots, m$.

Recall, we represent the isomorphism $\tau_{\dot{M}_{d_i,r_i}}$ underlying $\underline{\dot{M}}_{d_i,r_i}$ by choosing a suitable basis for $\dot{M}$ as the isomorphism $F^*\mathcal{O}^{\oplus r_i}_{\dot{\mathfrak{D}}^\infty_{\mathbb{C}_\infty}} \xrightarrow{\sim} \mathcal{O}^{\oplus r_i}_{\dot{\mathfrak{D}}^\infty_{\mathbb{C}_\infty}}$ followed by multiplication by A_{d_i,r_i}, where

$$A_{d_i,r_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ z^{-d_i} & 0 & \cdots & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_{r_i}(\mathcal{O}_{\dot{\mathfrak{D}}^\infty_{\mathbb{C}_\infty}}).$$

Notice that $A^{r_i}_{d_i,r_i} = z^{-d_i} \cdot \mathbf{1}_{r_i}$ where $\mathbf{1}_{r_i}$ denotes the identity matrix with r_i columns and r_i rows. Observe that by the purity condition, $\dot{M}$ extends to a locally free coherent sheaf $\dot{\mathcal{M}}$ of rank r over $\mathfrak{D}^\infty_{\mathbb{C}_\infty}$, with $\tau_{\dot{\mathcal{M}}} : F^*\dot{\mathcal{M}} \to \dot{\mathcal{M}}$, such that

$$\tau^n_{\dot{M}} := \tau_{\dot{M}} \circ F^*(\tau_{\dot{M}}) \circ \ldots \circ (F^*)^{n-1}(\tau_{\dot{M}}) : (F^*)^n\dot{M} \to \dot{M}$$

induces an isomorphism

$$\tau^n_{\dot{\mathcal{M}}} : (F^*)^n\dot{\mathcal{M}}_{\infty_{\mathbb{C}_\infty}} \xrightarrow{\sim} \dot{\mathcal{M}}(-l\cdot\infty)_{\infty_{\mathbb{C}_\infty}}$$

of the stalks of $\dot{\mathcal{M}}$ at $\infty_{\mathbb{C}_\infty}$. We can balance out the pole of degree $-l$ at $\infty_{\mathbb{C}_\infty}$ by multiplying $\tau^n_{\dot{\mathcal{M}}}$ with z^{-l}. Thus $(z^{-l}\tau^n_{\dot{\mathcal{M}}})^{r_i}$ extends to a well-defined isomorphism $(F^*)^{nr_i}\dot{\mathcal{M}} \xrightarrow{\sim} \dot{\mathcal{M}}$ at $\infty_{\mathbb{C}_\infty}$, implying the same for

$$(z^{-l}\tau^n_{\dot{M}_{d_i,r_i}})^{r_i} = z^{-lr_i}(\tau^{r_i}_{\dot{M}_{d_i,r_i}})^n = z^{-lr_i}z^{-nd_i} = z^{-lr_i-nd_i} \quad \text{for } i = 1, \ldots, m.$$

Therefore the exponent $-lr_i - nd_i$ must be equal to zero. Since $\gcd(d_i, r_i) = 1$ and $r_i > 0$ we conclude that $d_i = -l$ and $r_i = n$ for $i = 1, \ldots, m$. As the ranks of $\underline{\dot{M}}$ and $\underline{\dot{M}}_{-l,n}$ are r and n, respectively, we obtain the desired decomposition $\underline{\dot{M}} \cong \underline{\dot{M}}_{-l,n}^{\oplus r/n}$. □

4.2.3 From sub-Q-Hodge-Pink structures to dual Anderson sub-A-motives

Clearly there are pure Q-Hodge-Pink structures that do not come from pure rigid analytically trivial dual Anderson A-motives over $\mathbb{C}_\infty$. In this subsection, we fix such a dual Anderson A-motive $\underline{\mathrm{M}} \in \mathrm{Ob}(\mathscr{PRDA}_+^I)$ of rank r, dimension d and weight $\frac{l}{n}$ over $\mathbb{C}_\infty$ and denote its associated pure Q-Hodge-Pink structure $\mathcal{D}(\underline{\mathrm{M}})$ by $\underline{H}$. We prove that in the special case that $\underline{H}'$ is a strict pure sub-Q-Hodge-Pink structure of $\underline{H}$ of weight $\frac{d'}{r'}$ in the Tannakian category $\langle\langle \underline{H} \rangle\rangle$ generated by $\underline{H}$, there is a pure rigid analytically trivial dual sub-Anderson A-motive $\underline{\mathrm{M}}'$ of rank r', dimension d' and weight $\frac{d'}{r'}$ such that $\mathcal{D}(\underline{\mathrm{M}}') = \underline{H}'$.

We assume without loss of generality that $(H', W', \mathfrak{q}_{H'}) = \underline{H}' \subseteq \underline{H} := (H, W, \mathfrak{q}_H)$ is a pure sub-Q-Hodge-Pink structure of rank r' and weight $\frac{d'}{r'}$. The σ-invariants of the desired dual Anderson A-motive $\underline{\mathrm{M}}'$ must be $\mathrm{M}'(1)^\sigma := H' \cap \mathrm{M}(1)^\sigma$. This definition makes sense since if $\{\mathrm{m}_1, \ldots, \mathrm{m}_r\}$ generates $\mathrm{M}(1)^\sigma$ as an A-module, $\{\mathrm{m}_1, \ldots, \mathrm{m}_r\}$ is a Q-basis for H. So if a subset of $\{\mathrm{m}_1, \ldots, \mathrm{m}_r\}$ generates $\mathrm{M}'(1)^\sigma = H' \cap \mathrm{M}(1)^\sigma$ as an A-module, it also provides an Q-basis for H'.

Denote the rigid σ-sheaf over $A(\infty)$ on $\mathrm{Sp}\,\mathbb{C}_\infty$ obtained from the dual Anderson A-motive $\underline{\mathrm{M}}$ by $\underline{\mathcal{F}}_{\mathrm{M}}^{\mathrm{rig}} = (\mathcal{F}_{\mathrm{M}}^{\mathrm{rig}}, \sigma_{\mathcal{F}_{\mathrm{M}}^{\mathrm{rig}}})$. We first want to construct $\underline{\mathcal{F}}_{\mathrm{M}'}^{\mathrm{rig}}$ as a rigid sub-σ-sheaf of $\underline{\mathcal{F}}_{\mathrm{M}}^{\mathrm{rig}}$ over $A(\infty)$ on $\mathrm{Sp}\,\mathbb{C}_\infty$.

In order to do this, consider the locally free coherent sheaf $\mathcal{G}^{\mathrm{rig}}$ of $\mathcal{O}_{\mathbb{A}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$-modules of rank r' with global sections $\mathrm{M}'(1)^\sigma \otimes_A A(\infty)$ together with the $\mathcal{O}_{\mathbb{A}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$-module homomorphism $\sigma_{\mathcal{G}^{\mathrm{rig}}} := \mathrm{id} \otimes \varsigma^* : \varsigma^* \mathcal{G}^{\mathrm{rig}} \to \mathcal{G}^{\mathrm{rig}}$, which defines a rigid σ-sheaf $\underline{\mathcal{G}}^{\mathrm{rig}}$ of rank r' over $A(\infty)$ on $\mathrm{Sp}\,\mathbb{C}_\infty$. Moreover, we define an isomorphism

$$\eta_i := (\sigma_{\mathcal{G}^{\mathrm{rig}}} \circ \ldots \circ (\varsigma^*)^{i-1} \sigma_{\mathcal{G}^{\mathrm{rig}}}) \otimes \mathrm{id} : (\varsigma^*)^i \mathcal{G}^{\mathrm{rig}} \otimes \mathbb{C}_\infty[\![y_i]\!][\frac{1}{y_i}] \xrightarrow{\sim} \mathcal{G}^{\mathrm{rig}} \otimes \mathbb{C}_\infty[\![y_i]\!][\frac{1}{y_i}]$$

with $y_i := z - \zeta^{q^i}$. We put $x := \prod_{i \in \mathbb{N}} \left(1 - \frac{\zeta^{q^i}}{z}\right)$ and let $\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}$ be the $\mathcal{O}_{\mathbb{A}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$-submodule of $\mathcal{G}^{\mathrm{rig}}[x^{-1}]$ which coincides with $\mathcal{G}^{\mathrm{rig}}$ outside $z = \zeta^{q^i}$, $i > 0$, and at $z = \zeta^{q^i}$ satisfies

$$\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}} \otimes \mathbb{C}_\infty[\![z - \zeta^{q^i}]\!] = \eta_i^{-1}((F^*)^i(\mathfrak{q}_{H'})) \subseteq \eta_i^{-1}((F^*)^i(\mathfrak{p}_{H'})) = \mathcal{G}^{\mathrm{rig}} \otimes \mathbb{C}_\infty[\![z - \zeta^{q^i}]\!]$$

together with the $\mathcal{O}_{\mathbb{A}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$-module homomorphism $\sigma_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}} : \varsigma^* \mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}} \to \mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}$ induced by $\sigma_{\mathcal{G}^{\mathrm{rig}}}$. Note $\sigma_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}}$ is well-defined by construction. We have thus constructed a rigid sub-σ-sheaf $\underline{\mathcal{F}}_{\mathrm{M}'}^{\mathrm{rig}} := (\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}, \sigma_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}})$ of rank r' over $A(\infty)$ on $\mathrm{Sp}\,\mathbb{C}_\infty$, as desired.

Definition 4.2.10. We call $(\underline{\mathcal{F}}_{\mathrm{M}'}^{\mathrm{rig}}, \underline{\mathcal{G}}^{\mathrm{rig}})$ the *pair of rigid σ-sheaves over* $A(\infty)$ *on* $\mathrm{Sp}\,\mathbb{C}_\infty$ *associated with* $\underline{H}'$.

We want to add the purity condition, and therefore extend $\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}$ to a locally free coherent sheaf $\mathcal{M}'^{\mathrm{rig}}$ of $\mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$-modules on all of $\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}$ such that

$$\sigma^n_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}} := \sigma_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}} \circ \varsigma^*(\sigma_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}}) \circ \ldots \circ (\varsigma^*)^{n-1} \sigma_{\mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}} : (\varsigma^*)^n \mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}} \to \mathcal{F}_{\mathrm{M}'}^{\mathrm{rig}}$$

induces an isomorphism

$$\sigma^n_{\mathcal{M}'^{\mathrm{rig}}} : (\varsigma^*)^n \mathcal{M}'^{\mathrm{rig}}{}_{\infty_{\mathbb{C}_\infty}} \to \mathcal{M}'^{\mathrm{rig}}(l \cdot \infty)_{\infty_{\mathbb{C}_\infty}} := (\mathcal{M}'^{\mathrm{rig}} \otimes_{\mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}} \mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}(l \cdot \infty))_{\infty_{\mathbb{C}_\infty}}$$

of the stalks of $\mathcal{M}'^{\mathrm{rig}}$ at $\infty_{\mathbb{C}_\infty}$. We then apply the rigid analytic GAGA principle twice to find the desired dual Anderson sub-A-motive $(\mathrm{M}', \sigma_{\mathrm{M}'})$.

In order to do this, we observe that $\sigma_{\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}'}}$ induces an isomorphism of $\dot{D}(\zeta^{q^n})^{\infty}_{\mathbb{C}_\infty}$-modules by construction. Applying F^* to this isomorphism, we obtain a $\dot{D}(\zeta^{q^n})^{\infty}_{\mathbb{C}_\infty}$-module homomorphism $\tau_{M'} : F^*M' \xrightarrow{\sim} M'$, where we set

$$M' := \Gamma(\mathfrak{A}(\infty), \mathcal{F}^{\mathrm{rig}}_{\mathrm{M}'}) \otimes_{A(\infty)} \dot{D}(\zeta^{q^n})^{\infty}_{\mathbb{C}_\infty}.$$

We call $(M', \tau_{M'})$ the *F-module over $\dot{D}(\zeta^{q^n})^{\infty}_{\mathbb{C}_\infty}$ associated with $\underline{H}'$*. Let $\underline{\dot{M}}'$ be the corresponding F-module over $\dot{D}^{\infty}_{\mathbb{C}_\infty}$ by [Har10, Prop. 1.4.1].

Similarly, we construct an F-module $\underline{\dot{N}}' = (\dot{N}', \tau_{\dot{N}'})$ over $\dot{D}^{\infty}_{\mathbb{C}_\infty}$ from the rigid σ-sheaf $\underline{\mathcal{G}}^{\mathrm{rig}}$ over $A(\infty)$ on $\operatorname{Sp}\mathbb{C}_\infty$.

Definition 4.2.11. We call $(\underline{\dot{M}}', \underline{\dot{N}}')$ the *pair of F-modules over $\dot{D}^{\infty}_{\mathbb{C}_\infty}$ associated with $\underline{H}'$*.

Lemma 4.2.12. *Let $(\underline{\dot{M}}', \underline{\dot{N}}')$ be the pair of F-modules over $\dot{D}^{\infty}_{\mathbb{C}_\infty}$ associated with $\underline{H}'$. Then $\underline{\dot{M}}'$ is isomorphic to $\underline{\dot{M}}'^{\oplus r'/n}_{-l,n}$.*

Proof. From Theorem 4.2.7 we obtain $\underline{\dot{M}}' \cong \oplus_{i=1}^m \underline{\dot{M}}'_{d'_i, r'_i}$ with integers d'_i, r'_i such that $r'_i > 0$ and $\gcd(d'_i, r'_i) = 1$ for $i = 1, \dots, m$.

Further, let $\underline{\dot{M}} \cong \dot{\mathcal{F}}(\underline{\mathrm{M}})$ be the F-module over $\dot{D}^{\infty}_{\mathbb{C}_\infty}$ associated with $\underline{\mathrm{M}}$, so that $\underline{\dot{M}} \cong \underline{\dot{M}}^{\oplus r/n}_{-l,n}$ by Proposition 4.2.9. From the assumption $(\mathrm{M}'(1)^\sigma, \mathfrak{q}_{H'}) \subseteq (\mathrm{M}(1)^\sigma, \mathfrak{q}_H)$ it follows that $\underline{\dot{M}}'$ is a sub-F-module over $\dot{D}^{\infty}_{\mathbb{C}_\infty}$ of $\underline{\dot{M}}$. Thus $d'_i \leq -\frac{l}{n} r'_i$ by [HP04, Prop. 8.5].

