A criterion for good reduction of Drinfeld modules and Anderson motives in terms of local shtukas

U. Hartl, S. Hüsken

Abstract

For an Anderson A-motive over a discretely valued field whose residue field has A-characteristic ε, we prove a criterion for good reduction in terms of its associated local shtuka at ε. This yields a criterion for good reduction of Drinfeld modules. Our criterion is the function-field analog of Grothendieck's [SGA 7, Proposition IX.5.13] and de Jong's [dJ98, 2.5] criterion for good reduction of an abelian variety over a discretely valued field with residue characteristic p in terms of its associated p-divisible group.
Mathematics Subject Classification (2000): 11G09, (14L05)

Contents

1 Introduction

We fix a finite field $\mathbb{F}$ with r elements and characteristic p. Let $\mathcal{C}$ be a smooth projective and geometrically irreducible curve over $\mathbb{F}$ with function field $Q = \mathbb{F}(\mathcal{C})$. Let $\infty \in \mathcal{C}$ be a closed point and let $A = \Gamma(\mathcal{C} \smallsetminus \{\infty\}, \mathcal{O}_\mathcal{C})$ be the $\mathbb{F}$-algebra of those rational functions on $\mathcal{C}$ which are regular outside ∞. For every $\mathbb{F}$-algebra R we let σ be the endomorphism of $A_R := A \otimes_\mathbb{F} R$ given by $\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{r,R} \colon a \otimes b \mapsto a \otimes b^r$ for $a \in A$ and $b \in R$.

Let o_L be a complete discrete valuation ring containing $\mathbb{F}$, with fraction field L, uniformizing parameter π, maximal ideal $\mathfrak{m}_L = (\pi)$ and residue field $\ell = o_L/\mathfrak{m}_L$. We assume that ℓ is a finite field extension of ℓ^p. This is equivalent to saying that ℓ has a finite p-basis over ℓ^p in the sense of [Bou81, § V.13, Definition 1]. It holds for example if ℓ is perfect, or if ℓ is a finitely generated field. Since every Anderson A-motive over L can be defined over a finitely generated subfield of L our restriction on ℓ is not serious. Let $c^* \colon A \to o_L$ be a homomorphism of $\mathbb{F}$-algebras such that the kernel of the composition $A \to o_L \twoheadrightarrow \ell$ is a *maximal* ideal ε in A. We say that *the residue field ℓ has finite A-characteristic ε*. We do not assume that $c^* \colon A \to o_L$ is injective. So L can have either generic A-characteristic $\ker c^* = (0)$ or finite A-characteristic $\ker c^* = \varepsilon$.

In the following we will consider various A_{o_L}-algebras. In all of them we consider the ideal generated by $\{a \otimes 1 - 1 \otimes c^*(a) \colon a \in A\} \subseteq A_{o_L}$. By abuse of notation we denote all these ideals by $\mathfrak{J}$.

By an *Anderson A-motive over L* we mean a pair $\underline{M} = (M, F_M)$ consisting of a locally free A_L-module M of finite rank, and an injective A_L-homomorphism $F_M \colon \sigma^* M \to M$ where $\sigma^* M := M \otimes_{A_L,\sigma} A_L$, such that $\operatorname{coker}(F_M)$ is a finite dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. We say that $\underline{M}$ has *good reduction over o_L* if there exists a locally free A_{o_L}-module $\mathcal{M}$ and an injective A_{o_L}-homomorphism $F_{\mathcal{M}} \colon \sigma^*\mathcal{M} \to \mathcal{M}$ such that $(\mathcal{M}, F_{\mathcal{M}}) \otimes_{A_{o_L}} A_L \cong \underline{M}$ and $\operatorname{coker}(F_{\mathcal{M}})$ is a finite free o_L-module which is annihilated by a power of $\mathfrak{J}$. We call $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ a *good model of $\underline{M}$*. In particular if $\underline{M} = \underline{M}(\varphi)$ is the Anderson A-motive associated with a Drinfeld A-module φ over L, then $\underline{M}$ has good reduction if and only if φ has good reduction; see Proposition 4.10.

Anderson A-motives are function-field analogs of abelian varieties. For an abelian variety $\mathcal{A}$ over a discretely valued field K with residue field of characteristic p there are criteria for good reduction in terms of local data. For a prime number $l \neq p$ the criterion of Néron-Ogg-Shavarevich [ST68, §1, Theorem 1] states that $\mathcal{A}$ has good reduction if and only if the l-adic Tate module $T_l\mathcal{A}$ of $\mathcal{A}$ is unramified as a $\operatorname{Gal}(K^{\mathrm{alg}}/K)$-representation. At the prime p the criterion of Grothendieck [SGA 7, Proposition IX.5.13] (for $\operatorname{char}(K) = 0$), respectively de Jong [dJ98, 2.5] (for $\operatorname{char}(K) = p$) states that $\mathcal{A}$ has good reduction if and only if the Barsotti-Tate group $\mathcal{A}[p^\infty]$ has good reduction.

These criteria have function-field analogs for Anderson A-motives. The analog of the Néron-Ogg-Shavarevich-criterion was proved by Gardeyn [Gar02, Theorem 1.1]. In this article we simultaneously prove the analog of Grothendieck's and de Jong's criterion. Here the function-field analogs of Barsotti-Tate groups are local shtukas [Har11, § 2.1] which are defined as follows. Let $A_{o_L,(\varepsilon,\pi)}$ be the (ε, π)-adic completion of A_{o_L}. An (*effective*) *local shtuka at ε over o_L* is a pair $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ consisting of a finite free $A_{o_L,(\varepsilon,\pi)}$-module $\hat{M}$ and an injective $A_{o_L,(\varepsilon,\pi)}$-homomorphism $F_{\hat{M}} \colon \sigma^*\hat{M} \to \hat{M}$ such that $\operatorname{coker}(F_{\hat{M}})$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$. The local shtuka associated with a good model $\underline{\mathcal{M}}$ of an Anderson A-motive is $\underline{\hat{M}}(\underline{\mathcal{M}}) := \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,(\varepsilon,\pi)}$. Strictly speaking effective local shtukas are the function field analogs of the F-crystals of Barsotti-Tate groups. The analogs of the latter are called *ε-divisible local Anderson-modules* and their category is equivalent to the category of effective local shtukas; see [HS15] for more details. Our analog of Grothendieck's and de Jong's reduction criterion is now the following

Corollary 6.6. *Let $\underline{M}$ be an Anderson A-motive over L. Then $\underline{M}$ has good reduction over o_L if and only if there is an effective local shtuka $\underline{\hat{M}}$ at ε over o_L and an isomorphism $\underline{M} \otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$.*

(In the body of the text we prove a slightly stronger statement.) This applies in particular if $\underline{M}$ is the Anderson A-motive associated with a Drinfeld module φ over L to give a criterion for good reduction of φ in terms of its associated local shtuka. The reformulation of this criterion in terms of the ε-divisible local Anderson-module of φ is given in [HS15].

Acknowledgements. We would like to thank the anonymous referee for his careful reading and for asking an interesting question which lead to the answer given in Remark 5.4. We also thank the Deutsche Forschungsgemeinschaft for supporting this research in form of SFB 878.

2 The base rings

Let o_L be an equi-characteristic complete discrete valuation ring containing the finite field $\mathbb{F}$, with quotient field $L = \operatorname{Frac}(o_L)$ and residue field $\ell = o_L/\mathfrak{m}_L$, where $\mathfrak{m}_L \subseteq o_L$ is the maximal ideal of o_L. We assume that ℓ is a *finite* field extension of $\ell^p := \{b^p \colon b \in \ell\}$. We fix a uniformizer $\pi = \pi_L$ of o_L and sometimes identify

o_L with $\ell[\![\pi]\!]$. Let $v = v_\pi = \mathrm{ord}_\pi(\cdot)$ be the discrete valuation on L normalized by $v(\pi) = 1$.

We assume that there is an o_L-valued point $c \in \mathcal{C}(o_L)$ such that the corresponding $\mathbb{F}$-morphism $c\colon \operatorname{Spec}(o_L) \to \mathcal{C}$ factors via $\mathcal{C} \smallsetminus \{\infty\} \subseteq \mathcal{C}$. Such a datum corresponds to a homomorphism of $\mathbb{F}$-algebras $c^*\colon A \to o_L$ which we call the *characteristic map*. We further assume that the closed point $V(\pi) \subseteq \operatorname{Spec}(o_L)$ is mapped to a closed point ε of $\operatorname{Spec}(A) \subseteq \mathcal{C}$. The latter is the kernel of the composition $A \to o_L \twoheadrightarrow \ell$. So, in accordance with Drinfeld's terminology [Dri76], we call ε the *residue characteristic* or *residual characteristic place of* Q. By continuity, the characteristic map $c^*\colon A \to o_L$ factors through a morphism of complete discrete valuation rings $A_\varepsilon \to o_L$ where A_ε is the completion of A at the characteristic place ε. Note that $A_\varepsilon \to o_L$ is injective if c^* is injective, and factors through A/ε if c^* is not injective.

Remark 2.1. Since A is a Dedekind domain there is a power ε^m which is a principal ideal in A. We fix a generator t of ε^m and frequently use the finite flat monomorphism of $\mathbb{F}$-algebras $\iota\colon \mathbb{F}[z] \to A, z \mapsto t$.

For any $\mathbb{F}$-algebra R we abbreviate $A_R := A \otimes_{\mathbb{F}} R$. In particular, $A_{o_L} \subseteq A_L$ is a noetherian integral domain, and by virtue of the equality $A_\ell \cong A_{o_L}/\pi A_{o_L}$ it follows that $\pi \in o_L$ is a prime element of A_{o_L}.

Definition 2.2. Let $A_{o_L,\pi}$ (resp., $A_{o_L,(\varepsilon,\pi)}$) be the completion of the o_L-algebra A_{o_L} for the π-adic topology (resp., the (ε,π)-adic topology).

By Krull's Theorem ([Bou67], III.3.2), the ring A_{o_L} is separated for both the π-adic and the (ε,π)-adic topology. The topological o_L-algebra $A_{o_L,\pi}$ is admissible in the sense of Raynaud, i.e. it is of topologically finite presentation and has no π-torsion. In particular, the L-algebra $A_{o_L,\pi}[1/\pi]$ is affinoid in the sense of rigid analytic geometry; see [Bos14, BL93a, BGR84].

For example if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $A = \mathbb{F}[z]$ then we have $A_{o_L} = o_L[z]$ and correspondingly $A_L = L[z]$. Let us specify that $\varepsilon = z\mathbb{F}[z]$. Our choice of a uniformizer π gives rise to an identification $o_L = \ell[\![\pi]\!]$. Consequently $o_L[\![z]\!] = \ell[\![\pi]\!][\![z]\!] = \ell[\![\pi, z]\!] = A_{o_L,(\varepsilon,\pi)}$. On the other hand, the π-adic completion of $o_L[z]$ equals $o_L\langle z\rangle := \{\sum_{i=0}^{\infty} b_i z^i\colon v(b_i) \to \infty (i \to \infty)\}$, and since $L\langle z\rangle = o_L\langle z\rangle \otimes_{o_L} L$, we may view $A_{o_L,\pi}[1/\pi]$ as a replacement, for general $\mathcal{C}$, of the Tate algebra $L\langle z\rangle$ of strictly convergent power series in one indeterminate z over L, which serves as coordinate ring for the one-dimensional affinoid unit ball in rigid analytic geometry.

There is a natural embedding $A_L \to A_{o_L,\pi}[1/\pi]$ which, for general $\mathcal{C}$, replaces the completion homomorphism $L[z] \to L\langle z\rangle$, and which itself can be regarded as a completion map with respect to the L-algebra norm-topology on the *reduced* affinoid L-algebra $A_{o_L,\pi}[1/\pi]$ and its restriction on A_L; see [Bos14, §1.4, Proposition 19]. Note that the canonical homomorphism $A_{o_L} \to A_{o_L,(\varepsilon,\pi)}$ factors uniquely via $A_{o_L,\pi}$, where the induced map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ identifies $A_{o_L,(\varepsilon,\pi)}$ with the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion of $A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is a regular integral domain, it is $(\varepsilon,\pi)A_{o_L,\pi}$-adically separated by Krull's theorem and $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is injective and flat.

Recall that there is a finite flat monomorphism of $\mathbb{F}$-algebras $\iota\colon \mathbb{F}[z] \to A$ which identifies the indeterminate z with the generator $t \in A$ of ε^m chosen in Remark 2.1. The o_L-algebra homomorphism $\iota \otimes \mathrm{id}\colon o_L[z] \to A_{o_L}$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu t^\nu \otimes a_\nu$, is finite flat, so that we obtain finite flat maps

$$o_L\langle z\rangle \to A_{o_L,\pi}, \quad L\langle z\rangle \to A_{o_L,\pi}[1/\pi], \quad o_L[\![z]\!] \to A_{o_L,(t,\pi)}, \quad \ell[z] \to A_\ell. \tag{2.1}$$

Here the (t,π)-adic completion $A_{o_L,(t,\pi)}$ of A_{o_L} equals $A_{o_L,(\varepsilon,\pi)}$ since $(\varepsilon,\pi)^m \subseteq (\varepsilon^m,\pi) = (t,\pi)$ in A_{o_L}.

Lemma 2.3. *If $A_{o_L,\varepsilon}$ denotes the ε-adic completion of A_{o_L}, the canonical map $A_{o_L,\varepsilon} \to A_{o_L,(\varepsilon,\pi)}$ is an isomorphism.* □

3 Frobenius modules

The r-Frobenius $\mathrm{Frob}_r\colon o_L \to o_L$, $x \mapsto x^r$, gives rise to an endomorphism

$$\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_r\colon A_{o_L} \to A_{o_L}, \quad a \otimes x \mapsto a \otimes x^r,$$

which extends to give a map $\mathrm{id}_A \otimes \mathrm{Frob}_{r,L}\colon A_L \to A_L$ again denoted by σ. On the other hand, reducing mod π gives $\bar{\sigma} = \mathrm{id}_A \otimes \mathrm{Frob}_{r,\ell}\colon A_\ell \to A_\ell$. The latter is a finite flat endomorphism of the Dedekind domain A_ℓ, because ℓ is finite over ℓ^p. The map $\sigma\colon A_{o_L} \to A_{o_L}$ is π-adically and (ε, π)-adically continuous and therefore extends to give endomorphisms $A_{o_L,\pi} \to A_{o_L,\pi}$ and $A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$, again denoted by σ.

Lemma 3.1. *In the commutative diagram*

$$\begin{array}{ccccc} A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \\ \sigma \big\downarrow & & \sigma \big\downarrow & & \sigma \big\downarrow \\ A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \end{array}$$

both squares are cocartesian, and the vertical arrows are finite flat.

We let the proof be preceded by the following

Remark. Via the identification $o_L = \ell[\![\pi]\!]$, the r-Frobenius $\mathrm{Frob}_{r,o_L}\colon o_L \to o_L$ is mirrored by the map $\ell[\![\pi]\!] \to \ell[\![\pi]\!]$, $\sum_{\nu=0}^{\infty} a_\nu \pi^\nu \mapsto \sum_{\nu=0}^{\infty} a_\nu^r \pi^{r\nu}$. Choosing an ℓ^r-basis of ℓ and lifting it to a subset W of o_L, this implies $(\mathrm{Frob}_{r,o_L})_* o_L = \bigoplus_{i=0}^{r-1} \bigoplus_{w \in W} o_L\, w\pi^i$, so that $\mathrm{Frob}_{r,o_L}\colon o_L \to o_L$ is finite flat.

Proof of Lemma 3.1. By base change the remark implies that $\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_{r,o_L}\colon A_{o_L} \to A_{o_L}$ is finite flat, and that $A_{o_L} \otimes_{\sigma, A_{o_L}} A_{o_L,\pi}$ is a finite flat $A_{o_L,\pi}$-module and hence equals the π-adic completion of the A_{o_L}-module $\sigma_* A_{o_L}$. If we let $\mathfrak{a} = \sigma(\pi A_{o_L}) A_{o_L} = \pi^r A_{o_L}$ and $\mathfrak{b} = \pi A_{o_L}$, we get $\mathfrak{b}^r = \mathfrak{a} \subseteq \mathfrak{b}$. Consequently, by [Eis95, Lemma 7.14], the inverse systems $(A_{o_L}/\mathfrak{a}^n)_n$ and $(A_{o_L}/\mathfrak{b}^n)_n$ give the same limit, which shows that the square on the left is cocartesian, and that $\sigma\colon A_{o_L,\pi} \to A_{o_L,\pi}$ is finite flat. Similarly, we have $\sigma(\varepsilon, \pi) A_{o_L} = (\varepsilon, \pi^r) \subseteq (\varepsilon, \pi)$ as well as $(\varepsilon, \pi)^r \subseteq (\varepsilon, \pi^r)$, which proves that the displayed diagram qualifies $A_{o_L,(\varepsilon,\pi)}$ as tensor product $A_{o_L,(\varepsilon,\pi)} \otimes_{A_{o_L},\sigma} A_{o_L}$, and that $\sigma\colon A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$ is finite flat. $\square$

Finally, note that the embedding of o_L-algebras $\iota \otimes \mathrm{id}\colon o_L[z] \to A_{o_L}$ commutes with $\sigma\colon A_{o_L} \to A_{o_L}$ and the r-Frobenius lift of $o_L[z]$, given by $o_L[z] \to o_L[z]$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu a_\nu^r z^\nu$. Consequently, also the embeddings from (2.1) are Frobenius-equivariant.

Let B be an o_L-algebra together with a ring endomorphism $\sigma\colon B \to B$ such that σ and $\mathrm{Frob}_{r,o_L}\colon o_L \to o_L$ are compatible with the structure map $o_L \to B$. For example, B could be any of the base rings considered above.

Definition 3.2. We define the category $\mathrm{FMod}(B)$ of *Frobenius B-modules* (or simply *F-modules* over B) as follows:

- An object of $\mathrm{FMod}(B)$ is a pair $\underline{M} = (M, F)$ consisting of a B-module M which is locally free of finite rank, together with an *injective* B-linear map $F = F_M\colon \sigma^* M \to M$, where $\sigma^* M := M \otimes_{B,\sigma} B$.
- A *morphism* of Frobenius B-modules $(M, F_M) \to (N, F_N)$ is a B-linear map $\varphi\colon M \to N$ between the underlying B-modules such that φ is *F-equivariant*, i.e. such that $\varphi \circ F_M = F_N \circ \sigma^* \varphi$. It is called an *isomorphism* if φ is an isomorphism of the underlying B-modules.

Let B' be a flat B-algebra together with a ring endomorphism $\sigma\colon B' \to B'$ extending the Frobenius lift of B, as explained before. Then the exact functor $\cdot \otimes_B B'$ from B-modules to B'-modules yields a functor $\mathrm{FMod}(B) \to \mathrm{FMod}(B')$. If the structure map $B \to B'$ is, in addition, injective then the induced functor on $\mathrm{FMod}(B)$ is faithful since, given a map $f\colon M \to N$ of finite projective B-modules, restricting its image $f \otimes \mathrm{id}\colon M \otimes_B B' \to N \otimes_B B'$ to M gives back f. In particular, we obtain a natural commutative diagram of categories and faithful functors

$$\begin{array}{ccccc} \mathrm{FMod}(A_{o_L}) & \longrightarrow & \mathrm{FMod}(A_{o_L,\pi}) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{FMod}(A_L) & \longrightarrow & \mathrm{FMod}(A_{o_L,\pi}[1/\pi]) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \end{array}$$

Slightly abusing notation, we agree to write $\underline{M} \otimes_B B'$ for $(M \otimes_B B', F_M \otimes \mathrm{id}_{B'})$, whenever $\underline{M} = (M, F_M)$.