Since $\underline{H}'$ is supposed to be a strict subobject of $\underline{H}$ that is pure of weight $\frac{d}{r}$, $\frac{d'}{r'} = \frac{d}{r}$ must be satisfied. By construction, $\dot{M}' \subseteq \dot{N}'$ as $\dot{D}^{\infty}_{\mathbb{C}_\infty}$-submodules of $\dot{N}'[x^{-1}]$. Further,

$$(\dot{N}'/\dot{M}') \otimes_{\dot{D}^{\infty}_{\mathbb{C}_\infty}} \mathbb{C}_\infty[\![z-\zeta]\!] = \mathrm{M}(1)^\sigma \otimes_A \mathbb{C}_\infty[\![t-\theta]\!]/\mathfrak{q}_{H'} = \mathfrak{p}_{H'}/\mathfrak{q}_{H'} \cong \bigoplus_{i=1}^m \mathbb{C}_\infty[\![z-\zeta]\!]/(z-\zeta)^{-w_i}$$

where $w_1 \leq \ldots \leq w_m$ are the Hodge-Pink weights. Thus $\dot{N}'/\dot{M}'$ is supported at $z = \zeta^{q^i}, i \in \mathbb{Z}$, and has length $\sum_{i=1}^m w_i = d'$. Since the rank of $\underline{\dot{M}}'$ and $\underline{\dot{N}}'$ is r', we have an injection $i : \bigwedge^{r'} \underline{\dot{M}}' \hookrightarrow \bigwedge^{r'} \underline{\dot{N}}' \cong \underline{\mathcal{O}(0)}$ of sub-F-modules of rank 1 and therefore an isomorphism

$$(\mathcal{O}_{\dot{D}^{\infty}_{\mathbb{C}_\infty}} / \bigwedge^{r'} \dot{M}') \otimes_{\dot{D}^{\infty}_{\mathbb{C}_\infty}} \mathbb{C}_\infty[\![z-\zeta]\!] \cong \bigwedge^{r'} \mathfrak{p}_{H'} / \bigwedge^{r'} \mathfrak{q}_{H'}.$$

Hence, $\bigwedge^{r'} \dot{M} \overset{i}{\hookrightarrow} \mathcal{O}_{\dot{D}^{\infty}_{\mathbb{C}_\infty}}$ is the ideal sheaf of a F^*-invariant divisor Δ on $\dot{\mathfrak{D}}^{\infty}_{\mathbb{C}_\infty}$ supported on $\{z = \zeta^{q^i}, i \in \mathbb{Z}\}$ of length d'. Thus we see that for $1 \leq k < q^n$

$$(\mathcal{O}_{\dot{D}^{\infty}_{\mathbb{C}_\infty}} / \bigwedge^{r'} \dot{M}') \otimes_{\dot{D}^{\infty}_{\mathbb{C}_\infty}} \mathcal{O}_{\mathfrak{A}(\zeta^{q^n}, \zeta^k)}$$

is supported at $z = \zeta$ and we find that $\zeta \in \dot{D}^{\infty}_{\mathbb{C}_\infty}$ is a representative with multiplicity d'. By [Har10, Prop. 1.4.4] there is a function $f_\zeta \in \dot{D}^{\infty}_{\mathbb{C}_\infty}$ that has only zeros of order one at ζ^{q^i} for all $i \in \mathbb{Z}$. Therefore, Δ is also the divisor of $f_\zeta^{d'}$ so that multiplication by $f_\zeta^{d'}$ induces an isomorphism

$$\mathcal{O}(0) = \mathcal{O}_{\dot{D}^{\infty}_{\mathbb{C}_\infty}} \xrightarrow{\sim} \bigwedge^{r'} \dot{M}' \subseteq \mathcal{O}(0)$$

of the underlying vector bundles. Since $f_\zeta^{d'} \in \mathcal{O}(d')^\tau(\mathbb{C}_\infty)$, twisting with $\mathcal{O}(d')$ gives us:

$$\underline{\mathcal{O}(0)} \xrightarrow{\sim} \bigwedge^{r'} \underline{\dot{M}}'(d') \subseteq \underline{\mathcal{O}(d')},$$

so that $\underline{\mathcal{O}(-d')} \cong \bigwedge^{r'} \underline{\dot{M}}'$. Therefore $\deg(\underline{\dot{M}}') = -d'$. Then

$$-d' = \deg(\underline{\dot{M}}') = \sum d_i' \leq -\sum r_i' \cdot \frac{l}{n} = -r' \cdot \frac{d}{r} = -d'$$

By hypothesis $\gcd(d_i', r_i') = 1$ and $r_i' > 0$, so that $d_i = -l$ and $r_i = n$ for all i. This proves $\underline{\dot{M}}' \cong \underline{\dot{M}}'^{\oplus r'/n}_{-l,n}$ where r' is the rank of $\underline{\dot{M}}'$ by construction. □

Proposition 4.2.13. *Let $\underline{\mathrm{M}} \in \mathrm{Ob}(\mathscr{PRDA}_+^I)$ be a pure rigid analytically trivial Anderson A-motive over $\mathbb{C}_\infty$ of rank r, dimension d and weight $\frac{l}{n}$ and $\underline{H} := \mathcal{D}(\underline{\mathrm{M}})$ its associated pure Hodge-Pink structure. Suppose $\underline{H}'$ is a strict subobject of $\underline{H}$ in $\langle\langle \underline{H} \rangle\rangle$. Then there is a pure rigid analytically trivial dual Anderson sub-A-motive $\underline{\mathrm{M}}'$ of rank r', dimension d' and weight $\frac{l}{n}$ such that $\mathcal{D}(\underline{\mathrm{M}}') = \underline{H}'$.*

Proof. Let $(\underline{\dot{M}}', \underline{\dot{N}}')$ be the pair of F-modules over $\dot{D}^\infty_{\mathbb{C}_\infty}$ associated with $\underline{H}'$ such that we have $\underline{\dot{M}}' = (\dot{M}', \tau_{\dot{M}'}) \cong \underline{\dot{M}}'^{\oplus r'/n}_{-l,n}$ by the previous lemma. We may apply $\varsigma^* = (F^*)^{-1}$ to the inverse of $\tau_{\dot{M}'} : F^*\dot{M}' \xrightarrow{\sim} \dot{M}'$ and extend $\dot{M}'$ to a locally free coherent sheaf $\tilde{\mathcal{M}}'$ on $\mathfrak{D}^\infty_{\mathbb{C}_\infty}$ together with a homomorphism $\sigma_{\tilde{\mathcal{M}}'} : \varsigma^*\tilde{\mathcal{M}}' \to \tilde{\mathcal{M}}'(l \cdot \infty_{\mathbb{C}_\infty})$, which satisfies

$$\sigma^n_{\tilde{\mathcal{M}}'} : (\varsigma^*)^n \tilde{\mathcal{M}}'_{\infty_{\mathbb{C}_\infty}} \xrightarrow{\sim} \tilde{\mathcal{M}}'(l \cdot \infty_{\mathbb{C}_\infty})$$

at the stalks of $\tilde{\mathcal{M}}'$ at $\infty_{\mathbb{C}_\infty}$.

Further, let $(\underline{\mathcal{F}}^{\mathrm{rig}}_{\mathrm{M}'}, \underline{\mathcal{G}}^{\mathrm{rig}})$ be the pair of rigid σ-sheaves over $A(\infty)$ on $\mathrm{Sp}\,\mathbb{C}_\infty$ associated with $\underline{H}'$. We glue $\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}'}$ and $\tilde{\mathcal{M}}'$ to a locally free coherent sheaf on all of $\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}$ that we denote by $\mathcal{M}'^{\mathrm{rig}}$ and which satisfies $\mathcal{M}'^{\mathrm{rig}} \subseteq \mathcal{M}^{\mathrm{rig}}$ by construction. By Theorem 1.3.1, there is an algebraic locally free coherent sheaf $\mathcal{M}' \subseteq \mathcal{M}$ on $\mathbb{P}^1_{\mathbb{C}_\infty}$ and an algebraic homomorphism $\sigma_{\mathcal{M}'} : \varsigma^*\mathcal{M}' \to \mathcal{M}'(l \cdot \infty_{\mathbb{C}_\infty})$ that induces an isomorphism $\sigma^n_{\mathcal{M}'} : (\varsigma^*)^n \mathcal{M}'_{\infty_{\mathbb{C}_\infty}} \xrightarrow{\sim} \mathcal{M}'(l \cdot \infty_{\mathbb{C}_\infty})_{\infty_{\mathbb{C}_\infty}}$ of the stalks at $\infty_{\mathbb{C}_\infty}$. We consider the locally free coherent $\mathbb{C}_\infty[t]$-module $\mathrm{M}' := \Gamma(\mathbb{A}^1_{\mathbb{C}_\infty}, \mathcal{M}'|_{\mathbb{A}^1_{\mathbb{C}_\infty}}) \subseteq \mathrm{M}$ together with the induced $A_{\mathbb{C}_\infty}$-homomorphism $\sigma_{\mathrm{M}'} : \varsigma^*\mathrm{M}' \to \mathrm{M}'$, which exists by Theorem 1.3.1 (i). This gives us the desired dual sub-Anderson A-motive $\underline{\mathrm{M}}' = (\mathrm{M}', \sigma_{\mathrm{M}'})$ of $\underline{\mathrm{M}}$ such that $\mathcal{D}(\underline{\mathrm{M}}') = \underline{H}'$. □

4.2.4 The map μ is an isomorphism

We fix a pure dual t-motive $\underline{P}$ of rank r and weight $\frac{n}{l}$ over $\mathbb{C}_\infty$. Let $\underline{H} := \mathcal{T}(\underline{P})$ be the pure Q-Hodge-Pink structure assigned to $\underline{P}$. Consider the Q-group scheme homomorphism $\mu : G_{\underline{H}} \to \Gamma_{\underline{P}}$ from the Hodge-Pink group of $\underline{H}$ to the Galois group of $\underline{P}$ that we constructed in the previous section. We first prove that μ is faithfully flat through the equivalent conditions from Proposition 1.2.15 (i). Next we show that μ is a closed immersion. As desired, we then conclude that the Hodge-Pink group $G_{\underline{H}}$ and the Galois group $\Gamma_{\underline{P}}$ are isomorphic over Q.

Proposition 4.2.14. *Let $\underline{R}$ be a pure dual t-motive in $\langle\langle \underline{P} \rangle\rangle$. Then for each subobject $\underline{H}'$ of $\mathcal{T}(\underline{R})$ in $\langle\langle \underline{H} \rangle\rangle$ there exists a pure dual sub-t-motive $\underline{R}'$ of $\underline{R}$ such that $\underline{H}' = \mathcal{T}(\underline{R}')$.*

Proof. Suppose $\underline{R} = (\underline{\mathrm{N}}, i) \in \mathrm{Ob}(\mathscr{P}\mathscr{T}')$ so that $\underline{H}'$ is a subobject of

$$\mathcal{T}(\underline{R}) = \mathcal{D}(\underline{\mathrm{N}}) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-i}.$$

Hence, $\underline{H}' \otimes \mathcal{D}(\underline{\mathrm{C}})^i$ is a pure strict sub-Q-Hodge-Pink structure of $\mathcal{D}(\underline{\mathrm{N}})$. By Proposition 4.2.13, there is a pure rigid analytically trivial dual Anderson sub-A-motive $\underline{\mathrm{N}}'$ of $\underline{\mathrm{N}}$ such that $\mathcal{D}(\underline{\mathrm{N}}') = \underline{H}' \otimes \mathcal{D}(\underline{\mathrm{C}})^i$. We put $\underline{R}' := (\underline{\mathrm{N}}', i) \subset (\underline{\mathrm{N}}, i) = \mathrm{N}$ so then, as desired,

$$\mathcal{T}(\underline{R}') = \mathcal{D}(\underline{\mathrm{N}}') \otimes \mathcal{D}(\underline{\mathrm{C}})^{-i} \cong \underline{H}' \subseteq \mathcal{D}(\underline{\mathrm{N}}) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-i} = \mathcal{T}(\underline{R}). \qquad \square$$

Proposition 4.2.15. *The functor* $\mathcal{T}_{\underline{P}} : \langle\langle \underline{P} \rangle\rangle \to \langle\langle \underline{H} \rangle\rangle$ *is fully faithful.*

Proof. Let $\underline{\mathrm{M}}_1(i_1)$, and $\underline{\mathrm{M}}_2(i_2)$ be pure dual t-motives in $\langle\langle \underline{P} \rangle\rangle$. We want to show that to each $g \in \mathrm{Hom}_Q(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2)))$ there is a unique $f \in \mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2))$. By definition of the functor $\mathcal{D}$ we then find that the inner hom

$$\begin{aligned}
\mathcal{T}(\mathcal{H}om(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2))) &= \mathcal{T}(\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2})(N)) \\
&= \mathcal{D}(\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2})) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-N} \\
&\cong \mathcal{H}om(\mathcal{D}(\underline{\mathrm{M}}_1), \mathcal{D}(\underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2})) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-N} \\
&\cong \mathcal{H}om(\mathcal{D}(\underline{\mathrm{M}}_1) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-i_1}, \mathcal{D}(\underline{\mathrm{M}}_2) \otimes \mathcal{D}(\underline{\mathrm{C}})^{-i_2}) \\
&\cong \mathcal{H}om(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2))),
\end{aligned}$$

with N chosen sufficiently large, satisfies by the adjunction formula (1.3):

$$\begin{aligned}
\mathrm{Hom}_Q(\mathbb{1}_Q, \mathcal{H}om(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2)))) &\cong \mathrm{Hom}_Q(\mathbb{1}_Q \otimes \mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2))) \\
&\cong \mathrm{Hom}_Q(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2))).
\end{aligned}$$

Then $g \in \mathrm{Hom}_Q(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2)))$ corresponds under these isomorphisms to a unique strict subobject:

$$\underline{\mathbb{1}}_Q \hookrightarrow \mathcal{H}om(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2))) = \mathcal{T}(\mathcal{H}om(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2))), \quad \mathbb{1}_Q \mapsto g.$$

Using Proposition 4.2.13, there is a pure rigid analytically trivial dual Anderson sub-A-motive

$$\underline{\mathrm{M}}' \otimes \underline{\mathrm{C}}^N \hookrightarrow \mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2})$$

that satisfies $\mathcal{D}(\underline{\mathrm{M}}' \otimes \underline{\mathrm{C}}^N) = \mathcal{D}(\underline{\mathrm{M}}') \otimes \mathcal{D}(\underline{\mathrm{C}}^N) = \underline{\mathbb{1}}_Q \otimes \mathcal{D}(\underline{\mathrm{C}})^N \hookrightarrow \mathcal{D}(\mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2}))$. Then $\underline{\mathrm{M}}' = (\mathrm{M}, \sigma_{\mathrm{M}})$ must be a dual Anderson A-motive of rank 1 with $\mathrm{M}'(1)^\sigma = A$ such that σ_{M} extends to an isomorphism on all of $\mathbb{P}^1_{\mathbb{C}_\infty}$. We let $(\underline{\mathcal{F}}^{\mathrm{rig}}_{\mathrm{M}'}, \underline{\mathcal{G}}^{\mathrm{rig}})$ be the pair of rigid σ-sheaves over $A(\infty)$ on $\mathrm{Sp}\,\mathbb{C}_\infty$ associated with $\underline{\mathbb{1}}_Q$ (Definition 4.2.10). Taking a closer look at the construction of $(\underline{\mathcal{F}}^{\mathrm{rig}}_{\mathrm{M}'}, \underline{\mathcal{G}}^{\mathrm{rig}})$, we find that both $\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}'}$ and $\mathcal{G}^{\mathrm{rig}}$ have global sections $\mathrm{M}'(1)^\sigma \otimes_A A(\infty)$ and $\sigma_{\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}'}} = \sigma_{\mathcal{G}^{\mathrm{rig}}}$ holds because $\mathfrak{q}_{\underline{\mathbb{1}}_Q} = \mathfrak{p}_{\underline{\mathbb{1}}_Q}$. Since $\underline{\mathbb{1}}_Q$ is pure of weight 0, we may extend $\mathcal{F}^{\mathrm{rig}}_{\mathrm{M}'}$ to the locally free coherent sheaf $\mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$ on $\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}$ together with the natural isomorphism $\sigma_{\mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}} : \varsigma^* \mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}} \to \mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$. Applying the rigid analytic GAGA principle to $\mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}$ and $\sigma_{\mathcal{O}_{\mathbb{P}^{1,\mathrm{rig}}_{\mathbb{C}_\infty}}}$ as in the proof of Proposition 4.2.13, we conclude that $\underline{\mathrm{M}}'$ coincides with the pure dual Anderson A-motive $\underline{\mathbb{1}}_{A_k}$ of weight 0.