4 Anderson motives

Let $\mathfrak{J} \subseteq A_{o_L}$ be the ideal generated by $a \otimes 1 - 1 \otimes c^*(a)$ for all $a \in A$. For example, if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $A = \mathbb{F}[z]$, then $\mathfrak{J} = (z - \zeta) \subseteq o_L[z]$ where $\zeta = c^*(z)$. Note that the convention introduced in Remark 2.1 that $(z) = \varepsilon^m$ implies $\zeta \in \mathfrak{m}_L$. So $\zeta = 0$ if c^* is not injective. By abuse of notation we denote the ideal generated by $\mathfrak{J}$ in any A_{o_L}-algebra again by $\mathfrak{J}$. We consider the following variant of Anderson's [And86] t-motives.

Definition 4.1. An *Anderson A-motive over L* is an object $\underline{M} = (M, F_M) \in \mathrm{FMod}(A_L)$ such that $\mathrm{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. A *morphism* of Anderson A-motives is defined as a morphism inside $\mathrm{FMod}(A_L)$.

Since $\mathrm{Spec}(A_L)$ is of finite type over L, one can consider its rigid analytification $\mathrm{Spec}(A_L)^{\mathrm{an}}$; see [Bos14], [BGR84], [FP04]. In accordance with [BH07], we denote this rigid analytic L-space by $\mathfrak{A}(\infty)$. On the other hand, the formal completion of the o_L-scheme $X = \mathrm{Spec}(A_{o_L})$ along its special fiber $V(\pi)$ leads to the formal o_L-scheme $\mathfrak{X} = \mathrm{Spf}(A_{o_L,\pi})$; see [EGA, $\mathrm{I}_{\mathrm{new}}$, I.10.8.3]. Its associated rigid analytic space $\mathfrak{X}_{\mathrm{rig}}$ ([Bos14], [FP04]) is given by the affinoid L-space $\mathfrak{A}(1) := \mathrm{Sp}(A_{o_L,\pi}[1/\pi])$. This space can be regarded as the unit disc of the rigid analytic space $\mathfrak{A}(\infty)$ as it corresponds to "radius of convergence 1", hence the notation.

We study the following instance of rigid analytic τ-sheaves over $A_{o_L,\pi}[1/\pi]$, in the sense of [BH07].

Definition 4.2. An *analytic Anderson $A(1)$-motive* over L is an object $\underline{M} = (M, F_M) \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ such that $\mathrm{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. A *morphism* of analytic Anderson $A(1)$-motives is defined as a morphism in the category $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

Here the prefix "$A(1)$-" indicates that we are considering an analytic variant of Anderson A-motives over the rigid analytic "unit disc" $\mathfrak{A}(1)$ in $\mathrm{Spec}(A_L)$.

Proposition 4.3. *The natural functor* $\mathrm{FMod}(A_L) \to \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, $\underline{M} \mapsto \underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ *restricts to a functor* (*Anderson A-motives over L*) $\to$ (*analytic Anderson $A(1)$-motives over L*). □

Definition 4.4. (a) Let $\underline{M}_L \in \mathrm{FMod}(A_L)$ be an F-module over A_L. A *model* of $\underline{M}_L$ is a pair $(\underline{\mathcal{M}}, \alpha)$ consisting of an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ and an isomorphism $\alpha\colon \underline{M}_L \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_{o_L}} A_L$ inside $\mathrm{FMod}(A_L)$.

(b) Let $\underline{M}_L \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ be an F-module over $A_{o_L,\pi}[1/\pi]$. A *(formal) model* of $\underline{M}_L$ is a pair $(\underline{\mathcal{M}}, \alpha)$ consisting of an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ and an isomorphism $\alpha\colon \underline{M}_L \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi]$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

(c) In both cases a *morphism* of models $\beta\colon (\underline{\mathcal{M}}, \alpha) \to (\underline{\mathcal{M}}', \alpha')$ is an isomorphism $\beta\colon \underline{\mathcal{M}} \xrightarrow{\sim} \underline{\mathcal{M}}'$ of F-modules satisfying $\alpha' = \beta[1/\pi] \circ \alpha$. In particular the sets $\mathrm{Hom}\big((\underline{\mathcal{M}}, \alpha), (\underline{\mathcal{M}}', \alpha')\big)$ contain at most one element.

We will sometimes drop the α from the notation and simply speak of $\underline{\mathcal{M}}$ as a model of $\underline{M}_L$.

For every $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$, resp. $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ we can consider the reduction $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_\ell$, resp. $\underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_\ell$. Note, however, that this does *not* induce a functor from $\mathrm{FMod}(A_{o_L})$, resp. $\mathrm{FMod}(A_{o_L,\pi})$ to $\mathrm{FMod}(A_\ell)$, since the induced F-map need not be injective. This circumstance lies at the origin of our study of good models:

Definition 4.5. Let $\underline{\mathcal{M}}$ be a model of an F-module $\underline{M}_L$ over A_L, resp. over $A_{o_L,\pi}[1/\pi]$. Then $\underline{\mathcal{M}}$ is called a *good* model if $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ is an F-module over A_ℓ, i.e. if the induced A_ℓ-linear map

$$\bar{\sigma}^*(\mathcal{M}/\pi\mathcal{M}) = (\mathcal{M}/\pi\mathcal{M}) \otimes_{A_\ell,\bar{\sigma}} A_\ell \to \mathcal{M}/\pi\mathcal{M}$$

is injective.

If $\underline{M}_L$ is an (analytic) Anderson motive there is an alternative notion of good reduction as follows.

Definition 4.6. Let $\underline{\mathcal{M}}$ be a model of an Anderson A-motive $\underline{M}_L$, resp. of an analytic Anderson $A(1)$-motive $\underline{M}_L$. Then $\underline{\mathcal{M}}$ is called a *good model in the strong sense* if $\mathrm{coker}(F_\mathcal{M})$ is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$, for some $d \geq 0$. In this case we also say that $\underline{\mathcal{M}}$ has *good reduction over* o_L.

Theorem 4.7. *Let $\underline{\mathcal{M}}$ be a model of an Anderson A-motive, resp. of an analytic Anderson $A(1)$-motive $\underline{M}_L$. Then $\underline{\mathcal{M}}$ is a good model in the weak sense of Definition 4.5 if and only if it is a good model in the strong sense of Definition 4.6.*

Proof. Since $\sigma^*\mathcal{M}$ is locally free over A_{o_L}, resp. over $A_{o_L,\pi}$, the natural map $\sigma^*\mathcal{M} \to \sigma^* M_L$ is injective and hence $F_\mathcal{M}$ is injective because F_{M_L} is. We obtain a short eqact sequence

$$0 \longrightarrow \sigma^*\mathcal{M} \xrightarrow{F_\mathcal{M}} \mathcal{M} \longrightarrow \mathrm{coker}(F_\mathcal{M}) \longrightarrow 0\,. \tag{4.2}$$

Let $\underline{\mathcal{M}}$ be a good model in the strong sense. Tensoring the short exact sequence (4.2) with ℓ over o_L and using that $\mathrm{coker}(F_\mathcal{M})$ is supposed to be free over o_L shows that the induced A_ℓ-linear map $\bar{\sigma}^*(\mathcal{M}/\pi\mathcal{M}) \to \mathcal{M}/\pi\mathcal{M}$ remains injective. So $\underline{\mathcal{M}}$ is a good model in the weak sense.

Conversely suppose that $\underline{\mathcal{M}}$ is a good model in the weak sense. This time tensoring (4.2) with ℓ over o_L yields

$$0 \longrightarrow \mathrm{Tor}_1^{o_L}\big(\mathrm{coker} F_\mathcal{M}, \ell\big) \longrightarrow \sigma^*\mathcal{M} \otimes_{o_L} \ell \xrightarrow{F_\mathcal{M} \otimes \mathrm{id}_\ell} \mathcal{M} \otimes_{o_L} \ell \longrightarrow \mathrm{coker}(F_\mathcal{M}) \otimes_{o_L} \ell \longrightarrow 0\,.$$

By assumption $F_\mathcal{M} \otimes \mathrm{id}_\ell$ is injective, and so $0 = \mathrm{Tor}_1^{o_L}\big(\mathrm{coker} F_\mathcal{M}, \ell\big) = \{x \in \mathrm{coker}(F_\mathcal{M})\colon \pi x = 0\}$ and $\mathrm{coker}(F_\mathcal{M})$ is flat over o_L by [Eis95, Corollary 6.3]. This implies $\mathrm{coker}(F_\mathcal{M}) \hookrightarrow \mathrm{coker}(F_\mathcal{M}) \otimes_{o_L} L = \mathrm{coker}(F_{M_L})$. Since $\mathrm{coker}(F_{M_L})$ is annihilated by $\mathfrak{J}^d$ for some d, the same is true for $\mathrm{coker}(F_\mathcal{M})$ which therefore is a finitely generated $A_{o_L}/\mathfrak{J}^d$-module, resp. $A_{o_L,\pi}/\mathfrak{J}^d$-module, and a fortiori a finitely generated o_L-module. Being flat, $\mathrm{coker}(F_\mathcal{M})$ is a finite free o_L-module. Thus $\mathcal{M}$ is a good model in the strong sense. □

Remark 4.8. In [Gar03], Gardeyn develops a theory of semi-stable reduction of analytic Anderson $A(1)$-motives $\underline{M}_L$. He shows that after replacing L by a finite separable extension, $\underline{M}_L$ has a model $\underline{\mathcal{M}}$ such that the reduction $F_\mathcal{M} \otimes \mathrm{id}_\ell$ is not nilpotent [Gar03, Proposition 3.3]. If $\overline{\underline{\mathcal{M}}}' \subseteq \underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ is the maximal Frobenius A_ℓ-submodule with injective $F_{\overline{\mathcal{M}}'}$, he further shows that the support of $\mathrm{coker}(F_{\overline{\mathcal{M}}'})$ is a finite set $S \subseteq \mathrm{Spec}\, A_\ell$. After removing S from $\mathfrak{A}(1) := \mathrm{Sp}(A_{o_L,\pi}[1/\pi])$ one can lift $\overline{\underline{\mathcal{M}}}'$ to an F-submodule $\underline{\mathcal{M}}' \subseteq \underline{\mathcal{M}}|_{\mathfrak{A}(1)\smallsetminus S}$ which has good reduction in the weak sense of Definition 4.5; see [Gar03, Theorem 4.7]. As one sees from the following example, it is false in general that S is the zero locus of $\mathfrak{J}$ in $\mathrm{Spec}\, A_\ell$ and so we cannot expect that $\underline{\mathcal{M}}'$ has good reduction in the strong sense of Definition 4.6.

Let $A = \mathbb{F}[z]$ and $\zeta = c^*(z) \in \mathfrak{m}_L$. Then $\mathfrak{J} = (z-\zeta)$. Let $\mathcal{M} = o_L\langle z\rangle^{\oplus 2}$ and $F_{\mathcal{M}} = \left(\begin{smallmatrix} 0 & \pi(z-\zeta) \\ \pi & z-1 \end{smallmatrix}\right)$. Then $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ is a model of the analytic Anderson $A(1)$-motive $\underline{M}_L := \underline{\mathcal{M}} \otimes_{o_L} L$. The reduction $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}} = \big(\ell[z]^{\oplus 2}, \left(\begin{smallmatrix} 0 & 0 \\ 0 & z-1 \end{smallmatrix}\right)\big)$ contains the maximal Frobenius A_ℓ-submodule $\overline{\underline{\mathcal{M}}}' = \ell[z] \cdot \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, whose Frobenius is $F_{\overline{\mathcal{M}}'} = z-1$. So $S = V(z-1) \neq V(z) = V(\mathfrak{J})$.

Proposition 4.9. *If $\underline{M}_L$ is an Anderson A-Motive over L having a (good) model $\underline{\mathcal{M}}$ then its analytification $\underline{M}_L \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ is an analytic Anderson $A(1)$-motive having the (good) model $\widehat{\underline{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ and the reduction $\widehat{\underline{\mathcal{M}}}/\pi\widehat{\underline{\mathcal{M}}}$ of $\widehat{\underline{\mathcal{M}}}$ is canonically isomorphic to the reduction $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ of $\underline{\mathcal{M}}$.*

Proof. The statement without the properties of being a good model is obvious. From the isomorphism $\widehat{\underline{\mathcal{M}}}/\pi\widehat{\underline{\mathcal{M}}} \xrightarrow{\sim} \underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ it follows that $\underline{\mathcal{M}}$ is a good model in the sense of Definition 4.5 if and only if $\widehat{\underline{\mathcal{M}}}$ is a good model in the sense of Definition 4.5. □

Let us also mention the following result of Gardeyn on good reduction of Drinfeld A-modules.

Proposition 4.10. *Let $\varphi\colon A \to L[\tau]$ be a Drinfeld A-module over L; see [Dri76] or [Mat96]. Let $\underline{M} = \underline{M}(\varphi)$ be the associated Anderson A-motive; see [And86, §4.1] or [Gar02, §8.1]. Then the following are equivalent:*

(i) φ has good reduction over o_L, i.e. φ is isomorphic over L to a Drinfeld A-module $\psi\colon A \to L[\tau]$ satisfying $\psi(A) \subseteq o_L[\tau]$ such that the reduction $\overline{\psi}\colon A \to o_L[\tau] \twoheadrightarrow \ell[\tau]$ is a Drinfeld A-module over ℓ of the same rank as ψ and φ.

(ii) $\underline{M}$ has good reduction over o_L in the weak and strong senses of Definitions 4.6 and 4.5.

Proof. Gardeyn [Gar02, Theorem 8.1] proved that φ has good reduction over o_L if and only if $\underline{M}$ has a good model in the weak sense. So the proposition follows from Theorem 4.7. □

5 Local shtukas and analytic Anderson motives

Anderson A-motives can be viewed as function-field analogs of abelian varieties. Barsotti-Tate groups, which can be associated with abelian varieties over $\mathbb{Z}_p$-schemes, have effective local shtukas as function-field analogs.

Definition 5.1. An (*effective*) *local shtuka at ε over o_L* is an object $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}}) \in \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)})$ such that $\mathrm{coker}(F_{\hat{M}})$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$.

Remark 5.2. If the residue field $\mathbb{F}_\varepsilon = A/\varepsilon$ of ε is larger than $\mathbb{F}$, i.e. if the degree $d_\varepsilon := [\mathbb{F}_\varepsilon : \mathbb{F}] > 1$, the ring $A_{o_L,(\varepsilon,\pi)}$ is not an integral domain but a product $A_{o_L,(\varepsilon,\pi)} = \prod_{i \in \mathbb{Z}/d_\varepsilon\mathbb{Z}} A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_i$ of integral domains. To describe this product decomposition, note that $A_{o_L,(\varepsilon,\pi)} = \varprojlim_n A_{o_L}/\varepsilon^n = \varprojlim_n (A/\varepsilon^n) \otimes_{\mathbb{F}} o_L = A_\varepsilon \widehat{\otimes}_{\mathbb{F}} o_L$. By Cohen's structure theorem $A_\varepsilon \cong \mathbb{F}_\varepsilon[\![z_\varepsilon]\!]$ for a uniformizer z_ε of A at ε. Then $\mathfrak{a}_i = (\alpha \otimes 1 - 1 \otimes c^*(\alpha)^{r^i} : \alpha \in \mathbb{F}_\varepsilon \subseteq A_\varepsilon)$, where we use that $c^*\colon A \to o_L$ factors through $c^*\colon A_\varepsilon \to o_L$. The factors $A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_i$ are isomorphic to $o_L[\![z_\varepsilon]\!]$ and hence are integral domains. They are cyclically permuted by σ because $\sigma(\mathfrak{a}_i) = \mathfrak{a}_{i+1}$. By [BH11, Proposition 8.8] the functor $(\hat{M}, F_{\hat{M}}) \mapsto (\hat{M}/\mathfrak{a}_0\hat{M}, (F_{\hat{M}})^{d_\varepsilon})$ is an equivalence between the category of effective local shtukas at ε over o_L as in Definition 5.1 and the category of pairs $(\hat{M}_0, \widetilde{F}_{\hat{M}})$ where $\hat{M}_0$ is a free module of finite rank over $A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_0$ and $\widetilde{F}_{\hat{M}}\colon (\sigma^{d_\varepsilon})^*\hat{M}_0 \to \hat{M}_0$ is injective with $\mathrm{coker}(\widetilde{F}_{\hat{M}})$ being a finite free o_L-module. In [Har09, Har11] these pairs $(\hat{M}_0, \widetilde{F}_{\hat{M}})$ are called (*effective*) *local shtukas*.

The following criterion for good reduction of analytic Anderson $A(1)$-motives can be regarded as a *good-reduction Local-Global Principle at the characteristic place.*

Theorem 5.3. *Let $\underline{M}_L = (M_L, F_{M_L})$ be an analytic Anderson $A(1)$-motive over L such that* $\operatorname{coker}(F_{M_L})$ *is annihilated by $\mathfrak{J}^d$ for some d. Then the following assertions are equivalent:*

(i) $\underline{M}_L$ admits a good model in the strong sense of Definition 4.6.

(ii) There is an effective local shtuka $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ at ε over o_L such that $\operatorname{coker}(F_{\hat{M}})$ *is annihilated by $\mathfrak{J}^d$, and an isomorphism $\underline{M}_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$ in* $\mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])$.

Proof. 1. In order to show that (ii) implies (i), let $f\colon M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \hat{M} \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] =: \hat{M}[1/\pi]$ be an F-equivariant isomorphism of $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-modules as in (ii). We have canonical F-equivariant $A_{o_L,\pi}$-linear maps

$$i\colon M_L \to M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi], \qquad j\colon \hat{M} \to \hat{M}[1/\pi]$$

where i (resp., j) is injective since M_L (resp., $\hat{M}$) is flat. Consider the $A_{o_L,\pi}$-module $\mathcal{M} = \operatorname{im}(i) \cap f^{-1}(\operatorname{im}(j))$. We will show that $\mathcal{M}$ is a good model of $\underline{M}_L$. The inclusion $\mathcal{M} \hookrightarrow M_L$ gives rise to an $A_{o_L,\pi}[1/\pi]$-linear embedding $\mathcal{M}[1/\pi] \hookrightarrow M_L[1/\pi] \cong M_L$, which is in fact an isomorphism, because if $m \in M_L$ there is an $s \geq 0$ such that $\pi^s f(m \otimes 1) \in \operatorname{im}(j)$, i.e. $\pi^s m \in \mathcal{M}$.