By twisting with the dual Tate t-motive, we obtain a pure dual sub-t-motive

$$\underline{\mathbb{1}}_{A_k}(0) \hookrightarrow \mathcal{H}om(\underline{\mathrm{M}}_1, \underline{\mathrm{M}}_2 \otimes \underline{\mathrm{C}}^{N+i_1-i_2})(N) = \mathcal{H}om(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2))$$

such that $\mathcal{T}(\mathbb{1}_{A_k}(0)) = \mathbb{1}_Q \hookrightarrow \mathcal{H}om(\mathcal{T}(\underline{\mathrm{M}}_1(i_1)), \mathcal{T}(\underline{\mathrm{M}}_2(i_2)))$. This corresponds under the isomorphisms given by the adjunction formula:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\mathbb{1}_{A_k}(0), \mathcal{H}om(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2))) &\cong \mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\mathbb{1}_{A_k}(0) \otimes \underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2)) \\ &\cong \mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2))\end{aligned}$$

to an $f \in \mathrm{Hom}_{\mathscr{P}\mathscr{T}'}(\underline{\mathrm{M}}_1(i_1), \underline{\mathrm{M}}_2(i_2)$ such that $\mathcal{T}(f) = g$. □

By applying Proposition 1.2.15 (i) we get the following:

Corollary 4.2.16. *The Q-group scheme homomorphism $\mu : G_{\underline{H}} \to \Gamma_{\underline{P}}$ is faithfully flat.*

Proposition 4.2.17. *(i) There are closed immersions $G_{\underline{H}} \subseteq \mathrm{Cent}_{\mathrm{GL}(\varpi(\underline{H}))}(\mathrm{End}(\underline{H}))$ and $\Gamma_{\underline{P}} \subseteq \mathrm{Cent}_{\mathrm{GL}(\omega(\underline{P}))}(\mathrm{End}(\underline{P}))$.*

(ii) The Q-group scheme homomorphism $\mu : G_{\underline{H}} \to \Gamma_{\underline{P}}$ is a closed immersion.

Proof. (i) Each $g \in G_{\underline{H}}$ commutes with endomorphisms of $\underline{H}$ because the following diagram is commutative for all $f \in \mathrm{End}(\underline{H})$:

$$\begin{tikzcd} \varpi(\underline{H}) \arrow[r, "g_{\underline{H}}"] \arrow[d, "\varpi(f)"'] & \varpi(\underline{H}) \arrow[d, "\varpi(f)"] \\ \varpi(\underline{H}) \arrow[r, "g_{\underline{H}}"] & \varpi(\underline{H}). \end{tikzcd}$$

Similarly, each $\gamma \in \Gamma_{\underline{P}}$ commutes with endomorphisms of $\underline{P}$ as for all $f \in \mathrm{End}(\underline{P})$ there is a commutative diagram:

$$\begin{tikzcd} \omega(\underline{P}) \arrow[r, "\gamma_{\underline{P}}"] \arrow[d, "\omega(f)"'] & \omega(\underline{P}) \arrow[d, "\omega(f)"] \\ \omega(\underline{P}) \arrow[r, "\gamma_{\underline{P}}"] & \omega(\underline{P}). \end{tikzcd}$$

(ii) By (i), we have that $G_{\underline{H}} \subseteq \mathrm{Cent}_{\mathrm{GL}(\varpi(\underline{H}))}(\mathrm{End}(\underline{H}))$ is a closed Q-subgroup scheme of $\mathrm{GL}(\varpi(\underline{H}))$. By Proposition 4.1.33 we have $\mathrm{GL}(\omega(\underline{P})) \cong \mathrm{GL}(\varpi(\underline{H}))$ so that we may also view $\Gamma_{\underline{P}} \subseteq \mathrm{Cent}_{\mathrm{GL}(\omega(\underline{P}))}(\mathrm{End}(\underline{P}))$ as a closed Q-subgroup scheme of $\mathrm{GL}(\varpi(\underline{H}))$. By the following commutative diagram

$$\begin{tikzcd} G_{\underline{H}} \arrow[rr, "\mu"] \arrow[dr, hook] & & \Gamma_{\underline{P}} \arrow[dl, hook'] \\ & \mathrm{GL}(\varpi(\underline{H})), & \end{tikzcd}$$

we see that μ must also be a closed immersion. □

We observe that from Proposition 1.2.15 follows "surjectivity" of $\mathcal{T}_{\underline{P}}$ as follows.

Corollary 4.2.18. *Every Hodge structure $\underline{H}'$ in $\langle\langle \underline{H} \rangle\rangle$ is isomorphic to a subquotient of an object of the form $\mathcal{T}_{\underline{P}}(\underline{P}')$, where $\underline{P}'$ is a pure dual t-motive in $\langle\langle \underline{P} \rangle\rangle$.*

Since μ is a closed immersion and faithfully flat, we may state our main result.

Theorem 4.2.19 (Hodge conjecture for function fields)**.** *Let $\underline{P}$ be a pure dual t-motive over $\mathbb{C}_\infty$ and $\underline{H} := \mathcal{T}(\underline{P})$ its associated Q-Hodge-Pink structure. Then the Q-group scheme homomorphism $\mu : G_{\underline{H}} \to \Gamma_{\underline{P}}$ is an isomorphism. Equivalently, $\mathcal{T}_{\underline{P}} : \langle\langle \underline{P} \rangle\rangle \to \langle\langle \underline{H} \rangle\rangle$ is an equivalence of categories.*

5. GROTHENDIECK'S PERIOD CONJECTURE FOR FUNCTION FIELDS

Having shown the Hodge conjecture in the last chapter when $A = \mathbb{F}_q[t]$, we may combine it with [Pap08, Thm. 5.2.2]. This yields the analog of Grothendieck's period conjecture for abelian varieties.

Theorem 5.0.20 (Grothendieck's period conjecture for function fields)**.** *Let $\underline{\mathrm{M}}$ be a pure rigid analytically trivial Anderson A-motive of rank r over $\overline{Q} \subset \overline{Q_\infty}$ in $\mathscr{PRDA}^I_+$ and $G_{\underline{\mathrm{M}}}$ its associated Hodge-Pink group. Suppose that $\Phi \in \mathrm{GL}_r(\overline{Q}(t)) \cap \mathrm{Mat}_{r\times r}(\overline{Q}[t])$ represents σ_{M} with respect to a $k[t]$-basis for M. Then there is a rigid analytic trivialization Ψ of Φ in $\mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_{r\times r}(\mathbb{E})$ and*

$$\mathrm{tr.\,deg}_{\overline{Q}}\ \overline{Q}(\Psi(\theta)_{ij} \mid 1 \leq i, j \leq r) = \dim G_{\underline{\mathrm{M}}}.$$

Being interested in the transcendence degree of the entries $\Psi(\theta)_{i,j}$, defined as in the theorem above, we want to compute the Hodge-Pink group $G_{\underline{\mathrm{M}}}$ of a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ over $\overline{Q}$. As seen in Proposition 4.2.17, we have

$$G_{\underline{\mathrm{M}}} \cong \Gamma_{\underline{\mathrm{M}}} \subseteq \mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{QEnd}(\underline{\mathrm{M}})), \tag{5.1}$$

where $\Gamma_{\underline{\mathrm{M}}}$ is the Galois group of $\underline{\mathrm{M}}$.

In the first section, we study a pure rigid analytically trivial dual Anderson A-motive $\underline{\mathrm{M}}$ of rank r over $k \subset \mathbb{C}_\infty$ that has sufficiently many complex multiplication through a commutative semisimple Q-algebra $E \subseteq \mathrm{QEnd}(\underline{\mathrm{M}})$ with $\dim_Q E = r$. With the help of (5.1), we may show that $\dim G_{\underline{\mathrm{M}}} = \dim \mathfrak{R}_{E/Q}\mathbb{G}_{m,E} = r$ holds if E/Q is either separable or purely inseparable. This implies

$$\mathrm{tr.deg}_{\overline{Q}}\ \overline{Q}(\Psi(\theta)_{i,j} \mid 1 \leq i, j \leq r) = r, \tag{5.2}$$

where $\Psi(\theta)_{i,j}$ are defined as above. The second section deals with the case that $\underline{\mathrm{M}}$ is a pure rigid analytically trivial dual Drinfeld $\mathbb{F}_q[t]$-motive of rank 2 over $\overline{Q}$. We first see that we may interpret the entries $\Psi(\theta)_{ij}$ as the periods and quasi-periods of the corresponding Drinfeld $\mathbb{F}_q[t]$-module and investigate next the transcendence degree of the entries of $\Psi(\theta)$ through the use of Theorem 5.0.20. Using the main result of [Pin97a] and (5.2), we obtain the precise analog of the conjecturally expected transcendence degree of the periods and quasi-periods of an elliptic curve over $\overline{\mathbb{Q}}$.

5.1 Dual Anderson A-motives with sufficiently many complex multiplication

Complex multiplication theory was first developed for elliptic curves, and later extended to the case of higher dimensional algebraic varieties. In this section, we define complex multiplication (CM) of Anderson A-modules and dual Anderson A-motives and determine the Hodge-Pink group of pure rigid analytically trivial dual Anderson A-motives of CM-type over k under some conditions. Throughout this section, we assume $A = \mathbb{F}_q[t]$ and $k \subseteq \mathbb{C}_\infty$ is a perfect and complete field that contains Q_∞ as necessary for the definition of rigid analytic triviality and uniformizability.

Definition 5.1.1. Let $\underline{\mathrm{M}}$ be a dual Anderson A-motive of rank r and dimension d over k with $r, d > 0$ and $\underline{E}$ its corresponding Anderson A-module.

(i) If there exists a commutative semisimple Q-algebra $E \subseteq \mathrm{QEnd}(\underline{\mathrm{M}})$ such that $\dim_Q E = r$, we say that $\underline{\mathrm{M}}$ (resp. $\underline{E}$) is *of CM-type* or *has sufficiently many complex multiplication through E.*

(ii) We say that $\underline{\mathrm{M}}$ (resp. $\underline{E}$) *has no complex multiplication* if $\dim_Q \mathrm{QEnd}(\underline{\mathrm{M}}) = 1$.

Theorem 5.1.2. *Let $\underline{\mathrm{M}} \in \mathrm{Ob}(\mathscr{PRDA}_+^I)$ be a pure rigid analytically trivial dual Anderson A-motive of rank r, dimension d and weight $\frac{d}{r}$ over k that has sufficiently many complex multiplication through E. Further, let $\underline{H} = (H, W, \mathfrak{q}_H)$ be its associated pure Q-Hodge-Pink structure with Hodge-Pink group $G_{\underline{H}}$. If E/Q is either separable or purely inseparable, then*

$$G_{\underline{H}} \cong \mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{QEnd}(\underline{\mathrm{M}})) \cong \mathfrak{R}_{E/Q}\mathbb{G}_{m,E}.$$

Proof. Our proof largely follows Pink's proof of [Pin97a, Thm. 10.3]. By choosing a basis $h_1, \dots, h_r$ of H, we can regard $\mathrm{QEnd}(\underline{\mathrm{M}}) \cong \mathrm{End}(\underline{H})$ and hence E as a subalgebra of $\mathrm{Mat}_{r\times r}(Q)$. In particular, E is a finite extension of Q. Also, as in the proof of Proposition 4.2.17, the Hodge-Pink group $G_{\underline{H}}$ is an algebraic Q-subgroup of $\mathrm{GL}_{r,Q} \cong \mathrm{GL}(H)$.

Applying [BH09, Lem. 7.2], we see that $E \subseteq \mathrm{Mat}_{r\times r}(Q)$ is isomorphic to Q^r as a (left) module over itself. Since $\dim_Q H = r$, we conclude that H is a free E-module of rank 1. Thus $\mathrm{Cent}_{\mathrm{End}(H)}(E) = E$ and, moreover,

$$G_{\underline{H}} \overset{4.2.17}{\subseteq} \mathrm{Cent}_{\mathrm{GL}(H)}(\mathrm{End}(\underline{H})) \overset{E\subseteq \mathrm{QEnd}(\underline{\mathrm{M}})}{\subseteq} \mathrm{Cent}_{\mathrm{GL}(H)}(E) \cong \mathfrak{R}_{E/Q}\mathbb{G}_{m,E}.$$

We will first show that $G_{\mathrm{amb}} := \mathfrak{R}_{E/Q}\mathbb{G}_{m,E} \cong G_{\underline{H}}$ holds in the separable case and later apply it to a convenient pure Q-Hodge-Pink structure $\underline{H}'$ and $E' \subseteq \mathrm{End}(\underline{H}')$ with E'/Q separable to also prove equality if E/Q is purely inseparable.