2. In order to show that $\mathcal{M}$ is a finitely generated $A_{o_L,\pi}$-module we use the embedding $\iota\colon \mathbb{F}[z] \to A$ from Remark 2.1 and the induced maps $L\langle z\rangle \to A_{o_L,\pi}[1/\pi]$ and $o_L[\![z]\!] \to A_{o_L,(\varepsilon,\pi)}$ from (2.1). Let $(e_1, \ldots, e_m)$ be a basis of M_L over the principal ideal domain $L\langle z\rangle$. Furthermore, let $(d_1, \ldots, d_n)$ be a basis for $\hat{M}$ over the local ring $o_L[\![z]\!]$. Note that the basis $(e_1, \ldots, e_m)$ gives rise to an isomorphism $M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \cong o_L[\![z]\!][1/\pi]^{\oplus m}$. For every $\nu = 1, \ldots, n$ we consider $f^{-1}(d_\nu)$ and regard it as an element of the right-hand side of this isomorphism. We choose $N \geq 0$ big enough, such that $f^{-1}(\pi^N d_\nu) \in o_L[\![z]\!]^{\oplus m}$ for all ν, say

$$f^{-1}(\pi^N d_\nu) \;=\; (\rho_{\nu,1}, \ldots, \rho_{\nu,m})$$

where $\rho_{\nu,\mu} \in o_L[\![z]\!]$. Now let $x \in \mathcal{M}$. Via f we obtain $f(x) = \sum_\nu \lambda_\nu d_\nu$ in $\hat{M}$, with suitable $\lambda_\nu \in o_L[\![z]\!]$. Consequently $f(\pi^N x) = \sum_\nu \lambda_\nu(\pi^N d_\nu)$, so that the image of $\pi^N x$ in $o_L[\![z]\!]^{\oplus m}$ satisfies $\pi^N x = \sum_\mu (\sum_\nu \lambda_\nu \rho_{\nu,\mu}) e_\mu$. The appearing scalars $h_\mu = \sum_\nu \lambda_\nu \rho_{\nu,\mu}$ have, in fact, to be elements of $L\langle z\rangle \cap o_L[\![z]\!] = o_L\langle z\rangle$. Inside M_L we may write $x = \pi^{-N}\pi^N x = \sum_\mu h_\mu \pi^{-N} e_\mu$, so that we may conclude

$$\mathcal{M} \subseteq \sum_\mu o_L\langle z\rangle \pi^{-N} e_\mu.$$

Being a submodule of a finitely generated module over a noetherian ring, $\mathcal{M}$ has to be a finitely generated $o_L\langle z\rangle$-module and hence a finitely generated $A_{o_L,\pi}$-module.

3. We claim that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free and hence free over $\ell[z]$, because it is finitely generated. Let $x \in \mathcal{M}$, and let $\lambda \in o_L\langle z\rangle$ be such that $\lambda \notin \pi o_L\langle z\rangle$ and $\lambda x \in \pi\mathcal{M}$, say $\lambda x = \pi y$ for some $y \in \mathcal{M}$. In order to prove that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free we must show that $x \in \pi\mathcal{M}$. First suppose that $\lambda \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$. We consider $\pi^{-1}x \in M_L$. In fact, this element lies in $\mathcal{M}$, since we have $f(\pi^{-1}x) = \lambda^{-1} f(y) \in \hat{M}$. Consequently $x = \pi(\pi^{-1}x) \in \pi\mathcal{M}$.

Let us next assume that $\lambda = z^n$ and show that $z^n x \in \pi\mathcal{M}$ implies $x \in \pi\mathcal{M}$ for any $n \geq 0$. By induction, it suffices to consider the case $n = 1$. So suppose $zx \in \pi\mathcal{M}$, say $zx = \pi y$. Let $f(x) = \sum_\nu \beta_\nu d_\nu$, where $(d_1, \ldots, d_n)$ is the finite $o_L[\![z]\!]$-basis of $\hat{M}$ fixed before. The relation $zx = \pi y$ implies that $\pi \mid z\beta_\nu$ for every index ν, so that $\pi \mid \beta_\nu$ for every ν. Therefore $\pi^{-1}x \in M_L$ necessarily maps via f to an element of $\hat{M}$, i.e. $x \in \pi\mathcal{M}$.

Finally we treat the case for general $\lambda = \sum_s \lambda_s z^s$ and suppose that $\lambda \notin o_L[\![z]\!]^\times$, that is $\pi \mid \lambda_0$. This means we find $\lambda' \in o_L[z]$ and $\lambda'' \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$ such that $\lambda = \pi\lambda' + z^N\lambda''$ for some $N \geq 1$. We have $\pi y = \lambda x = \pi\lambda' x + z^N\lambda'' x$. In particular $z^N\lambda'' x = \pi(y - \lambda' x) \in \pi\mathcal{M}$ and by the above $\lambda'' x \in \pi\mathcal{M}$ and $x \in \pi\mathcal{M}$.

Thus we have proved that $\mathcal{M}/\pi\mathcal{M}$ is free over $\ell[z]$. It follows that $\mathcal{M}/\pi\mathcal{M}$ is locally free of finite rank over A_ℓ.

4. We claim that $\mathcal{M}$ is locally free of finite rank over $A_{o_L,\pi}$. Since it is finitely generated it only remains to show that $\mathcal{M}$ is flat over $A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is π-adically complete and separated, $\pi A_{o_L,\pi}$ is contained in the Jacobson radical $\mathfrak{j}(A_{o_L,\pi})$ by [Mat86, Theorem 8.2], and the $A_{o_L,\pi}$-module $\mathcal{M}$ is finitely generated, so that $\mathcal{M}$ is π-adically *ideally Hausdorff* in the sense of [Bou67, III.5.1]. In the preceding step we have shown that $\mathcal{M}/\pi\mathcal{M}$ is flat over $A_\ell \cong A_{o_L,\pi}/\pi A_{o_L,\pi}$, and we know that $\mathcal{M}$ has no π-torsion, so that the canonical map $\pi A_{o_L,\pi} \otimes_{A_{o_L,\pi}} \mathcal{M} \to \pi\mathcal{M}$ is an isomorphism. Therefore, by Bourbaki's Flatness Criterion [Bou67, § III.5.2, Théorème 1(iii)], we may conclude that $\mathcal{M}$ is indeed flat over $A_{o_L,\pi}$.

5. We note that $\sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$ because the functor σ^* is exact by Lemma 3.1. By the F-equivariance of f we obtain a Frobenius $F_\mathcal{M}\colon \sigma^*\mathcal{M} \to \mathcal{M}$. It is injective because F_{M_L} is. We set $\underline{\mathcal{M}} := (\mathcal{M}, F_\mathcal{M})$.

6. Next we claim that $\mathfrak{J}^d\mathrm{coker}(F_\mathcal{M}) = 0$. Let $x = \sum_\nu h_\nu m_\nu \in \mathfrak{J}^d\mathcal{M}$ where $h_\nu \in \mathfrak{J}^d$ and $m_\nu \in \mathcal{M}$. Since $\mathrm{coker}(F_{M_L})$ is annihilated by $\mathfrak{J}^d$, there is a (unique) $y \in \sigma^* M_L$ such that $x = \sum_\nu h_\nu m_\nu = F_{M_L}(y)$. We have to show that $y \in \sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$. So it remains to see that $(\sigma^* f)(y) \in \mathrm{im}(\sigma^* j)$. Indeed, inside $\hat{M}[1/\pi]$ we have $f(x) = f(F_{M_L}(y)) = F_{\hat{M}}((\sigma^* f)(y))$. On the other hand, the linearity of f and j gives that $f(x) = \sum_\nu h_\nu f(m_\nu \otimes 1) = j(y')$ for some $y' \in \mathfrak{J}^d\hat{M} \subseteq \mathrm{im}(F_{\hat{M}})$, say $y' = F_{\hat{M}}(y'')$ for a $y'' \in \sigma^*\hat{M}$. Thus $f(x) = F_{\hat{M}}((\sigma^* j)(y''))$. So finally, since $F_{\hat{M}}\colon \sigma^*\hat{M}[1/\pi] \to \hat{M}[1/\pi]$ is injective, we obtain that $(\sigma^* f)(y) = (\sigma^* j)(y'')$, as desired.

7. Finally we show that the kernel V of $\overline{F}\colon \sigma^*(\mathcal{M}/\pi\mathcal{M}) \to \mathcal{M}/\pi\mathcal{M}$ is trivial. This implies that $\underline{\mathcal{M}}$ is a good model of $\underline{M}_L$ in the weak sense of Definition 4.5, which is enough by Theorem 4.7.

We have already shown that $\mathfrak{J}^d\mathcal{M} \subseteq \mathrm{im}(F_\mathcal{M})$. Since $(z - \zeta) \in \mathfrak{J}$ for $\zeta := c^*(z) \in o_L$ we have a chain of $o_L\langle z\rangle$-modules $(z-\zeta)^d\mathcal{M} \subseteq \mathrm{im}(F_\mathcal{M}) \subseteq \mathcal{M}$. The element $\zeta \in o_L$ is zero mod π, and we obtain

$$z^d(\mathcal{M}/\pi\mathcal{M}) \subseteq \mathrm{im}(\overline{F}) \subseteq \mathcal{M}/\pi\mathcal{M}. \tag{5.3}$$

We know that $\mathcal{M}/\pi\mathcal{M}$ is finite free over $\ell[z]$. Therefore the middle term $W := \mathrm{im}(\overline{F})$ in the latter chain has full rank inside $\mathcal{M}/\pi\mathcal{M}$. Finally, taking ranks in the (split) short exact sequence of finite free $\ell[z]$-modules

$$0 \to V \to \sigma^*(\mathcal{M}/\pi\mathcal{M}) \xrightarrow{\overline{F}} W \to 0$$

accomplishes the proof that V indeed is trivial.

8. Conversely, in order to show that (i) implies (ii), suppose that $(\underline{\mathcal{M}}, \alpha)$ is a good model of $\underline{M}_L$. We define

$$\underline{\hat{M}} = \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)},$$

i.e. $\underline{\hat{M}}$ equals the completion of $\underline{\mathcal{M}}$ for the $(\varepsilon,\pi)A_{o_L,\pi}$-adic topology. It is clear that the F-equivariant isomorphism $\alpha\colon M_L \xrightarrow{\sim} \mathcal{M}[1/\pi]$ of $A_{o_L,\pi}[1/\pi]$-modules gives rise to a natural F-equivariant $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-linear isomorphism $M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \hat{M}[1/\pi]$.

We claim that $\underline{\hat{M}}$ is a local shtuka. Indeed, by base change, $\hat{M}$ is again locally free of finite rank. Furthermore, since the completion map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is Frobenius-equivariant and flat, we obtain an injective map $\hat{M} \otimes_{(A_{o_L,(\varepsilon,\pi)}),\sigma} A_{o_L,(\varepsilon,\pi)} \to \hat{M}$. Let C' be its cokernel, and let $C = \mathrm{coker}(F_\mathcal{M})$, i.e. $C' \cong C \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$. Since C is annihilated by $\mathfrak{J}^d$ the module C' equals C and it is finite free over o_L. Thus $\underline{\hat{M}}$ is an effective local shtuka over o_L. □

Remark 5.4. Steps 1-4 in the previous proof suggest that there is an equivalence of categories

$$\mathcal{F}\colon \left\{ \begin{array}{l} \text{finite locally free} \\ A_{o_L,\pi}\text{-modules } \mathcal{M} \end{array} \right\} \xleftrightarrow{\ \sim\ } \left\{ \begin{array}{l} \text{triples } (M_L, \hat{M}, f) \text{ consisting of} \\ \bullet \text{ a finite locally free } A_{o_L,\pi}[1/\pi]\text{-module } M_L, \\ \bullet \text{ a finite locally free } A_{o_L,(\varepsilon,\pi)}\text{-module } \hat{M}, \text{ and} \\ \bullet \text{ an isomorphism of } A_{o_L,(\varepsilon,\pi)}[1/\pi]\text{-modules} \\ \quad f\colon M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \hat{M} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi] \end{array} \right\}$$

$$\mathcal{M} \longmapsto \big(\mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi],\ \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)},\ \mathrm{id}_{\mathcal{M}\otimes A_{o_L,(\varepsilon,\pi)}[1/\pi]}\big)\,,$$

where on the right a morphism $\underline{h} = (h_L, \hat{h})\colon (M_L, \hat{M}, f) \to (M'_L, \hat{M}', f')$ consists of a morphism $h_L\colon M_L \to M'_L$ and a morphism $\hat{h}\colon \hat{M} \to \hat{M}'$ such that $f' \circ (h_L \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}[1/\pi]}) = (\hat{h} \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}[1/\pi]}) \circ f$.

However, *this is false* as can be seen from the following example, where we take $A = \mathbb{F}[z]$. We choose an element $a \in \ell[\![z]\!] \subseteq \ell(\!(z)\!)$ such that $a \notin \ell(z)$, and we let $\Delta = \left(\begin{smallmatrix} 1 & \pi^{-1}a \\ 0 & \pi^{-1} \end{smallmatrix}\right)$. Set $M_L = L\langle z\rangle^{\oplus 2}$, $\hat{M} = \Delta \cdot o_L[\![z]\!]^{\oplus 2}$ and $f = \mathrm{id}_{o_L[\![z]\!][1/\pi]^2}$. Then $\Delta^{-1} = \left(\begin{smallmatrix} 1 & -a \\ 0 & \pi \end{smallmatrix}\right) \in o_L[\![z]\!]^{2\times 2}$ and

$$o_L[\![z]\!]^{\oplus 2} \;=\; \Delta\cdot\Delta^{-1} o_L[\![z]\!]^{\oplus 2} \;\subseteq\; \hat{M} \;\subseteq\; \pi^{-1} o_L[\![z]\!]^{\oplus 2}\,.$$

If there was a finite free $A_{o_L,\pi}$-module $\mathcal{M}$ with $(h_L, \hat{h})\colon \mathcal{F}(\mathcal{M}) \xrightarrow{\sim} (M_L, \hat{M}, f)$, then it had to satisfy $\mathcal{M} \cong M_L \cap \hat{M}$ with h_L and $\hat{h}$ induced from the inclusions $M_L \cap \hat{M} \subseteq M_L$ and $M_L \cap \hat{M} \subseteq \hat{M}$. So we may take directly $\mathcal{M} := M_L \cap \hat{M}$. It satisfies $o_L\langle z\rangle^{\oplus 2} \subseteq \mathcal{M} \subseteq \pi^{-1} o_L\langle z\rangle^{\oplus 2}$. We claim that, in fact, the first inclusion is an equality. Namely let $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi^{-1}v_0+v' \\ \pi^{-1}w_0+w' \end{smallmatrix}\right) \in \mathcal{M}$ with $v_0, w_0 \in \ell[z]$ and $v', w' \in o_L\langle z\rangle$. Then $\Delta^{-1}\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi^{-1}v_0+v'-\pi^{-1}aw_0-aw' \\ w_0+\pi w' \end{smallmatrix}\right) \in o_L[\![z]\!]^{\oplus 2}$. This implies $v_0 = aw_0$ in $\ell[\![z]\!]$. If $w_0 \neq 0$ we get $a = v_0/w_0 \in \ell(z)$ in contradiction to our assumption. So $w_0 = v_0 = 0$ and $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) \in o_L\langle z\rangle^{\oplus 2}$. This proves our claim that $\mathcal{M} = o_L\langle z\rangle^{\oplus 2}$. We conclude that $\mathcal{F}(\mathcal{M}) \ncong (M_L, \hat{M}, f)$ and $\mathcal{F}$ is not an equivalence of categories.

After this example the following result is even more surprising.

Corollary 5.5. *Let $\underline{M}_L$ be an analytic Anderson $A(1)$-motive over L. Then there is an equivalence of categories*

$$\left\{ \begin{array}{l} \textit{good models } (\underline{\mathcal{M}}, \alpha) \textit{ of } \underline{M}_L \textit{ in the} \\ \textit{sense of Definitions 4.6 and 4.5} \end{array} \right\} \xleftrightarrow{\ \sim\ } \left\{ \begin{array}{l} \textit{pairs } (\underline{\hat{M}}, f) \textit{ consisting of} \\ \bullet \textit{ a local shtuka } \underline{\hat{M}} \textit{ at } \varepsilon \textit{ over } o_L, \textit{ and} \\ \bullet \textit{ an isomorphism in } \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \\ \quad f\colon \underline{M}_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi] \end{array} \right\}$$

$$(\underline{\mathcal{M}}, \alpha) \longmapsto (\underline{\mathcal{M}}, \alpha) \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}\,,$$

where on the right-hand side a morphism *of pairs $\hat{\beta}\colon (\underline{\hat{M}}, f) \xrightarrow{\sim} (\underline{\hat{M}}', f')$ is defined to be an isomorphism of local shtukas $\hat{\beta}\colon \underline{\hat{M}} \xrightarrow{\sim} \underline{\hat{M}}'$ satisfying $f' = \hat{\beta} \circ f$.*

Proof. Suppose that $(\underline{\mathcal{M}}, \alpha)$ is a good model of $\underline{M}_L$. In the proof of 5.3 we have seen that its completion $\underline{\hat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ is a local shtuka at ε. The F-equivariant isomorphism $\alpha\colon M_L \xrightarrow{\sim} \mathcal{M}[1/\pi]$ of $A_{o_L,\pi}[1/\pi]$-modules induces the isomorphism

$$f := \alpha \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}[1/\pi]}\colon M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \hat{\mathcal{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$$

which is F-equivariant, and satisfies $\mathcal{M} = f(M_L) \cap \hat{\mathcal{M}}$, because $A_{o_L,\pi} = A_{o_L,\pi}[1/\pi] \cap A_{o_L,(\varepsilon,\pi)}$.

To see that this functor is fully faithful let $(\underline{\mathcal{M}}, \alpha)$ and $(\underline{\mathcal{M}}', \alpha')$ be good models of $\underline{M}_L$ and let $\hat{\beta}\colon (\underline{\hat{\mathcal{M}}}, f) := (\underline{\mathcal{M}}, \alpha) \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)} \xrightarrow{\sim} (\underline{\hat{\mathcal{M}}}', f') := (\underline{\mathcal{M}}', \alpha') \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ be an isomorphism. This means $f' = \hat{\beta} \circ f$. Applying $\mathcal{M} = f(M_L) \cap \hat{\mathcal{M}}$ and $\mathcal{M}' = f'(M_L) \cap \hat{\mathcal{M}}'$ we see that $\hat{\beta}(\mathcal{M}) = \mathcal{M}'$. Therefore $\beta := \hat{\beta}|_{\mathcal{M}} : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ is the desired isomorphism satisfying $\beta \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}} = \hat{\beta}$. This implies $\alpha' = \beta \circ \alpha$ and the F-equivariance of β, and hence $\beta\colon (\underline{\mathcal{M}}, \alpha) \xrightarrow{\sim} (\underline{\mathcal{M}}', \alpha')$.

To prove essential surjectivity, let a local shtuka $\underline{\hat{M}}$ together with an isomorphism $f\colon \underline{M}_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi]$ be given. It remains to show that the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion $\underline{\hat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ of the good model $\mathcal{M} = M_L \cap f^{-1}(\hat{M})$ gained in the proof of 5.3 gives back $\underline{\hat{M}}$. Then we take α as the canonical isomorphism $\mathrm{id}\colon \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi] \xrightarrow{\sim} M_L$. By construction of $\underline{\mathcal{M}}$, the map f restricts to an embedding $\mathcal{M} \hookrightarrow \hat{M}$, which in turn induces an F-equivariant and $A_{o_L,(\varepsilon,\pi)}$-linear map $\psi := f|_{\hat{\mathcal{M}}}\colon \hat{\mathcal{M}} \to \hat{M}$, which becomes an isomorphism after inverting π. Our aim is to show that already the map ψ is an isomorphism $(\underline{\mathcal{M}}, \mathrm{id}) \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)} \xrightarrow{\sim} (\underline{\hat{M}}, f)$. According to Remark 5.4 we have to use the Frobenius morphisms $F_{\hat{\mathcal{M}}}$ and $F_{\hat{M}}$ in an essential way.