Sublemma 5.1.3. *Let $\underline{H}$ be a pure Q-Hodge-Pink structure of rank r and weight $-\frac{d}{r}$, and $E \subseteq \mathrm{End}(\underline{H})$ a commutative semisimple subalgebra such that $\dim_Q E = r$. If E/Q is separable, then $\underline{H} \in \mathscr{Hodge}_Q^{\mathrm{sha}}$ and $G_{\underline{H}} \cong G_{\mathrm{amb}} = \mathfrak{R}_{E/Q}\mathbb{G}_{m,E}$.*

Proof. We fix an algebraic closure Q^{sep} of Q and let $\Sigma := \mathrm{Hom}_Q(E, Q^{\mathrm{sep}})$. We have a decomposition

$$H \otimes_Q Q^{\mathrm{sep}} \cong \bigoplus_{\sigma\in\Sigma} H \otimes_{E,\sigma} Q^{\mathrm{sep}} = \bigoplus_{\sigma\in\Sigma} Q^{\mathrm{sep}}$$

because H is a free E-module of rank 1, so that we get a corresponding inclusion

$$G_{\underline{H},Q^{\mathrm{sep}}} = G_{\underline{H}} \times_Q Q^{\mathrm{sep}} \subset G_{\mathrm{amb}} \times_Q Q^{\mathrm{sep}} \cong \prod_{\sigma\in\Sigma} \mathrm{GL}_{1,Q^{\mathrm{sep}}} = \prod_{\sigma\in\Sigma} \mathbb{G}_{m,Q^{\mathrm{sep}}}. \tag{5.3}$$

We want to show that this is an equality.

At first, consider the unipotent radical $R_u G_{\underline{H},Q^{\mathrm{sep}}}$ of $G_{\underline{H},Q^{\mathrm{sep}}}$ and the image of $R_u G_{\underline{H},Q^{\mathrm{sep}}} \subseteq G_{\underline{H},Q^{\mathrm{sep}}} \subseteq \prod \mathbb{G}_{m,Q^{\mathrm{sep}}}$ in each factor $\mathbb{G}_{m,Q^{\mathrm{sep}}}$, which is a quotient of $R_u G_{\underline{H},Q^{\mathrm{sep}}}$. By definition $R_u G_{\underline{H},Q^{\mathrm{sep}}}$ is unipotent. That is, each $x \in R_u G_{\underline{H},Q^{\mathrm{sep}}}$ is unipotent, which is equivalent to $x^{p^t} = 1$ for some $t \geq 0$ in characteristic p for all $x \in R_u G_{\underline{H},Q^{\mathrm{sep}}}$ (see [Hum75, Ch. VI §15.1]). This property is shared by any quotient of $R_u G_{\underline{H},Q^{\mathrm{sep}}}$; in particular, we see that the image of $R_u G_{\underline{H},Q^{\mathrm{sep}}}$ in $\mathbb{G}_{m,Q^{\mathrm{sep}}}$ is also unipotent. By [Hum75, §19.5] $\mathbb{G}_{m,Q^{\mathrm{sep}}}$ is reductive; that is, its unitpotent radical $R_u\mathbb{G}_{m,Q^{\mathrm{sep}}}$ is trivial, so that the projection of $R_u G_{\underline{H},Q^{\mathrm{sep}}}$ to each $\mathbb{G}_{m,Q^{\mathrm{sep}}}$

is trivial. Hence, we find that $R_u G_{\underline{H},Q^{\mathrm{sep}}} = (1)$ holds. By Definition 1.1.16, $G_{\underline{H}}$ is reductive and from [Pin97a, Prop. 9.8] we deduce that $\underline{H}$ is strongly Hodge-Pink additive.

Thus $G_{\underline{H},Q^{\mathrm{sep}}}$ is generated by the images of all $G_{\underline{H}}(Q^{\mathrm{sep}}) \rtimes \mathrm{Gal}(Q^{\mathrm{sep}}/Q)$-conjugates of cocharacters $\mathbb{G}_{m,Q^{\mathrm{sep}}} \to G_{\underline{H},Q^{\mathrm{sep}}}$ of $G_{\underline{H},Q^{\mathrm{sep}}}$ by [Pin97a, Thm. 9.11]. The weights of such a Hodge-Pink cocharacter $\lambda : \mathbb{G}_{m,Q^{\mathrm{sep}}} \to G_{\underline{H},Q^{\mathrm{sep}}}$ are the elementary divisors of $\mathfrak{q}_H$ relative to $\mathfrak{p}_H$. This means, if we choose integers $e_+ \geq e_-$ such that

$$(z-\zeta)^{e_+}\mathfrak{p}_H \subset \mathfrak{q}_H \subset (z-\zeta)^{e_-}\mathfrak{p}_H$$

and

$$\mathfrak{q}_H/(z-\zeta)^{e_+}\mathfrak{p}_H \cong \bigoplus_{i=1}^{n} \mathbb{C}_\infty[\![t-\theta]\!]/(z-\zeta)^{e_+ + w_i},$$

then the weights of λ are the Hodge-Pink weights $w_1, \ldots, w_n$.

Since $\underline{H}$ is pure of weight $-\frac{d}{r}$, we have $\deg_{\mathfrak{q}}(\underline{H}) = -d$ by the semistability condition. This means, $\mathfrak{q}_H$ is of $\mathbb{C}$-codimension d in $\mathfrak{p}_H$. Hence, there is a non-trivial weight d_0 of λ with $1 \leq d_0 \leq d$. Pulling back by λ allows us to associate with $G_{\underline{H},Q^{\mathrm{sep}}} \hookrightarrow \prod_{\sigma\in\Sigma} \mathbb{G}_{m,Q^{\mathrm{sep}}}$ in (5.3) a cocharacter

$$\lambda' : \mathbb{G}_{m,Q^{\mathrm{sep}}} \to G_{\underline{H},Q^{\mathrm{sep}}} \hookrightarrow \prod_{\sigma\in\Sigma} \mathbb{G}_{m,Q^{\mathrm{sep}}}$$

of $\prod_{\sigma\in\Sigma} \mathbb{G}_{m,Q^{\mathrm{sep}}}$. Since $d_0 \neq 0$, there is a $\sigma_0 \in \Sigma$ such that the composition of λ' with the projection to its factor in $\prod_{\sigma\in\Sigma} \mathbb{G}_{m,Q^{\mathrm{sep}}}$ is non-trivial, providing us a non-trivial cocharacter $\mathbb{G}_{m,Q^{\mathrm{sep}}} \to \mathbb{G}_{m,Q^{\mathrm{sep}}}$ of $\mathbb{G}_{m,Q^{\mathrm{sep}}}$.

Any non-trivial cocharacter of $\mathbb{G}_{m,Q^{\mathrm{sep}}}$ that maps x to x^n, $n \in \mathbb{Z} - \{0\}$, must be surjective. The image of its $\mathrm{Gal}(Q^{\mathrm{sep}}/Q)$-conjugates is the whole group $\prod_{\sigma\in\Sigma} \mathbb{G}_{m,Q^{\mathrm{sep}}} \cong G_{\mathrm{amb}} \times_Q Q^{\mathrm{sep}}$, giving us the desired result. □

We will now consider the case when E/Q is not separable.

Write $q := [E:Q]_i$ for the purely inseparable degree of E/Q. We define a pure Q-Hodge-Pink structure $\underline{H}' = (H', W', \mathfrak{q}_{H'})$ by setting $\underline{H} =: \mathrm{Frob}_{q,*}\,\underline{H}'$. By Proposition 3.3.4 (ii), $\underline{H}'$ has rank $r' := \frac{r}{q}$ and is pure of weight $-\frac{d}{r'}$ since $\deg_{\mathfrak{q}}(\underline{H}) = \deg_{\mathfrak{q}}(\underline{H}')$. Its endomorphism ring is $\mathrm{End}(\underline{H}') = \mathrm{Frob}_{q,\mathrm{End}(\underline{H})}(\mathrm{End}(\underline{H}))$. We find a commutative semisimple subalgebra $E' := E^q := \mathrm{Frob}_{q,E}(E) \subseteq \mathrm{End}(\underline{H}')$ with

$$\dim_Q E' = \frac{\dim_Q E}{q} = \frac{r}{q} = r'.$$

Let $G_{\underline{H}'}$ denote the Hodge-Pink group of $\underline{H}'$. Since E'/Q is separable, $G_{\underline{H}'} \cong \mathfrak{R}_{E'/Q}\mathbb{G}_{m,E'}$ by Lemma 5.1.3. Again from [BH09, Lem. 7.2], it follows that H' is a free E'-module of rank 1, so we may identify H' with E' from now on. Further, define a pure Q-Hodge-Pink structure

$$\underline{\tilde{H}} := (\tilde{H}, \tilde{W}, \mathfrak{q}_{\tilde{H}}) := \mathrm{Frob}_q^*\,\underline{H} = \mathrm{Frob}_q^*\,\mathrm{Frob}_{q,*}\,\underline{H}'.$$

Then we see by definition of the Frobenius pullback and Frobenius pushforward that

$$(\tilde{H}, \tilde{W}, \mathfrak{q}_{\tilde{H}}) = (E' \otimes_{Q^q} Q, W \otimes_{Q^q} Q, \mathfrak{q}_{H'} \otimes_{\mathbb{C}_\infty[\![(z-\zeta)^q]\!]} \mathbb{C}_\infty[\![z-\zeta]\!]),$$

where $Q^q := \mathrm{Frob}_{q,Q}(Q)$ and Q, resp. $\mathbb{C}_\infty[\![z-\zeta]\!]$, acts on the second factor of each tensor product. We put

$$\tilde{G}_{\mathrm{amb}} := G_{\mathrm{amb}} \times_{Q,\mathrm{Frob}_{q,\mathrm{Spec}\,Q}} Q \cong \mathfrak{R}_{\tilde{H}/Q}\mathbb{G}_{m,\tilde{H}}.$$

By Proposition 3.4.5 we have $G_{\underline{\tilde{H}}} \cong G_{\underline{H}} \times_{Q,\mathrm{Frob}_{q,\mathrm{Spec}\,Q}} Q$; hence it suffices to show $G_{\underline{\tilde{H}}} \cong \tilde{G}_{\mathrm{amb}}$.

Without loss of generality we may assume that the uniformizer $z \in Q_\infty$ already lies in Q. There is an isomorphism

$$\begin{array}{rcl} \tilde{H} = E' \otimes_{Q^q} Q & \to & E'[t]\,/(t^q), \\ z \otimes 1 - 1 \otimes z & \mapsto & t. \end{array}$$

The elements of its group of "1-units"

$$U := \{u = 1 + \sum_{i=1}^{\infty} u_i t^i \in (E'[t]\,/(t^q))^\times = \tilde{H}^\times\}$$

are unipotent since

$$\left(\sum_{i=1}^{\infty} u_i t^i\right)^q = \sum_{i=1}^{\infty} u_i^q t^{iq} = 0 \quad \text{in } (E'[t]\,/(t^q))^\times$$

and because they are invertible they define unipotent automorphisms of $\tilde{H}$. Thus we may regard U as a unipotent linear algebraic group of $\mathrm{GL}(E')$ (cf. [Hum75, Ch. VI §15]). We then have a unique decomposition $\mathfrak{R}_{\tilde{H}/E'}\mathbb{G}_{m,\tilde{H}} = \mathbb{G}_{m,E'} \times_{\mathrm{Spec}\,E'} U$ and correspondingly

$$\tilde{G}_{\mathrm{amb}} \cong \mathfrak{R}_{\tilde{H}/Q}\mathbb{G}_{m,\tilde{H}} \cong \mathfrak{R}_{E'/Q}\mathbb{G}_{m,E'} \times_{\mathrm{Spec}\,Q} \mathfrak{R}_{E'/Q}U. \tag{5.4}$$

Let p_i, $i = 1, 2$, denote the projection to the i^{th} factor. The first factor $p_1(\tilde{G}_{\mathrm{amb}}) = \mathfrak{R}_{E'/Q}\mathbb{G}_{m,E'}$ is the image of $\tilde{G}_{\mathrm{amb}}$ in $\mathrm{Aut}_Q(H')$, so $p_1(G_{\underline{\tilde{H}}}) = \mathfrak{R}_{E'/Q}\mathbb{G}_{m,E'} \overset{\text{Lem. 5.1.3}}{\cong} G_{\underline{H'}}$ is reductive since E'/Q is separable and thus a torus [Bor91, §21.11]. Furthermore, $p_2(\tilde{G}_{\mathrm{amb}})$ is unipotent and hence any connected algebraic subgroup of $\tilde{G}_{\mathrm{amb}}$ must decompose accordingly. Therefore, if we prove that $G_{\underline{\tilde{H}}}$ also surjects to the second factor $\mathfrak{R}_{E'/Q}U$ in (5.4), equality holds in $G_{\underline{\tilde{H}}} \subseteq \tilde{G}_{\mathrm{amb}}$.

Let us consider the projection homomorphism

$$\psi : \tilde{G}_{\mathrm{amb}} \to V := (\mathfrak{R}_{E'/Q}U)/(\mathfrak{R}_{E'/Q}U)^p \cong \mathfrak{R}_{E'/Q}(U/U^p).$$

We are done if we show that the restriction of ψ to $G_{\underline{\tilde{H}}}$ is surjective. In order to do this we investigate now the structure of the quotient U/U^p. We will see in Sublemma 5.1.7 that it is isomorphic to a direct sum of copies of $\mathbb{G}_{a,E'}$. If we assume that ψ is not surjective, then there must be a direct summand $\mathfrak{R}_{E'/Q}\mathbb{G}_{a,E'}$ of V in which the image of $G_{\underline{\tilde{H}}}$ is zero under $\psi|_{G_{\underline{\tilde{H}}}}$. We will show that this leads to a contradiction.

Definition 5.1.4 (Cf. [Hum75, Ch. VI §15]). We regard an element $u = 1 + \sum_{i=1}^{\infty} u_i t^i$ of U as a power series in $\mathbb{Z}[u_i; i \geq 1][\![t]\!]$, so that its *formal logarithm*

$$\log u := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot (u-1)^k \in \mathbb{Q}[u_i; i \geq 1][\![t]\!] \tag{5.5}$$

defines a nilpotent automorphism of $\tilde{H}$. We express $\log u$ in terms of t and set

$$\log u =: \sum_{i=1}^{\infty} L_i \frac{(-1)^{i-1}}{i} t^i.$$

Sublemma 5.1.5. *For every $i \geq 1$, we have*

$$L_i \;=\; \sum_{\underline{k}} (-1)^{|k|-i} \frac{i}{|k|} \binom{|k|}{k_1, \dots, k_i} u_1^{k_1} \cdot \ldots \cdot u_i^{k_i},$$

where $|k| := \sum_{\nu \geq 1} k_\nu$ and the sum is taken over $\{\underline{k} = (k_\nu)_{\nu \geq 1} \mid k_\nu \in \mathbb{N}$ for all $\nu \geq 1\}$.

Proof. Using multiple binomial coefficients one can compute $(u-1)^k$ as follows

$$(u-1)^k = \sum_{\underline{k}} \binom{|k|}{k_1, k_2, \dots} u_1^{k_1} \cdot u_2^{k_2} \cdot \ldots \cdot t^{\underline{i}}$$

where $\underline{i} = \sum_{\nu \geq 1} \nu k_\nu$ and the sum runs over $\{\underline{k} = (k_\nu)_{\nu \geq 1} \mid k_\nu \in \mathbb{N}$ for all $\nu \geq 1\}$.