We know that $\mathcal{M}$ is finite free over $o_L\langle z\rangle$ and that $\mathrm{rk}_{o_L[\![z]\!]}(\hat{\mathcal{M}}) = \mathrm{rk}_{o_L[\![z]\!]}(\hat{M}) =: s$. We fix an $o_L[\![z]\!]$-basis $\mathfrak{B}$ (resp., $\mathfrak{C}$) of $\hat{\mathcal{M}}$ (resp., of $\hat{M}$) and let $\mathbf{A} = {}_{\mathfrak{C}}[\psi]_{\mathfrak{B}} \in o_L[\![z]\!]^{s\times s}$ be the matrix which describes ψ with respect to $\mathfrak{B}$ and $\mathfrak{C}$. Likewise, we let

$$\mathbf{T} = {}_{\mathfrak{B}}[F_{\hat{\mathcal{M}}}]_{\sigma^*\mathfrak{B}}, \qquad \mathbf{T}' = {}_{\mathfrak{C}}[F_{\hat{M}}]_{\sigma^*\mathfrak{C}}$$

be the matrices corresponding to $F_{\hat{\mathcal{M}}}$ and $F_{\hat{M}}$, so that $\mathbf{AT} = \mathbf{T}'\sigma(\mathbf{A})$ by virtue of the F-equivariance of ψ. In order to see that ψ is an isomorphism, we need to show that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. To begin with, an elementary application of the Weierstraß Division Theorem for $o_L[\![z]\!]$ ([Bou67, VII.3.8.5]) shows that the kernel of the epimorphism $o_L[\![z]\!] \to o_L$, $z \mapsto \zeta$, is generated by $z - \zeta$, so that the latter is a prime element of $o_L[\![z]\!]$. Furthermore, recall that $o_L[\![z]\!]$, being a regular local ring, is factorial ([Mat86], 20.3). We know that $\underline{\hat{\mathcal{M}}}$ is a local shtuka, so that $F_{\hat{\mathcal{M}}}$ becomes an isomorphism after inverting $z - \zeta$ which means that $\det(\mathbf{T})^{-1}$ lies in $o_L[\![z]\!][\frac{1}{z-\zeta}]$. Say we have a relation $(z-\zeta)^e = \det(\mathbf{T})u$ in $o_L[\![z]\!]$, for some $e \geq 0$ and some $u \in o_L[\![z]\!]$. By a comparison of powers of $z - \zeta$, we may assume that u is not divisible by $z - \zeta$. In this equation there is only one prime element of $o_L[\![z]\!]$ occurring on both sides, which, by factoriality, implies that u has to be a unit in $o_L[\![z]\!]$. Let $(z-\zeta)^{e'} = \det(\mathbf{T}')u'$ be the corresponding relation for the local shtuka $\hat{M}$, with a unit $u' \in o_L[\![z]\!]^\times$ and some suitable $e' \geq 0$. Since $\hat{\mathcal{M}} \to \hat{M}$ becomes an isomorphism after inverting π, we see that $\det(\mathbf{A}) \in o_L[\![z]\!][1/\pi]^\times$. Note that the natural reduction-mod-z map $o_L[\![z]\!] \to o_L$, $h \mapsto h(0)$, induces an epimorphism of abelian groups $o_L[\![z]\!][\frac{1}{\pi}]^\times \to L^\times$, so that the absolute term $\delta := \det(\mathbf{A})(0)$ of $\det(\mathbf{A})$ lies in $L^\times$. By virtue of the relations derived above, the equation $\det(\mathbf{A})\det(\mathbf{T}) = \det(\mathbf{T}')\sigma(\det(\mathbf{A}))$ yields

$$\det(\mathbf{A})u^{-1}(z-\zeta)^e = u'^{-1}(z-\zeta)^{e'}\sigma(\det(\mathbf{A}))$$

which modulo z gives $\delta^{q-1} = \frac{u'(0)}{u(0)}(-\zeta)^{e-e'}$ in $L^\times$. Suppose for a moment that $e = e'$. In this case it follows at once that δ is a unit in o_L, so that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. Therefore it remains to verify that our assumption $e = e'$ is justified. This can be seen as follows: The reduction-mod-π map $o_L[\![z]\!] \to \ell[\![z]\!]$ is an epimorphism with kernel $\pi o_L[\![z]\!]$, and via applying the functor $\cdot \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$ to $F_{\hat{M}}\colon \sigma^*\hat{M} \to \hat{M}$ we obtain a commutative diagram

$$\begin{array}{ccc}
\sigma^*\hat{M} = \hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!] & \longrightarrow & \hat{M} \\
\downarrow & & \downarrow \\
\bar{\sigma}^*\hat{M}/\pi\hat{M} = \hat{M}/\pi\hat{M} \otimes_{\ell[\![z]\!],\bar{\sigma}} \ell[\![z]\!] & \longrightarrow & \hat{M}/\pi\hat{M}
\end{array}$$

where in the upper row (resp., the bottom row) both modules are finite free of the same rank over $o_L[\![z]\!]$ (resp., over $\ell[\![z]\!]$) and the arrow is given by $F_{\hat{M}}$ (resp., by $\bar{F} = F_{\hat{M}} \otimes \mathrm{id}_{\ell[\![z]\!]}$). The reduced matrix $\overline{\mathbf{T}'} \in \ell[\![z]\!]^{s\times s}$ describes the map $\bar{F}$ with respect to the $\ell[\![z]\!]$-bases $\overline{\sigma^*\mathfrak{C}} = \bar{\sigma}^*\bar{\mathfrak{C}}$ of $\bar{\sigma}^*\hat{M}/\pi\hat{M}$ and $\bar{\mathfrak{C}}$ of $\hat{M}/\pi\hat{M}$ respectively, and from what we have seen before, we derive the relation $\det(\overline{\mathbf{T}'})\overline{u'} = z^{e'}$, i.e. $e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'}))$, the latter being true since $\overline{u'} \in \ell[\![z]\!]^{\times}$. In particular we have $\det(\overline{\mathbf{T}'}) \in \ell[\![z]\!] - \{0\}$. A similar observation for the local shtuka $\hat{\mathcal{M}}$ instead of $\hat{M}$ shows that $e = \mathrm{ord}_z(\det(\overline{\mathbf{T}}))$. Let $C = \mathrm{coker}(F_{\hat{\mathcal{M}}})$ and $C' = \mathrm{coker}(F_{\hat{M}})$. Multiplication with the matrix $\overline{\mathbf{T}'}$ gives rise to a finite presentation $\ell[\![z]\!]^s \to \ell[\![z]\!]^s \to C'/\pi C' \to 0$. Taking determinants in an equation of the form $\mathbf{S}_1\overline{\mathbf{T}'}\mathbf{S}_2 = \mathrm{Diag}(a_1, ..., a_d, 0, 0, ..., 0)$, where $\mathbf{S}_1, \mathbf{S}_2 \in \mathrm{Gl}_s(\ell[\![z]\!])$ are suitable matrices such that $a_1, ..., a_d \in \ell[\![z]\!] - \{0\}$ are the elementary divisors of $\overline{\mathbf{T}'}$ (see [Bou81], VII.4.5.1), yields that necessarily $d = s$, so that $C'/\pi C'$ is a torsion $\ell[\![z]\!]$-module and

$$C'/\pi C' \cong \ell[\![z]\!]/a_1\ell[\![z]\!] \oplus ... \oplus \ell[\![z]\!]/a_s\ell[\![z]\!] \cong \ell^{n_1} \oplus ... \oplus \ell^{n_s}$$

where $n_j = \mathrm{ord}_z(a_j)$ and $\sum_j n_j = e'$, i.e. $e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'})) = \mathrm{rk}_\ell(C'/\pi C') = \mathrm{rk}_{o_L}(C')$, the latter equation being valid since $C'/\pi C' \cong C' \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$. Finally, imitating this argument for the local shtuka $\hat{\mathcal{M}}$ yields that $e = \mathrm{ord}_z(\det(\overline{\mathbf{T}})) = \mathrm{rk}_\ell(C/\pi C) = \mathrm{rk}_{o_L}(C)$. So it remains to show that $\mathrm{rk}_{o_L}(C) = \mathrm{rk}_{o_L}(C')$. Indeed, we know that $\psi\colon \hat{\mathcal{M}} \to \hat{M}$ gives back f in the generic fiber, which means that ψ is an isomorphism after inverting π. Therefore, inverting π in the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma^*(\hat{\mathcal{M}}) & \longrightarrow & \hat{\mathcal{M}} & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma^*\psi} & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\ 0 & \longrightarrow & \sigma^*\hat{M} & \longrightarrow & \hat{M} & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

exhibits $(\sigma^*\psi)[1/\pi] = \sigma^*(\psi[1/\pi])$ and $\psi[1/\pi]$ as $o_L[\![z]\!][1/\pi]$-linear isomorphisms, so that the Snake Lemma yields $C'[1/\pi] \cong C[1/\pi]$, and we obtain $\mathrm{rk}_{o_L}(C') = \dim_L(C'[1/\pi]) = \dim_L(C[1/\pi]) = \mathrm{rk}_{o_L}(C)$, as desired. $\square$

6 The reduction criterion for Anderson motives

Definition 6.1. (a) Let $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$. Following Gardeyn [Gar03], $\underline{\mathcal{M}}$ is called A_{o_L}*-maximal* if for every $\underline{\mathcal{N}} \in \mathrm{FMod}(A_{o_L})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L})}(\underline{\mathcal{N}}, \underline{\mathcal{M}}) \to \mathrm{Hom}_{\mathrm{FMod}(A_L)}(\underline{\mathcal{N}}[1/\pi], \underline{\mathcal{M}}[1/\pi])$$

is surjective (and hence bijective).