Therefore a monomial $u_1^{k_1} \dots u_i^{k_i}$ is in L_i if $i = \sum_{\nu \geq 1} \nu k_\nu$ and $u_1^{k_1} \dots u_i^{k_i}$ with $k := \sum_{\nu \geq 1} k_\nu$ in L_i must come from the k^{th} term in (5.5). Summing this up, the coefficient of $u_1^{k_1} \dots u_i^{k_i}$ in L_i is by definition of $\log u$

$$\frac{i}{(-1)^{i-1}} \cdot \frac{(-1)^{k-1}}{k} \cdot \binom{k}{k_1, k_2, \dots, k_i} = (-1)^{k-i} \frac{i}{k} \cdot \binom{k}{k_1, k_2, \dots, k_i}. \qquad \square$$

Sublemma 5.1.6 ([Pin97a, Lem. 10.15])**.** *For every $1 \leq i \leq n$ the map $L_i : U \to \mathbb{G}_{a,E'}$*

$$u = 1 + \sum_{i=1}^{\infty} u_i t^i \mapsto L_i(u_1, \dots, u_i)$$

is a homomorphism of algebraic groups.

Write $I := \{i \in 1, \dots, q-1 | p \nmid i\}$ and let q' denote the cardinality of I.

Sublemma 5.1.7 ([Pin97a, Lem. 10.16])**.** *The homomorphism $L' : U \to \mathbb{G}_{a,E'}^{\oplus q'}$*

$$L'(1 + \sum_{i=1}^{\infty} u_i t^i) := (L_i(u_1, \dots, u_i))_{i \in I}$$

induces an isomorphism of algebraic groups $U/U^p \to \mathbb{G}_{a,E'}^{\oplus q'}$.

This provides us the desired result that

$$V = (\mathfrak{R}_{E'/Q} U)/(\mathfrak{R}_{E'/Q} U)^p \cong \mathfrak{R}_{E'/Q}(U/U^p) \cong \mathfrak{R}_{E'/Q} \mathbb{G}_{a,E'}^{\oplus q'}. \tag{5.6}$$

As mentioned earlier we assume to the contrary that

$$\psi|_{G_{\underline{\tilde{H}}}} : G_{\underline{\tilde{H}}} \to V \cong \mathfrak{R}_{E'/Q} \mathbb{G}_{a,E'}^{\oplus q'}.$$

is not surjective so that there must be a direct summand of $V \cong (\mathfrak{R}_{E'/Q} \mathbb{G}_{a,E'})^{\oplus q'}$ in which $G_{\underline{\tilde{H}}}$ gets mapped to zero under $\psi|_{G_{\underline{\tilde{H}}}}$. This is equivalent to saying that there is a non-zero homomorphism of algebraic groups $\varphi : V \to \mathbb{G}_{a,Q}$ such that $\psi|_{G_{\underline{\tilde{H}}}}(G_{\underline{\tilde{H}}}) \subseteq \ker(\varphi)$. Further, $\mathbb{G}_{a,Q}$ is isomorphic to the subgroup $U_{2,Q} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ of $\mathrm{GL}_{2,Q}$. Via

$$G_{\underline{\tilde{H}}} \hookrightarrow V \xrightarrow{\varphi} U_{2,Q} \to \mathrm{GL}(H_\varphi) \quad \text{with } H_\varphi := Q^2$$

we define a representation of $G_{\underline{\tilde{H}}}$ on H_φ. The category of representations of $G_{\underline{\tilde{H}}}$ is equivalent to $\langle\langle\underline{\tilde{H}}\rangle\rangle$, so that we get a corresponding pure Q-Hodge-Pink structure $\underline{H}_\varphi = (H_\varphi, W_\varphi, \mathfrak{q}_{H_\varphi})$ in $\langle\langle\underline{\tilde{H}}\rangle\rangle$. Since $\mathrm{im}(G_{\underline{\tilde{H}}}) \subseteq \ker\varphi$ we have

$$G_{\underline{\tilde{H}}} \hookrightarrow \ker\varphi \xrightarrow{\varphi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \mathrm{GL}(\mathbb{1}_Q^{\oplus 2}),$$

and thus $\underline{H}_\varphi \cong \mathbb{1}_Q^{\oplus 2}$; in particular, $\mathfrak{q}_{H_\varphi} = \mathfrak{p}_{H_\varphi}$.

We will now state a slight modification of Proposition 3.4.4 (ii), and apply it to the above defined pure Q-Hodge-Pink structure $\underline{H}_\varphi$ in $\langle\langle\underline{\tilde{H}}\rangle\rangle$.

Sublemma 5.1.8. *Let $\gamma \in \tilde{G}_{\mathrm{amb}}(\mathbb{C}_\infty(\!(z-\zeta)\!))$ such that $\mathfrak{q}_{\tilde{H}} = \gamma\mathfrak{p}_{\tilde{H}}$ and ρ' the associated representation of $\tilde{G}_{\mathrm{amb}}$ on the Q-vector space $\tilde{H}'$ underlying an object $\underline{\tilde{H}}' = (\tilde{H}', \tilde{W}', \mathfrak{q}_{\tilde{H}'})$ in $\langle\langle\underline{\tilde{H}}\rangle\rangle$. Then $\mathfrak{q}_{\tilde{H}'} = \rho'(\gamma)\mathfrak{p}_{\tilde{H}'}$.*

Proof. Cf. the proof of [Pin97a, Prop. 6.3 (b)]. □

The associated representation of $\underline{H}_\varphi$ is by definition $\phi \circ \psi : \tilde{G}_{\mathrm{amb}} \to V \to Q^2$, implying

$$\mathfrak{q}_{H_\varphi} = \phi(\psi(\gamma))\mathfrak{p}_{H_\varphi} = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \phi(\psi(\gamma)) \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}_\infty[\![z-\zeta]\!]}.$$

We want to get a contradiction to $\mathfrak{q}_{H_\varphi} = \mathfrak{p}_{H_\varphi}$, which will be the case if $\varphi(\psi(\gamma)) \notin \mathbb{C}_\infty[\![z-\zeta]\!]$.

We will now investigate the lattices associated to $\underline{H}'$ and $\underline{\tilde{H}}$ to find such a $\gamma \in \tilde{G}_{\mathrm{amb}}(\mathbb{C}_\infty(\!(z-\zeta)\!))$.

Fix an inclusion $Q^{\mathrm{sep}} \hookrightarrow \mathbb{C}_\infty[\![z-\zeta]\!]$ and set $\Sigma := \mathrm{Hom}_Q(E', Q^{\mathrm{sep}})$ as in the separable case. Recall that $H' = E'$ as E'-modules, so that we get a decomposition of $\mathfrak{p}_{H'}$:

$$\begin{aligned} \mathfrak{p}_{H'} &= E' \otimes_Q \mathbb{C}_\infty[\![z-\zeta]\!] \cong (E' \otimes_Q Q^{\mathrm{sep}}) \otimes_{Q^{\mathrm{sep}}} \mathbb{C}_\infty[\![z-\zeta]\!] \\ &\cong (\bigoplus_{\sigma\in\Sigma} E' \otimes_{E',\sigma} Q^{\mathrm{sep}}) \otimes_{Q^{\mathrm{sep}}} \mathbb{C}_\infty[\![z-\zeta]\!] \cong (\bigoplus_{\sigma\in\Sigma} Q^{\mathrm{sep}}) \otimes_{Q^{\mathrm{sep}}} \mathbb{C}_\infty[\![z-\zeta]\!] \\ &\cong \bigoplus_{\sigma\in\Sigma} \mathbb{C}_\infty[\![z-\zeta]\!]. \end{aligned}$$

From Proposition 3.3.4 we know that $\deg_{\mathfrak{q}}(\underline{H}') = \deg_{\mathfrak{q}}(\underline{H}) = -d$, so $\mathfrak{q}_{H'}$ is of $\mathbb{C}$-codimension d in $\mathfrak{p}_{H'}$ by the semistability condition. Using the above decomposition, we can write $\mathfrak{q}_{H'} = \pi \cdot \mathfrak{p}_{H'}$ with $\pi = (\pi_\sigma)_{\sigma\in\Sigma} \in \mathfrak{p}_{H'}$, such that

$$\pi_\sigma = \begin{cases} (z-\zeta)^{d_{\sigma_l}} & \text{if } \sigma = \sigma_l \\ 1 & \text{otherwise,} \end{cases}$$

where the $\sigma_l \in \Sigma$ are fixed for all $l \in \{1, \dots, m\}$ with $\sigma_l \neq \sigma_{l'}$ for $l \neq l'$ and $m \in \{1, \dots, d\}$ such that $\sum_{l=1}^m d_{\sigma_l} = d$ holds.

Similarly to $E' \otimes_{Q^q} Q \cong E'[t]/(t^q)$, we have an isomorphism

$$\begin{aligned} \mathfrak{p}_{\tilde{H}} = \mathfrak{p}_{H'} \otimes_{\mathbb{C}_\infty[\![(z-\zeta)^q]\!]} \mathbb{C}_\infty[\![z-\zeta]\!] &\to \mathfrak{p}_{H'}[t]/(t^q) \\ (z-\zeta) \otimes 1 - 1 \otimes (z-\zeta) &\mapsto t, \end{aligned}$$

and $\mathfrak{q}_{\tilde{H}}$ is an ideal in $\mathfrak{p}_{H'}[t]/(t^q)$. Furthermore,

$$\begin{aligned} \tilde{G}_{\mathrm{amb}}(\mathbb{C}_\infty(\!(z-\zeta)\!)) &\cong (\mathfrak{R}_{\tilde{H}/Q}\mathbb{G}_{m,\tilde{H}})(\mathbb{C}_\infty(\!(z-\zeta)\!)) = \mathbb{G}_m(\tilde{H} \otimes_Q \mathbb{C}_\infty(\!(z-\zeta)\!)) \\ &= \mathbb{G}_m((E'[z]/(t^q) \otimes_Q \mathbb{C}_\infty[\![z-\zeta]\!]) \otimes_{\mathbb{C}_\infty[\![z-\zeta]\!]} \mathbb{C}_\infty(\!(z-\zeta)\!)) \\ &= (\mathfrak{p}_{H'}[t]/(t^q) \otimes_{\mathbb{C}_\infty[\![z-\zeta]\!]} \mathbb{C}_\infty(\!(z-\zeta)\!))^\times. \end{aligned}$$

Sublemma 5.1.9. *Let* $\gamma := \pi + e_1 t + \ldots + e_d t^d \in \tilde{G}_{\mathrm{amb}}(\mathbb{C}_\infty(\!(z-\zeta)\!))$ *with*

$$e_r = (e_{r,\sigma}) \in \mathfrak{p}_{H'},\ r = 1, \ldots, d$$

defined as follows

$$e_{r,\sigma} := \begin{cases} \binom{d_{\sigma_l}}{r}(-1)^{r-1}(z-\zeta)^{d_{\sigma_l}-r} & \text{if } \sigma = \sigma_l \text{ and } r \le d_{\sigma_l} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathfrak{q}_{\tilde{H}} = \gamma \cdot \mathfrak{p}_{\tilde{H}}$.

Proof. We have $\mathfrak{q}_{\tilde{H}} = \mathfrak{q}_{H'} \otimes_{\mathbb{C}_\infty[\![z-\zeta]\!],\mathrm{Frob}_q} \mathbb{C}_\infty[\![z-\zeta]\!] = (\pi \cdot \mathfrak{p}_{H'}) \otimes_{\mathbb{C}_\infty[\![z-\zeta]\!],\mathrm{Frob}_q} \mathbb{C}_\infty[\![z-\zeta]\!]$, so it is enough to prove

$$\pi \otimes 1 = 1 \otimes \pi + \sum_{r=1}^{d} e_r((z-\zeta)\otimes 1 - 1 \otimes (z-\zeta))^r \tag{5.7}$$

in $\mathfrak{p}_{\tilde{H}} = \mathfrak{p}_{H'} \otimes_{\mathbb{C}_\infty[\![(z-\zeta)^q]\!]} \mathbb{C}_\infty[\![z-\zeta]\!]$.

There is a decomposition of $\mathfrak{p}_{\tilde{H}}$ according to the decomposition of $\mathfrak{p}_{H'}$:

$$\begin{aligned} \mathfrak{p}_{\tilde{H}} &= \mathfrak{p}_{H'} \otimes_{\mathbb{C}_\infty[\![(z-\zeta)^q]\!]} \mathbb{C}_\infty[\![z-\zeta]\!] \cong \bigoplus_{\sigma\in\Sigma} (\mathbb{C}_\infty[\![z-\zeta]\!] \otimes_{\mathbb{C}_\infty[\![(z-\zeta)^q]\!]} \mathbb{C}_\infty[\![z-\zeta]\!]) \\ &\cong \bigoplus_{\sigma\in\Sigma} (\mathbb{C}_\infty[\![z-\zeta]\!][t]/(t^q)) \end{aligned}$$

and (5.7) is equal to

$$\begin{aligned} &(z-\zeta)^{d_{\sigma_l}} \otimes 1 - 1 \otimes (z-\zeta)^{d_{\sigma_l}} \\ &= \sum_{r=1}^{d_{\sigma_l}} e_{r,d_{\sigma_l}} \cdot ((z-\zeta)\otimes 1 - 1 \otimes (z-\zeta))^r \\ &= \sum_{r=1}^{d_{\sigma_l}} \binom{d_{\sigma_l}}{r} (-1)^{r-1} (z-\zeta)^{d_{\sigma_l}-r} \cdot ((z-\zeta)\otimes 1 - 1 \otimes (z-\zeta))^r \end{aligned} \tag{5.8}$$

in the summand associated to σ_l, $l = 1, \ldots m$, in the others to $1 \otimes 1 = 1 \otimes 1$. (5.8) holds by the following:

Sublemma 5.1.10. *Let* $j \in \mathbb{N}^{>0}$. $(z^j \otimes 1 - 1 \otimes z^j)$ *in* $E' \otimes_{K^q} K$ *gets mapped to*

$$\sum_{r=1}^{j} (-1)^{r-1} \binom{j}{r} z^{j-r} t^r = (-1)^{j-1}(t-z)^j + z^j$$

in $E'[t]/(t^q)$.

Proof. We proceed by induction on j. The assertion is true for $j = 1$ since $(z^1 \otimes 1 - 1 \otimes z^1) \mapsto t$

holds by definition. We assume that the assertion is true for all $i \leq j$. Then we have as desired

$$\begin{aligned}
z^{j+1}\otimes 1 - 1\otimes z^{j+1} =& z^j \otimes z - 1\otimes z^{j+1} + z^{j+1}\otimes 1 - z^j\otimes z \\
=&(1\otimes z)(z^j\otimes 1 - 1\otimes z^j) + (z^j\otimes 1)(z\otimes 1 - 1\otimes z) \\
\mapsto&(z-t)\sum_{k=1}^{j}(-1)^{k-1}\binom{j}{k}z^{j-k}t^k + z^j t \\
=&(-1)^j t^{j+1} + z\sum_{k=1}^{j}(-1)^{k-1}\binom{j}{k}z^{j-k}t^k \\
&+ t\sum_{k=1}^{j}(-1)^k\binom{j}{k}z^{j-k}t^k + z^j t - (-1)^j t^{j+1} \\
=&(-1)^j t^{j+1} + z\sum_{k=1}^{j}(-1)^{k-1}\binom{j}{k}z^{j-k}t^k + t\sum_{k=0}^{j-1}(-1)^k\binom{j}{k}z^{j-k}t^k \\
=&(-1)^j t^{j+1} + z\sum_{k=1}^{j}(-1)^{k-1}\binom{j}{k}z^{j-k}t^k + \sum_{k=1}^{j}(-1)^{k-1}\binom{j}{k-1}z^{j+1-k}t^k \\
=&(-1)^j t^{j+1} + \sum_{k=1}^{j}(-1)^{k-1}(\binom{j}{k-1}+\binom{j}{k})z^{j+1-k}t^k \\
=&\sum_{k=1}^{j+1}(-1)^{1-k}\binom{j+1}{k}z^{j+1-k}t^k.
\end{aligned}$$

□

This proves Sublemma 5.1.9. □

So we have found a $\gamma \in \tilde{G}_{\mathrm{amb}}$ such that $\mathfrak{q}_{\tilde{H}} = \gamma \cdot \mathfrak{p}_{\tilde{H}}$. We will now calculate $\varphi(\psi(\gamma))$ and show that $\varphi(\psi(\gamma)) \notin \mathbb{C}_\infty[\![z-\zeta]\!]$ under the given assumption $\varphi \neq 0$. The projection of γ to the second factor $\mathfrak{R}_{E'/Q}U$ is $1 + \pi^{-1}\sum_{r=1}^{d} e_r \cdot t^r$, so that we get with the isomorphism $V \overset{(5.6)}{\cong} \mathfrak{R}_{E'/Q}\mathbb{G}_{a,E'}^{\oplus q'}$ induced by L':

$$\begin{aligned}
\psi(\gamma) &= L'(1+\pi^{-1}\sum_{r=1}^{d} e_r\cdot t^r) \\
&= (L_i(\pi^{-1}e_1,\dots,\pi^{-1}e_i))_{i\in I} \quad \text{with } e_r := 0 \text{ for } i > d \\
&= \left(\sum_{\underline{k}}(-1)^{|k|-i}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i} e_r^{k_r}\cdot\pi^{-|k|}\right)_{i\in I}
\end{aligned}$$

where $|k| := \sum_{\nu=1}^{i} k_\nu$ and the last sum is taken over $\{\underline{k} = (k_1,\dots,k_i) \mid k_\nu \in \mathbb{N} \text{ for all } 1 \leq \nu \leq 1\}$. Note that V is isomorphic to a direct sum of copies of $\mathbb{G}_{a,Q}$ and $\mathrm{End}_{\mathbb{F}_p,Q}(\mathbb{G}_{a,Q}) \cong Q[\tau_p]$. Furthermore, φ is a Q-linear form on $(E')^{\oplus q'}$ that is therefore a composition of an E'-linear form and $\mathrm{trace}_{E'/Q} : E' \to Q$. This means we can write φ as

$$\begin{aligned}
\mathfrak{R}_{E'/Q}\mathbb{G}_{a,E'}^{\oplus q'} &\to Q \\
x = (x_i)_{i\in I} &\mapsto \sum_{i\in I}\sum_{j\geq 0}\mathrm{trace}_{E'/Q}\left(\varphi_{i,j}\cdot x_i^{p^j}\right)
\end{aligned}$$

for some $\varphi_{i,j} \in E'$. We are now able to determine $\varphi(\psi(\gamma))$. Denote $|k| := \sum_{\nu=1}^{i} k_\nu$ so that $\varphi(\psi(\gamma))$

$$\begin{aligned}
&= \sum_{i\in I}\sum_{j\geq 0} \operatorname{trace}_{E'/Q}\left(\varphi_{i,j}\cdot\left(\sum_{\underline{k}}(-1)^{|k|-i}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i} e_r^{k_r}\cdot\pi^{-|k|}\right)^{p^j}\right)\\
&=: \sum_{i\in I}\sum_{j\geq 0}\sum_{l=1}^{m}\sigma_l(\varphi_{i,j})\left(\sum_{\underline{k}}(-1)^{|k|-i}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i} e_{r,\sigma_l}^{k_r}\cdot\pi_{\sigma_l}^{-|k|}\right)^{p^j}\\
&= \sum_{i\in I}\sum_{j\geq 0}\sum_{l=1}^{m}\sigma_l(\varphi_{i,j})\left(\sum_{\underline{k}}(-1)^{|k|-i}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i}\left(\binom{d_{\sigma_l}}{r}(-1)^{r-1}(z-\zeta)^{d_{\sigma_l}-r}\right)^{k_r}\pi_{\sigma_l}^{-|k|}\right)^{p^j}\\
&= \sum_{i\in I}\sum_{j\geq 0}\sum_{l=1}^{m}\sigma_l(\varphi_{i,j})\left(\sum_{\underline{k}}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i}\binom{d_{\sigma_l}}{r}^{k_r}(z-\zeta)^{\sum_{r=1}^{i}(d_{\sigma_l}k_r-rk_r)}(z-\zeta)^{-d_{\sigma_l}|k|}\right)^{p^j}\\
&= \sum_{i\in I}\sum_{j\geq 0}\sum_{l=1}^{m}\sigma_l(\varphi_{i,j})\left(\sum_{\underline{k}}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i}\binom{d_{\sigma_l}}{r}^{k_r}(z-\zeta)^{-i}\right)^{p^j}\\
&= \sum_{i\in I}\sum_{j\geq 0}\sum_{l=1}^{m}\sigma_l(\varphi_{i,j})\cdot(z-\zeta)^{-ip^j}\left(\sum_{\underline{k}}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i}\binom{d_{\sigma_l}}{r}^{k_r}\right)^{p^j}\\
&= \sum_{i\in I}\sum_{j\geq 0}(z-\zeta)^{-ip^j}\cdot\sum_{l=1}^{m}\sigma_l(\varphi_{i,j})\cdot\tilde{b}_{d_{\sigma_l},i,j}
\end{aligned}$$

with $\tilde{b}_{d_{\sigma_l},i,j} := \left(\sum_{\underline{k}}\frac{i}{|k|}\binom{|k|}{k_1,\dots,k_i}\prod_{r=1}^{i}\binom{d_{\sigma_l}}{r}^{k_r}\right)^{p^j}$ and the sum run over $\{\underline{k} = (k_1,\dots,k_i) \mid k_\nu \in \mathbb{N} \text{ for all } 1\leq\nu\leq 1\}$. . Every exponent $-ip^j$ $(i\in I,\ j\geq 0)$ of $(z-\zeta)$ just occurs one time for a σ_l, $l=1,\dots,m$, since $p\nmid i$ by definition of I. Unfortunately, as we do not know more about φ, the $\sigma_l(\varphi_{i,j})$ might be linearly dependent, so that we cannot make any statements about the order of $(z-\zeta)$ in the general case.

If there is just one Hodge-Pink slope d_{σ_l}, this is in particular satisfied when E/Q is purely inseparable, we can choose $i,\ j$ such that $\varphi_{i,j}\neq 0$ and $-ip^j$ maximal. Then

$$\operatorname{ord}_{(z-\zeta)}\varphi(\psi(\gamma)) = -ip^j < 0,$$

which contradicts $\varphi(\psi(\gamma)) \in \mathbb{C}_\infty[\![z-\zeta]\!]$, as desired. Hence $\psi|_{G_{\underline{\tilde{H}}}}$ is surjective and thus $G_{\underline{\tilde{H}}} \cong \tilde{G}_{\mathrm{amb}}$ and $G_{\underline{H}} \cong \mathfrak{R}_{E/Q}\mathbb{G}_{m,E}$ if E/Q is purely inseparable. □

With the help of Grothendieck's period conjecture for function fields and Lemma 1.1.19, we may directly deduce the following assertion about the transcendence degree of the entries of the period matrix of a dual Anderson A-motive over $\mathbb{C}_\infty$.

Theorem 5.1.11. *Let $\underline{\mathrm{M}} = (\mathrm{M},\sigma_{\mathrm{M}}) \in \mathrm{Ob}(\mathscr{PRDA}_+^I)$ be a pure rigid analytically trivial dual Anderson A-motive of rank r and weight $\frac{d}{r}$ over $\overline{Q}\subset\overline{Q_\infty}$ that has sufficiently many complex multiplication through E. Moreover, let $\Phi_{\mathbf{m}}$ represent σ_{M} with respect to a basis $\mathbf{m}$ for M and Ψ be a rigid analytic trivialization for $\Phi_{\mathbf{m}}$. If E/Q is either separable or purely inseparable, then*

$$\operatorname{tr.deg}_{\overline{Q}}\ \overline{Q}(\Psi(\theta)_{ij}\mid 1\leq i,j\leq r) = \dim G_{\underline{\mathrm{M}}} = \dim\mathfrak{R}_{E/Q}\mathbb{G}_{m,E} = r,$$

where $G_{\underline{\mathrm{M}}}$ is the Hodge-Pink group of $\underline{\mathrm{M}}$.

5.2 Periods and Quasi-Periods of Drinfeld $\mathbb{F}_q[t]$-modules

Drinfeld A-modules of rank 2 are analogous to elliptic curves. Motivated by this analogy, we first introduce the notion of quasi-periods of a Drinfeld A-module $\underline{E}$ through biderivations δ. We then define its period matrix $P_{\underline{E}} = (\int_{\lambda_i} \delta_j)$, which consists of the periods and quasi-periods of $\underline{E}$. Finally, we determine the transcendence degree of the periods and quasi-periods of a Drinfeld $\mathbb{F}_q[t]$-motive of rank 2 over $\overline{Q}$ through Grothendieck's period conjecture for function fields and Pink's main result of [Pin97a]. The result is the precise analog of the following:

Conjecture 5.2.1 (Grothendieck's period conjecture for elliptic curves [DMOS82, Rem. 1.8])**.** Let E be an elliptic curve over $\overline{\mathbb{Q}}$, $P = (\int_{\lambda_i} \delta_j)$ its period matrix and G_E the Hodge group of E. Then

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(\int_{\lambda_i} \delta_j) = \dim G_E = \begin{cases} 2 & \text{if } E \text{ is of CM-type,} \\ 4 & \text{otherwise.} \end{cases}$$

Chudnovsky was able to give a proof of this conjecture when an elliptic curve is of CM-type [Chu84, Thm. 1.16].

De Rham cohomology for Drinfeld A-modules

We first introduce the de Rham cohomology realization of a Drinfeld A-module through universal additive extensions. This construction goes back to Deligne and parallels the classical de Rham theory for elliptic curves and higher dimension abelian varieties. When $A = \mathbb{F}_q[t]$, we remark that this coincides with the definition of the first de Rham cohomology group of a dual Anderson A-motive as given in Remark 4.1.32. Next we give an alternative construction of the first de Rham cohomology group, which is related to some kind of path integration. This idea is due to Anderson and was further developed by Yu and Gekeler. With the help of biderviations, we may define such "path integrals" and further quasi-periods of Drinfeld A-modules.

We let A be the ring of integers of an arbitrary function field Q and fix a Drinfeld A-module $\underline{E} = (E, \varphi)$ of rank r over k where (k, γ) is an A-field that contains $\mathbb{F}_q$.

Definition 5.2.2. We set

$$H^1_{\mathrm{DR}}(\underline{E}, k) := \operatorname{Ext}^\sharp(\underline{E}, \mathbb{G}_{a,k});$$

that is, the group of classes of short exact sequences of algebraic groups over k

$$0 \longrightarrow \mathbb{G}_{a,k} \longrightarrow E^* \longrightarrow E \longrightarrow 0 \tag{5.9}$$

together with an additional splitting s of the short exact sequence

$$0 \to k \overset{s}{\leftrightarrows} \mathrm{T}_0 E^* \to \mathrm{T}_0 E \to 0$$

of tangent spaces at the identity (equipped with the tautological A-action).

Remark 5.2.3. When $A = \mathbb{F}_q[t]$, we let $(\mathrm{M}, \sigma_{\mathrm{M}})$ be the dual Drinfeld $\mathbb{F}_q[t]$-motive corresponding to a Drinfeld A-motive $\underline{E}$ over $\mathbb{C}_\infty$. We want to see that the definition of $H^1_{\mathrm{DR}}(\underline{E}, \mathbb{C}_\infty)$ coincides with the one of $H^1(\mathcal{M}^*(\underline{E}), \mathbb{C}_\infty)$ made in Remark 4.1.32. We put

$$\check{\mathfrak{p}}_H := \tau_{\mathrm{M}}(F^*\mathcal{M}_*(E)) \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] \cong \mathcal{M}_*(E)(1)^\tau \otimes_A \mathbb{C}_\infty[\![t-\theta]\!],$$

where $\mathcal{M}_*(E)(1)^\tau$ are the τ-invariants of the Drinfeld $\mathbb{F}_q[t]$-motive $\mathcal{M}_*(\underline{E})$ assigned to $\underline{E}$ [And86, §2.3]. Using [And86, Cor. 1.12.1], we find

$$\begin{aligned}\mathfrak{p}_H := \mathrm{M} \otimes_{\mathbb{C}_\infty[t]} \mathbb{C}_\infty[\![t-\theta]\!] &\cong \mathrm{M}(1)^\sigma \otimes_A \mathbb{C}_\infty[\![t-\theta]\!] \\ &\cong \Lambda_{\underline{E}} \otimes_A \mathbb{C}_\infty[\![t-\theta]\!] \\ &\cong \mathrm{Hom}_{\mathbb{C}_\infty[\![t-\theta]\!]}(\mathcal{M}_*(E)(1)^\tau \otimes_A \mathbb{C}_\infty[\![t-\theta]\!], \mathbb{C}_\infty[\![t-\theta]\!]\,\mathrm{dt}) \\ &\cong \mathrm{Hom}_{\mathbb{C}_\infty[\![t-\theta]\!]}(\check{\mathfrak{p}}_H, \mathbb{C}_\infty[\![t-\theta]\!]\,\mathrm{dt}),\end{aligned}$$

where $\Lambda_{\underline{E}}$ is the period lattice of $\underline{E}$. Therefore, $\mathfrak{p}_H/(t-\theta)\mathfrak{p}_H \cong \mathrm{Hom}_{\mathbb{C}_\infty}(\check{\mathfrak{p}}_H/(t-\theta)\check{\mathfrak{p}}_H, \mathbb{C}_\infty\,\mathrm{dt})$ and

$$H^1_{\mathrm{DR}}(\underline{E}, \mathbb{C}_\infty) \overset{[\text{Tha04},\S 7.5]}{\cong} H^1_{\mathrm{DR}}(\mathcal{M}_*(E), \mathbb{C}_\infty) \cong H^1_{\mathrm{DR}}(\mathrm{M}, \mathbb{C}_\infty).$$

Definition 5.2.4. We define the A_k-module $N(E) := \{\mathrm{m} \in \mathcal{M}^*(E) \mid \mathrm{T}_0\mathrm{m} = 0\}$. That is, $N(E) \cong k[\sigma]\sigma$ by choosing a coordinate system ρ for E and defining $\sigma : \mathcal{M}^*(E) \to \mathcal{M}^*(E)$ to be the ς^*-linear map induced by $\sigma_{\mathcal{M}^*(E)}$.