(b) An object $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called $A_{o_L,\pi}$*-maximal* if for every $\underline{\mathcal{N}}' \in \mathrm{FMod}(A_{o_L,\pi})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi})}(\underline{\mathcal{N}}', \underline{\mathcal{M}}') \to \mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi}[1/\pi])}(\underline{\mathcal{N}}'[1/\pi], \underline{\mathcal{M}}'[1/\pi])$$

is surjective (and hence bijective).

(c) Let $\underline{M} \in \mathrm{FMod}(A_L)$. An object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ is called an A_{o_L}*-maximal model* for $\underline{M}$ if $\underline{\mathcal{M}}[1/\pi] \cong \underline{M}$ inside $\mathrm{FMod}(A_L)$ (i.e. $\underline{\mathcal{M}}$ is a *model* for $\underline{M}$) and if $\underline{\mathcal{M}}$ is A_{o_L}-maximal. Correspondingly, given $\underline{M}' \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, an object $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called an $A_{o_L,\pi}$*-maximal model* for $\underline{M}'$ if $\underline{\mathcal{M}}'[1/\pi] \cong \underline{M}'$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ and if $\underline{\mathcal{M}}'$ is $A_{o_L,\pi}$-maximal.

The existence of (A_{o_L}- and $A_{o_L,\pi}$-)maximal models has been established in [Gar03].

Proposition 6.2 ([Gar03, Proposition 2.13])**.** *Let $\underline{M} \in \mathrm{FMod}(A_L)$. Then the following assertions hold:*

(i) $\underline{M}$ admits an A_{o_L}-maximal model, which is unique up to unique isomorphism.

(ii) If a model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ of $\underline{M}$ is good in the weak sense of Definition 4.5, then it is A_{o_L}-maximal.

The next proposition is a variant of Gardeyn's theory of maximal models.

Proposition 6.3. *The following assertions hold:*

(i) Every $\underline{M} \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *admits a maximal model, which is unique up to unique isomorphism.*

(ii) If $\underline{M} \in \mathrm{FMod}(A_L)$ *is given and if* $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ *is an* A_{o_L}*-maximal model of* $\underline{M}$ *then* $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \in \mathrm{FMod}(A_{o_L,\pi})$ *is an* $A_{o_L,\pi}$*-maximal model of* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

(iii) Let $\underline{M} \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *and let* $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ *be a model of* $\underline{M}$*. If* $\underline{\mathcal{M}}$ *is a good model in the weak sense of Definition 4.5, then it is* $A_{o_L,\pi}$*-maximal.*

Proof. For (i) (resp. (ii); resp. (iii)), see [Gar03], 3.3(i) (resp. 3.4(i); resp. 2.13(ii)). Note that strictly speaking Gardeyn proves these statements for the rings $\Gamma(\mathfrak{A}(\infty), \mathcal{O}_{\mathfrak{A}(\infty)})$ instead of $A_{o_L,\pi}[1/\pi]$ and $\Gamma(\mathfrak{A}(\infty), \mathcal{O}_{\mathfrak{A}(\infty)}) \cap A_{o_L,\pi}$ instead of $A_{o_L,\pi}$. His arguments carry over literally to our rings. □

We may conclude:

Proposition 6.4. *In the weak sense of Definition 4.5 a Frobenius* A_L*-module* $\underline{M}$ *admits a good model over* A_{o_L} *if and only if* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *admits a good model over* $A_{o_L,\pi}$*. If this is the case, the functor* $(\underline{\mathcal{M}}, \alpha) \mapsto (\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi},\ \alpha \otimes \mathrm{id}_{A_{o_L,\pi}[1/\pi]})$ *is an equivalence of categories between the good models of* $\underline{M}$ *and the good models of* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$.

Proof. First suppose that $\underline{M}$ admits a good model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$. It follows that $\underline{\mathcal{M}}$ is an A_{o_L}-maximal model of $\underline{M}$. Furthermore, its image $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ inside $\mathrm{FMod}(A_{o_L,\pi})$ is an $A_{o_L,\pi}$-maximal model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$. Since the reduction of $\underline{\mathcal{M}}$ is canonically isomorphic to the reduction of $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ by Proposition 4.9, it follows that the latter is a good model.

Conversely, suppose that $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\widehat{\underline{\mathcal{M}}} \in \mathrm{FMod}(A_{o_L,\pi})$. Necessarily $\widehat{\underline{\mathcal{M}}}$ is a maximal model by Proposition 6.3(iii). We know that there is an A_{o_L}-maximal model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ of $\underline{M}$ such that $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \cong \widehat{\underline{\mathcal{M}}}$, and that the reduction of $\widehat{\underline{\mathcal{M}}}$ is canonically isomorphic to the reduction of $\underline{\mathcal{M}}$ by Propositions 6.2, 6.3(ii) and 4.9. Since $\widehat{\underline{\mathcal{M}}}$ is a good model, so is $\underline{\mathcal{M}}$. This proves the first statement and it also proves essential surjectivity of the functor.

To prove full faithfulness let $(\underline{\mathcal{M}}, \alpha)$ and $(\underline{\mathcal{M}}', \alpha')$ be good models of $\underline{M}$ and let $\hat{\beta} : \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \xrightarrow{\sim} \underline{\mathcal{M}}' \otimes_{A_{o_L}} A_{o_L,\pi}$ be an isomorphism in $\mathrm{FMod}(A_{o_L,\pi})$ satisfying $\alpha' \otimes \mathrm{id} = \hat{\beta} \circ (\alpha \otimes \mathrm{id})$. Since $A_{o_L} = A_L \cap A_{o_L,\pi}$ inside $A_{o_L,\pi}[1/\pi]$, we can recover $\mathcal{M}$ as $\mathcal{M} = \alpha(M) \cap \mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi}$. This implies $\hat{\beta}(\mathcal{M}) = \mathcal{M}'$ and $\beta := \hat{\beta}|_{\mathcal{M}}$ is the desired isomorphism $\beta \colon \underline{\mathcal{M}} \xrightarrow{\sim} \underline{\mathcal{M}}$ with $\alpha' = \beta \circ \alpha$. This proves full faithfulness. □

For Anderson A-motives Proposition 6.4 and Theorem 4.7 imply the following

Corollary 6.5. *Let* $\underline{M}$ *be an Anderson* A*-motive over* L*. Then in the strong sense of Definition 4.6,* $\underline{M}$ *admits a good model* $\underline{\mathcal{M}}$ *if and only if the associated analytic Anderson* $A(1)$*-motive* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ *admits a good model* $\underline{\mathcal{M}}'$*. If this is the case, the functor* $(\underline{\mathcal{M}}, \alpha) \mapsto (\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi},\ \alpha \otimes \mathrm{id}_{A_{o_L,\pi}[1/\pi]})$ *is an equivalence of categories between the good models of* $\underline{M}$ *and the good models of* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$. □

This corollary together with Theorem 5.3 and Corollary 5.5 implies the following criterion for good reduction of Anderson A-motives, which can be regarded as an analog of the reduction criteria for abelian varieties of Grothendieck [SGA 7, Proposition IX.5.13] and de Jong [dJ98, 2.5].

Corollary 6.6. *Let* $\underline{M}$ *be an Anderson* A*-motive over* L *such that* $\mathrm{coker}(F_{\underline{M}})$ *is annihilated by* $\mathfrak{J}^d$ *for some* d*. Then the following assertions are equivalent:*

(i) $\underline{M}$ *admits a good model* $(\underline{\mathcal{M}}, \alpha)$ *in the strong sense of Definition 4.6, i.e. there is an object* $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ *such that* $\mathrm{coker}(F_{\underline{\mathcal{M}}})$ *is a finite free* o_L*-module and is annihilated by* $\mathfrak{J}^d$*, together with an isomorphism* $\alpha \colon \underline{M} \xrightarrow{\sim} \underline{\mathcal{M}}[1/\pi]$ *inside* $\mathrm{FMod}(A_L)$*;*

(ii) There is an effective local shtuka $\underline{\hat{M}}$ at ε over o_L such that $\operatorname{coker}(F_{\hat{M}})$ *is annihilated by $\mathfrak{J}^d$, and an isomorphism $\underline{M} \otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}}[1/\pi]$ inside* $\mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])$.

Moreover, there is an equivalence of categories

$$\left\{\begin{array}{l}\textit{good models } (\underline{\mathcal{M}},\alpha) \textit{ of } \underline{M} \textit{ in the}\\ \textit{sense of Definitions 4.6 and 4.5}\end{array}\right\} \xleftrightarrow{\;\sim\;} \left\{\begin{array}{l}\textit{pairs } (\underline{\hat{M}}, f) \textit{ consisting of}\\ \bullet \textit{ a local shtuka } \underline{\hat{M}} \textit{ at } \varepsilon \textit{ over } o_L, \textit{ and}\\ \bullet \textit{ an isomorphism in } \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])\\ \quad f\colon \underline{M} \otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi]\end{array}\right\}$$

$$(\underline{\mathcal{M}},\alpha) \longmapsto (\underline{\mathcal{M}},\alpha) \otimes_{A_{o_L}} A_{o_L,(\varepsilon,\pi)}\,,$$

where on the right-hand side a morphism *of pairs $\hat{\beta}\colon (\underline{\hat{M}}, f) \xrightarrow{\sim} (\underline{\hat{M}}', f')$ is defined to be an isomorphism of local shtukas $\hat{\beta}\colon \underline{\hat{M}} \xrightarrow{\sim} \underline{\hat{M}}'$ satisfying $f' = \hat{\beta} \circ f$.* □

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Urs Hartl
Universität Münster
Mathematisches Institut
Einsteinstr. 62
D – 48149 Münster
Germany
www.math.uni-muenster.de/u/urs.hartl/

Simon Hüsken
Universität Münster
Mathematisches Institut
Einsteinstr. 62
D – 48149 Münster
Germany