(i) An $\mathbb{F}_q$-linear *biderivation of A into $N(E)$* is an $\mathbb{F}_q$-linear map $\delta : A \to N(E)$, $\delta(a) \mapsto \delta_a$ such that

$$\delta_{ab} = \gamma(a)\delta_a + \delta_a \circ \varphi_b.$$

We denote the vector space of biderivations of A into $N(E)$ by $D(\underline{E}, k)$.

(ii) A biderivation δ is called *inner* if there is an $\mathrm{m} \in \mathcal{M}^*(E)$ such that

$$\delta_a = \gamma(a)\mathrm{m} - \mathrm{m} \circ \varphi_a$$

Further δ is called *strictly inner* or *exact* if $\mathrm{m} \in N(E)$. We denote the subspace of strictly inner biderivations of A into $N(E)$ by $D_{\mathrm{si}}(\underline{E}, k)$.

For $\delta \in D(\underline{E}, k)$ and $a \in A$, we put

$$\varphi_a^\delta := \begin{pmatrix} \gamma(a) & \delta_a \\ 0 & \varphi_a \end{pmatrix}.$$

Note that we have

$$\varphi_a^\delta \varphi_b^\delta = \begin{pmatrix} \gamma(a)\gamma(b) & \gamma(a)\delta_a + \delta_a \circ \varphi_b \\ 0 & \varphi_a\varphi_b \end{pmatrix} = \varphi_{ab}^\delta.$$

We denote the class of additive extensions of E

$$0 \longrightarrow \mathbb{G}_{a,k} \longrightarrow \mathbb{G}_{a,k} \oplus E \longrightarrow E \longrightarrow 0,$$

where A acts on $\mathbb{G}_{a,k}$ via γ and on $\mathbb{G}_{a,k} \oplus E$ via φ_a^δ, by $[\delta]$. Because $\delta \in N(E)$, the induced action of A on $\mathrm{T}_0(\mathbb{G}_{a,k} \oplus E)$ is given by

$$\mathrm{T}_0\varphi_a^\delta = \begin{pmatrix} \gamma(a) & 0 \\ 0 & \gamma(a) \end{pmatrix},$$

so that the sequence of tangent spaces at the identity induced by $[\delta]$ splits canonically.

Theorem 5.2.5 (Anderson [Gos94, Thm. 1.5.4 and Thm. 1.5.6]). *(i) The map*

$$(\delta \mapsto [\delta]) : D(\underline{E}, k) \to \operatorname{Ext}^\sharp(\underline{E}, \mathbb{G}_{a,k})$$

induces an isomorphism from $D(\underline{E}, k)/D_{\mathrm{si}}(\underline{E}, k)$ to $H^1_{\mathrm{DR}}(\underline{E}, k)$.

(ii) $H^1_{\mathrm{DR}}(\underline{E}, k)$ is a k-vector space of dimension r.

Thus we may identify $H^1_{\mathrm{DR}}(\underline{E}, k)$ and $D(\underline{E}, k)/D_{\mathrm{si}}(\underline{E}, k)$ from now on. Let us now give the examples of biderivations, which play a role when we define the period matrix of a Drinfeld $\mathbb{F}_q[t]$-module.

Example 5.2.6. (i) Consider the $\mathbb{F}_q$-linear map $\delta^{(0)} : A \to N(E)$ that is defined by $\delta^{(0)}_a = \varphi_a - \gamma(a)$. Then

$$\begin{aligned} \delta^{(0)}_{ab} &= \varphi_{ab} - \gamma(a)\gamma(b) = \varphi_a \circ \varphi_b - \gamma(ab) = \gamma(a)(\varphi_b - \gamma(b)) + (\varphi_a - \gamma(a)) \circ \varphi_b \\ &= \gamma(a)\delta^{(0)}_b + \delta^{(0)}_a \circ \varphi_b; \end{aligned}$$

that is, $\delta^{(0)}$ is a biderivation of A into $N(E)$.

(ii) When $A = \mathbb{F}_q[t]$, the Drinfeld $\mathbb{F}_q[t]$-module (E, φ) is given by

$$\varphi_t = \theta + \alpha_1\tau + \ldots + \alpha_r\tau^r \in k[\tau] \cong \operatorname{End}(E),$$

where we write $\theta := \gamma(t)$. We define an $\mathbb{F}_q$-linear map $\delta^{(i)} : A \to N(E)$ by setting $\delta^{(i)}_t := \tau^i$ for $1 \le i \le r-1$. Since

$$\begin{aligned} \theta\delta^{(i)}_t + \delta^{(i)}_t \circ \varphi_t &= \theta\tau^i + \tau^i(\theta + \alpha_1\tau + \ldots + \alpha_r\tau^r) \\ &= (\theta + \theta^{(i)})\tau^i + \alpha_1^{(i)}\tau^{i+1} + \ldots + \alpha_r^{(i)}\tau^{i+r} \\ &= ((\theta + \theta^{(i)}) + \alpha_1^{(i)}\tau + \ldots + \alpha_r^{(i)}\tau^r)\tau^i \\ &= \delta^{(i)}_{t^2}, \end{aligned}$$

we find that $\delta^{(i)}$ is a biderivation of A into $N(E)$. Clearly, $\delta^{(0)}, \ldots, \delta^{(r-1)}$ provide a basis for $H^1_{\mathrm{DR}}(\underline{E}, k)$.

For any non-constant a in A, the exponential function $\exp_{\underline{E}}$ attached to the Drinfeld A-module $\underline{E}$ is the unique solution of the algebraic differential equation

$$\exp_{\underline{E}}(\gamma(a)z) = \varphi_a(\exp_{\underline{E}}(z)).$$

Similarly, a biderivation gives rise to a quasi-periodic function.

Lemma 5.2.7 ([Gos94, Lem. 1.5.8 and Lem. 1.5.12]). *Let δ be a biderivation of A into $N(E)$. Then there is a unique entire $\mathbb{F}_q$-linear solution $F_\delta(z)$ of the algebraic differential equation*

$$\gamma(a)F_\delta(z) - F_\delta(\gamma(a)z) = \delta_a\left(\exp_{\underline{E}}(z)\right), \tag{5.10}$$

which is independent of a. For any non-constant $a \in A$,

$$\begin{aligned} F_\delta(z) &= \gamma(a) F_\delta\left(\frac{z}{\gamma(a)}\right) - \delta_a\left(\exp_{\underline{E}}\left(\frac{z}{\gamma(a)}\right)\right) \\ &= \gamma(a)^2 F_\delta\left(\frac{z}{\gamma(a)^2}\right) - \gamma(a)\delta_a\left(\exp_{\underline{E}}\left(\frac{z}{\gamma(a)^2}\right)\right)\delta_a\left(\exp_{\underline{E}}\left(\frac{z}{\gamma(a)}\right)\right) \\ &\vdots \\ &= -\sum_{j=0}^{\infty} \gamma(a)^j \delta_a\left(\exp_{\underline{E}}\left(\frac{z}{\gamma(a)^{j+1}}\right)\right). \end{aligned} \tag{5.11}$$

Such a solution $F_\delta(z)$ of (5.10) is *quasi-periodic* with respect to the period lattice $\Lambda_{\underline{E}}$, meaning that the following always holds:

(a) $F_\delta(z+\lambda) = F_\delta(z) + F_\delta(\lambda)$ for $z \in k$ and $\lambda \in \Lambda_{\underline{E}}$,

(b) $F_\delta(\lambda)$ is A-linear in $\lambda \in \Lambda_{\underline{E}}$.

We change notation to define the quasi-periods of $\underline{E}$ through "path integration".

Definition 5.2.8. Let δ be a biderivation in $D(\underline{E})$ and $\lambda \in \Lambda_{\underline{E}}$ a period of $\underline{E}$. Then we write formally

$$\int_\lambda \delta := F_\delta(\lambda)$$

and call $\int_\lambda \delta$ a *quasi-period of $\underline{E}$*.

Remark 5.2.9 (De Rham isomorphism). Let $\underline{E}$ be a Drinfeld module of rank r over $\mathbb{C}_\infty$. The "path integral" induces a pairing

$$\begin{aligned} H_{\mathrm{B}}(\underline{E}, A) \times H_{\mathrm{DR}}(\underline{E}, \mathbb{C}_\infty) &\to \mathbb{C}_\infty \\ (\lambda, \delta) &\mapsto \int_\lambda \delta. \end{aligned}$$

The induced *de Rham map* $\mathrm{DR} : H^1_{\mathrm{DR}}(\underline{E}, \mathbb{C}_\infty) \to H^1_{\mathrm{B}}(\underline{E}, \mathbb{C}_\infty)$ is an isomorphism by [Gek89, Thm. 5.14] in happy analogy with the classical case. In particular, this shows

$$\dim_{\mathbb{C}_\infty} H^1_{\mathrm{DR}}(\underline{E}, \mathbb{C}_\infty) = r.$$

For X an abelian variety over $k \subset \mathbb{C}$ there is a canonical isomorphism

$$H^1_{\mathrm{DR}}(X) \otimes_k \mathbb{C} \cong H^1_{\mathrm{B}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

whose defining matrix P_X is the period matrix of X (cf. [DMOS82, §I.1]). By the previous remark, the period matrix of a Drinfeld A-module over $k \subset \mathbb{C}_\infty$ similarly gives rise to the de Rham isomorphism.

Definition 5.2.10. Let λ_i, $1 \le i \le r$, be a basis of the period lattice $\Lambda_{\underline{E}}$ and δ_j, $1 \le j \le r$, a basis for $H^1_{\mathrm{DR}}(\underline{E}, k)$. We define the *period matrix $P_{\underline{E}}$ of $\underline{E}$* to be

$$P_{\underline{E}} := \left(\int_{\lambda_i} \delta_j \mid 1 \le i, j \le r\right).$$

Example 5.2.11. (i) Let δ be an exact biderivation of A into $N(E)$, that is, there is an $\mathrm{m} \in N(E)$ such that $\delta_a = \gamma(a)\mathrm{m} - \mathrm{m} \circ \varphi_a$. Then

$$\begin{aligned}\delta_a\left(\exp_{\underline{E}}(z)\right) &= \gamma(a)\mathrm{m}\left(\exp_{\underline{E}}(z)\right) - (\mathrm{m} \circ \varphi_a)\left(\exp_{\underline{E}}(z)\right) \\ &= \gamma(a)\mathrm{m}\left(\exp_{\underline{E}}(z)\right) - \mathrm{m}\left(\exp_{\underline{E}}\left(\gamma(a)z\right)\right)\end{aligned}$$

and the corresponding solution is $F_\delta(z) = \mathrm{m}(\exp_{\underline{E}}(z))$. Therefore,

$$\int_\lambda \delta = F_\delta(\lambda) = 0 \quad \text{for } \lambda \in \Lambda_{\underline{E}}.$$

(ii) Consider the biderivation $\delta^{(0)}$ of A into $N(E)$, which is given by $\delta_a^{(0)} = \varphi_a - \gamma(a)$. Obviously,

$$\begin{aligned}\delta_a^{(0)}\left(\exp_{\underline{E}}(z)\right) &= \varphi_a\left(\exp_{\underline{E}}(z)\right) - \gamma(a)\exp_{\underline{E}}(z) \\ &= \exp_{\underline{E}}\left(\gamma(a)z\right) - \gamma(a)\exp_{\underline{E}}(z) + \gamma(a)z - \gamma(a)z \\ &= \gamma(a)\left(z - \exp_{\underline{E}}(z)\right) - \left(\gamma(a)z - \exp_{\underline{E}}\left(\gamma(a)z\right)\right),\end{aligned}$$

so the unique solution of (5.10) is $F_{\delta^{(0)}}(z) = z - \exp_{\underline{E}}(z)$. Hence,

$$\int_\lambda \delta^{(0)} = F_{\delta^{(0)}}(\lambda) = \lambda \quad \text{for } \lambda \in \Lambda_{\underline{E}},$$

and the periods are contained in the quasi-periods of $\underline{E}$.

(ii) Let $A = \mathbb{F}_q[t]$ and consider the biderivation $\delta^{(i)}$ of A into $N(E)$, which is given by $\delta_t^{(i)} = \tau^i$, $1 \le i \le r-1$. By (5.11), the unique solution $F_{\delta^{(i)}}(z)$ of (5.10) is

$$F_{\delta^{(i)}}(z) = -\sum_{j=0}^{\infty} \theta^j \exp_{\underline{E}}\left(\frac{z}{\theta^{j+1}}\right)^{(i)}.$$

Therefore,

$$\int_\lambda \delta^{(0)} = F_{\delta^{(i)}}(\lambda) = -f_\lambda^{(i)}(\theta) \quad \text{for } \lambda \in \Lambda_{\underline{E}},$$

where $f_\lambda^{(i)}(t) = \sum_{n=0}^\infty \exp_{\underline{E}}(\frac{\lambda}{\theta^{n+1}})^{(i)} t^n$ is the i-fold twist of the Anderson generating function associated with the period λ (see Example 4.1.25).

Recall that the scattering matrix of a pure uniformizable Drinfeld $\mathbb{F}_q[t]$-module $\underline{E}$ of rank r over k is given by

$$\tilde{\Psi}_{ij} := -\sum_{k=0}^{\infty} \exp_{\underline{E}}\left(\frac{\lambda_j}{t^{k+1}}\right)^{(i-1)} t^k, \quad 1 \le i, j \le r,$$

with respect to the $k[t]$-basis $\{1, \ldots, \tau^{r-1}\}$ of $\mathcal{M}^*(E)$ (see (4.3)). We have seen in Example 4.1.25 how $\tilde{\Psi}$ also gives rise to the rigid analytic trivialization Ψ of the corresponding dual Drinfeld $\mathbb{F}_q[t]$-motive. In the next subsection, we will further investigate the relations between the rigid analytic trivialization Ψ and the periods and quasi-periods of $\underline{E}$ if $\underline{E}$ is of rank 2.

Grothendieck's period conjecture for Drinfeld $\mathbb{F}_q[t]$-modules

We assume $A = \mathbb{F}_q[t]$ and fix now a Drinfeld $\mathbb{F}_q[t]$-module (E, φ) of rank 2 over $\overline{Q}$. We first study the relations between its period matrix $P_{\underline{E}}$ and the rigid analytic trivialization of the corresponding dual Drinfeld $\mathbb{F}_q[t]$-motive $(\mathrm{M}, \sigma_{\mathrm{M}})$. We choose a coordinate system ρ, so that we may assume without loss of generality $E = \mathbb{G}_{a,k}$, and φ is determined by

$$\varphi_t = \theta + \alpha_1\tau + \alpha_2\tau^2 \in \overline{Q}[\tau] \cong \mathrm{End}(E).$$

Further, by [CP08, Rem. 3.4.2], we may assume without loss of generality $\alpha_2 = 1$. We write $\exp_{\underline{E}} = \sum_{i=0}^{\infty} e_i\tau^i$ where $e_0 = 1$. In Example 4.1.25, we have associated the Anderson generating function

$$\begin{aligned} f_\lambda(t) \;&:=\; \sum_{n=0}^{\infty} \exp_{\underline{E}}\left(\frac{\lambda}{\theta^{n+1}}\right) t^n = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} e_i \left(\frac{\lambda}{\theta^{n+1}}\right)^{(i)} t^n = \sum_{i=0}^{\infty} \frac{e_i\lambda^{(i)}}{\theta^{(i)}} \cdot \sum_{n=0}^{\infty}\left(\frac{t}{\theta^{(i)}}\right)^n \\ &=\; \sum_{i=0}^{\infty} \frac{e_i\lambda^{(i)}}{\theta^{(i)}} \cdot \frac{1}{1-\frac{t}{\theta^{(i)}}} = \sum_{i=0}^{\infty} \frac{e_i\lambda^{(i)}}{\theta^{(i)}} \cdot \frac{\theta^{(i)}}{\theta^{(i)}-t} = \sum_{i=0}^{\infty} \frac{e_i\lambda^{(i)}}{\theta^{(i)}-t} \in \mathbb{T} = k\langle t\rangle \end{aligned}$$

with a period $\lambda \in \Lambda_{\underline{E}}$ through the corresponding convergent t-division tower. Thus $f_\lambda(t)$ has a simple pole at $t = \theta^{(i)}$ with residue $-e_i\lambda^{(i)}$ for $i = 0, 1, \ldots$. By the same arguments as in (4.9),

$$\alpha_1 f_\lambda^{(1)}(t) + f_\lambda^{(2)}(t) = (t-\theta) f_\lambda(t)$$

and because $f_\lambda(t)$ coverges outside $\{\theta^{-(i)} \mid i \in \mathbb{N}\}$,

$$\alpha_1 f_\lambda^{(1)}(\theta) + f_\lambda^{(2)}(\theta) = -\lambda \tag{5.12}$$

holds. Moreover,

$$\Phi_{\mathbf{m}} := \begin{pmatrix} 0 & 1 \\ (t-\theta) & -\alpha_1^{(-1)} \end{pmatrix}$$

represents σ_{M} with respect to the basis $\mathbf{m} = (1, \sigma)^{\mathrm{tr}}$ for M. We pick a basis $\{\lambda_1, \lambda_2\}$ of periods of $\underline{E}$ and denote $f_{\lambda_i}(t)$ by f_i for $i = 1, 2$. We then put

$$A := \begin{pmatrix} \alpha_1^{(-1)} & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\overline{Q}) \quad \text{and} \quad \tilde{\Psi} := -\begin{pmatrix} f_1 & f_1^{(1)} \\ f_2 & f_2^{(1)} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{T}),$$

which gives rise to the matrix

$$\Theta := \tilde{\Psi}^{(1)} \cdot A^{(1)} = -\begin{pmatrix} \alpha_1 f_1^{(1)} + f_1^{(2)} & f_1^{(1)} \\ \alpha_1 f_2^{(1)} + f_2^{(2)} & f_2^{(1)} \end{pmatrix} = -\begin{pmatrix} (t-\theta)f_1 & f_1^{(1)} \\ (t-\theta)f_2 & f_2^{(1)} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{T})$$

such that the vector

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} := \Theta \cdot \mathbf{m}$$

comprises an $\mathbb{F}_q[t]$-basis for $\mathrm{M}(1)^\sigma$ and $\Theta = \Theta^{(-1)} \cdot \Phi_{\mathbf{m}}$ holds. Hence,

$$\begin{aligned} \Psi \;&:=\; \Theta^{-1} = \frac{1}{f_1^{(1)}(t-\theta)f_2 - (t-\theta)f_1 f_2^{(1)}} \cdot \begin{pmatrix} -f_2^{(1)} & f_1^{(1)} \\ (t-\theta)f_2 & -(t-\theta)f_1 \end{pmatrix} \\ &\in\; \mathrm{GL}_2(\mathbb{T}) \cap \mathrm{Mat}_{2\times 2}(\mathbb{E}) \end{aligned}$$

is a rigid analytic trivialization of $\Phi_{\mathbf{m}}$, that is $\Psi^{(-1)} = \Phi_{\mathbf{m}}\Psi$. We let $\{\delta^{(0)}, \delta^{(1)}\}$ be the basis of $H^1_{\mathrm{DR}}(\underline{E}, \mathbb{C}_\infty)$ as defined in Example 5.2.6. By Example 5.2.11, the period matrix of $\underline{E}$ is given by

$$P_{\underline{E}} = \begin{pmatrix} \int_{\lambda_1} \delta^{(0)} & \int_{\lambda_1} \delta^{(1)} \\ \int_{\lambda_2} \delta^{(0)} & \int_{\lambda_2} \delta^{(1)} \end{pmatrix} = \begin{pmatrix} \lambda_1 & -f^{(1)}_{\lambda_1}(\theta) \\ \lambda_2 & -f^{(1)}_{\lambda_2}(\theta) \end{pmatrix} \overset{(5.12)}{=} \Theta(\theta).$$

We let $\tilde{\pi} \in \overline{Q}$ be the period of the dual Carlitz $\mathbb{F}_q[t]$-module over $\overline{Q}$. Using the *Legendre relation*

$$f^{(1)}_{\lambda_1}(\theta)\lambda_2 - \lambda_1 f^{(1)}_{\lambda_2}(\theta) = \tilde{\pi}$$

proved by Anderson [Tha04, Thm. 6.4.6], we then obtain

$$\Psi(\theta) \overset{(5.12)}{=} -\frac{1}{\tilde{\pi}} \cdot \begin{pmatrix} -f^{(1)}_{\lambda_2}(\theta) & f^{(1)}_{\lambda_1}(\theta) \\ -\lambda_2 & \lambda_1 \end{pmatrix}.$$

We may thus conclude that

$$\begin{aligned} \overline{Q}\Big(\int_{\lambda_i} \delta^{(j-1)} \mid 1 \le i,j \le 2\Big) &= \overline{Q}\left(\lambda_1, \lambda_2, f^{(1)}_{\lambda_1}(\theta), f^{(1)}_{\lambda_2}(\theta)\right) \\ &= \overline{Q}(\Psi(\theta)_{ij} \mid 1 \le i,j \le 2). \end{aligned} \tag{5.13}$$

An elliptic curve X has no complex multiplication if $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}$. By the following lemma, we find that the same holds for Drinfeld $\mathbb{F}_q[t]$-modules of rank 2.

Lemma 5.2.12. *Let* $\underline{\mathrm{M}} \in \mathrm{Ob}(\mathscr{PRDA}^I_+)$ *be a dual Anderson A-motive of rank* 2 *over* $\overline{Q}$. *If* $\underline{\mathrm{M}}$ *has no complex multiplication then* $\mathrm{QEnd}(\underline{\mathrm{M}}) = Q$.

Proof. For the purpose of deriving a contradiction, assume that $\dim_Q \mathrm{QEnd}(\underline{\mathrm{M}}) \ge 2$. Hence there is an $f \in \mathrm{QEnd}(\underline{\mathrm{M}})$ such that $f \notin Q \cdot \mathrm{id}$. Then $Q\,[f]$ is a commutative semisimple subalgebra of $\mathrm{QEnd}(\underline{\mathrm{M}})$ with $\dim_Q Q\,[f] \ge 2$. By assumption we have $2 = \dim_Q Q\,[f]$ and $\underline{\mathrm{M}}$ has sufficiently many complex multiplication through $Q\,[f]$. This gives us a contradiction and proves the lemma. □

Combining this with the following main result of [Pin97a], we may determine the Hodge-Pink group of a Drinfeld $\mathbb{F}_q[t]$-module of rank 2 that has no complex multiplication.

Theorem 5.2.13 ([Pin97a, Thm. 10.3])**.** *Let* $\underline{E}$ *be a Drinfeld* $\mathbb{F}_q[t]$*-module of rank* r *over* $k \subseteq \mathbb{C}_\infty$ *and* $G_{\underline{E}}$ *its Hodge-Pink group. Then*

$$G_{\underline{E}} = \mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{QEnd}(\underline{E})).$$

Remark 5.2.14. This result can be obtained in a different way through use of the proven Hodge conjecture (Theorem 4.2.19) and the Mumford-Tate conjecture [Pin97c, Thm. 0.2]. Let $\underline{\mathrm{M}} = (\mathrm{M}, \sigma_{\mathrm{M}}) \in \mathrm{Ob}(\mathscr{PRDA}^I_+)$ be a pure rigid analytically trivial dual Anderson A-motive of rank r over k, M_a its a-adic completion and $\sigma : \mathrm{M}_a \to \mathrm{M}_a$ the $\varsigma^*_{A(a)}$-linear map induced by σ_{M} (see Section 2.5). The σ-invariants of M_a are the A_a-module $\mathrm{M}_a^\sigma := \{\mathrm{m} \in \mathrm{M}_a \mid \sigma(\mathrm{m}) = \mathrm{m}\}$ of rank r. Then the *a-adic cohomology realization of* $\underline{\mathrm{M}}$ is given by $H_a(\underline{\mathrm{M}}) := \mathrm{M}_a^\sigma \otimes_{A_a} Q_a$. Suppose now that $\underline{\mathrm{M}} = \mathcal{M}^*(\underline{E})$ and consider the strictly full rigid abelian tensor subcategory $\langle\!\langle \underline{P} \rangle\!\rangle$ of $\mathscr{PT}'$ generated by the pure dual t-motive $\underline{P} := \mathcal{P}'(\underline{\mathrm{M}})$. Together with the fiber functor

$$\big((\mathrm{N}, \sigma_{\mathrm{N}})(i) \mapsto H_a(\underline{\mathrm{N}}) \otimes_{Q_a} H_a(\underline{\mathrm{C}})^{-i}\big) : \langle\!\langle \underline{P} \rangle\!\rangle \to \mathscr{V}ec_{Q_a},$$

it is a Tannakian category over Q_a. The *absolute Galois group* $\mathrm{Gal}(\overline{k}/k)$ acts on M_a^σ, which induces a map

$$\Gamma_{\underline{P}} \times_Q Q_a \leftarrow \mathrm{Gal}(\overline{k}/k).$$

We know from Proposition 4.2.17 that $\Gamma_{\underline{P}}$ is a closed subgroup scheme of $\mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{End}(\underline{P}))$ and hence

$$\Gamma_{\underline{P}} \times_Q Q_a \subseteq \mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{End}(\underline{P})) \times_Q Q_a \cong \mathrm{Cent}_{\mathrm{GL}_{r,Q_a}}(\mathrm{End}(\underline{P})).$$

Note that $\mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{End}(\underline{P})) \cong \mathfrak{R}_{\mathrm{End}(\underline{P})/Q}\, \mathrm{GL}_{r,Q}$ is irreducible and therefore connected. We assume that

$$\Gamma_{\underline{P}} \subsetneq \mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{End}(\underline{P}))$$

is a proper closed subgroup scheme, which contradicts that the image of

$$\mathrm{Gal}(\overline{k}/k) \to \mathrm{Cent}_{\mathrm{GL}_{r,Q_a}}(\mathrm{End}(\underline{P}))$$

is open [Pin97c, Thm. 0.2]. Hence,

$$\Gamma_{\underline{P}} = \mathrm{Cent}_{\mathrm{GL}_{r,Q}}(\mathrm{End}(\underline{P})),$$

and Theorem 5.2.13 follows from the Hodge conjecture.

Corollary 5.2.15. *Let $\underline{E}$ be a Drinfeld $\mathbb{F}_q[t]$-module of rank 2 over $\overline{Q}$ that has no complex multiplication and $G_{\underline{E}}$ its Hodge-Pink group. Then*

$$G_{\underline{E}} = \mathrm{Cent}_{\mathrm{GL}_{2,Q}}(\mathrm{QEnd}(\underline{E})) = \mathrm{GL}_{2,Q}\,.$$

In particular, $\dim G_{\underline{E}} = 4$.

Recall that we have determined the dimension of the Hodge-Pink group of $\underline{E}$ in Theorem 5.1.11 when $\underline{E}$ has complex multiplication under some conditions. Putting this together with the previous Corollary deduced from [Pin97a, Thm. 10.3], Grothendieck's period conjecture for function fields yields the following:

Theorem 5.2.16 (Grothendieck's period conjecture for Drinfeld $\mathbb{F}_q[t]$-modules of rank 2)**.** *Let $\underline{E}$ be a Drinfeld $\mathbb{F}_q[t]$-module over $\overline{Q} \subset \overline{Q_\infty}$, $P = (\int_{\lambda_i} \delta_j)$ its period matrix and $G_{\underline{E}}$ the Hodge-Pink group of $\underline{E}$. Then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}\, \overline{\mathbb{Q}}(\int_{\lambda_i} \delta_j) \overset{(5.13)}{=} \dim G_{\underline{E}} = \begin{cases} 2 & \textit{if } \underline{E} \textit{ is of CM-type,}^1 \\ 4 & \textit{otherwise.} \end{cases}$$

Remark 5.2.17. (i) Thiéry gives a proof of Grothendieck's period conjecture for Drinfeld $\mathbb{F}_q[t]$-modules of rank 2 that have complex multiplication in [Thi92].

(ii) Papanikolas and Chang are able to show Grothendieck's period conjecture for Drinfeld $\mathbb{F}_q[t]$-modules of rank 2 that have complex multiplication if the characteristic p is odd [CP08, Thm. 3.4.1]. Their proof proceeds by showing that the canonical representation $\phi : \Gamma_{\underline{P}} \hookrightarrow \mathrm{GL}(P(1)^\sigma)$ of the Galois group $\Gamma_{\underline{P}}$ (see [CP08, Thm. 3.3.1]) is absolutely irreducible.

(ii) Using the ring homomorphism $i^* : \mathbb{F}_q[t] \to A$, $a \mapsto t$, one can show that Theorem 5.2.16 holds for Drinfeld A-modules over $\overline{Q}$ where A is the ring of integers of an arbitrary function field Q.

[1] If $\underline{E}$ has sufficiently many complex multiplication through E, then $\dim_Q E = 2$. Hence, E/Q is either separable or purely inseparable and we may apply Theorem 5.1.11.

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