The Picard Functor

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Definition

Let S be a base scheme and let $f : X \longrightarrow S$ be a morphism of schemes. Consider the contravariant functor from the category of S-schemes to the category of abelian groups

$$P_{X/S} : (\mathrm{Sch}/S) \longrightarrow (\mathrm{Ab}), \qquad S' \longmapsto \mathrm{Pic}(X \times_S S') = \mathrm{H}^1(X \times_S S', \mathcal{O}^*_{X \times_S S'}).$$

The *relative Picard functor* is the (fppf)-sheaf associated to the functor $P_{X/S}$.

$$\mathrm{Pic}_{X/S} = \mathrm{R}^1 f_* \mathbb{G}_m \qquad \text{as (fppf)-sheaves on } S.$$

This means $\mathrm{Pic}_{X/S}$ is a contravariant functor from (Sch/S) to (Ab) such that, for each S-scheme T and for each morphism $T' \longrightarrow T$ which is either faithfully flat and of finite presentation, i.e. (fppf) or a Zariski-covering, the following sequence is exact:

$$\mathrm{Pic}_{X/S}(T) \longrightarrow \mathrm{Pic}_{X/S}(T') \rightrightarrows \mathrm{Pic}_{X/S}(T' \times_T T').$$

Every Element of $\mathrm{Pic}_{X/S}(T)$ for a quasi-compact S-scheme T can be given by a line bundle $\mathcal{L}'$ on $X \times_S T'$ for some scheme T' which is (fppf) over T. Furthermore there exists an (fppf)-morphism $\widetilde{T} \longrightarrow T' \times_T T'$, such that the pullbacks with respect to the two projections $\widetilde{T} \longrightarrow T'$ are isomorphic.

Now consider the case:

(∗) Let f be quasi-compact and quasi-separated with a section $x : S \longrightarrow X$ and let $f_*\mathcal{O}_X = \mathcal{O}_S$ universally, i.e. still valid after any base change. This holds for example if f is proper and flat with geometrically reduced and irreducible fibers.

Then $\mathrm{Pic}_{X/S}$ is the contravariant Functor from (Sch/S) to (Ab) given by

$$\begin{aligned} S' \longmapsto & \left\{ \begin{array}{l} \text{Isomclasses of } (\mathcal{L}, \lambda) : \mathcal{L} \text{ line bundle on } X \times_S S' \\ \lambda : \mathcal{O}_{S'} \xrightarrow{\sim} (x \times \mathrm{id}_{S'})^* \mathcal{L} \quad \text{rigidification} \end{array} \right\} \\ = & \ \mathrm{Pic}(X \times_S S') / \mathrm{Pic}(S'). \end{aligned}$$

The rigidification λ has two effects. It kills all line bundles coming from S' and secondly it causes the automorphism group of $(\mathcal{L}, \lambda)$ to be trivial.

Representability

The functor $\operatorname{Pic}_{X/S}$ is called *representable* if there exists an S-scheme P such that there is an isomorphism of functors

$$\operatorname{Pic}_{X/S} \;\cong\; \operatorname{Hom}_{(\mathrm{Sch}/S)}(\,\bullet\,, P) \;=:\; P(\,\bullet\,)\,.$$

In the case $(*)$ this means, that there exists a rigidified line bundle $(\mathcal{P},\rho) \in \operatorname{Pic}_{X/S}(P)$ called the Poincaré-bundle, with the universal property (Yoneda Lemma):

for every S-scheme S' and for every line bundle $(\mathcal{L},\lambda) \in \operatorname{Pic}_{X/S}(S')$ there exists a unique morphism $g : S' \longrightarrow P$ with

$$(\mathcal{L},\lambda) \;\cong\; (\,\mathrm{id}_X \times g)^*(\mathcal{P},\rho)\,.$$

Concerning the representability there is the following theorem.

Theorem 1. (Grothendieck)
Let $f : X \longrightarrow S$ be projective and finitely presented, flat with geometrically reduced and irreducible fibers. Then $\operatorname{Pic}_{X/S}$ is representable by a separated S-scheme which is locally of finite presentation over S.

Proof: cf. [FGA, n° 232, Thm. 3.1], [BLR, Thm. 8.2.1]
1. One introduces *effective, relative Cartier divisors* D on X over S, i.e. D is a closed subscheme of X, flat over S, which in each fibre is an effective Cartier divisor. One considers the contravariant functor

$$\begin{aligned} \operatorname{Div}_{X/S}: \;\; (\mathrm{Sch}/S) &\longrightarrow (\mathrm{Sets}) \\ S' &\longmapsto \Big\{\text{effective relative Cartier divisors on } \; X\times_S S'/S'\Big\}\,. \end{aligned}$$

There is a morphism of functors $\operatorname{Div}_{X/S} \longrightarrow \operatorname{Pic}_{X/S}$ sending $D \mapsto \mathcal{O}_X(D)$, which is shown to be relatively representable, i.e. for each S-scheme S' the morphism

$$\operatorname{Div}_{X/S} \times_{\operatorname{Pic}_{X/S}} S' \;\longrightarrow\; S'$$

is a morphism of schemes. So the divisors inducing a given line bundle in $\operatorname{Pic}_{X/S}(S')$ are parameterized by a scheme.

2. One shows the representability of the functor $\operatorname{Div}_{X/S}$ using the existence of the Hilbert-scheme. This is the hardest part of the proof.

3. For a fixed $\Phi \in \mathbb{Q}[t]$ one considers the subfunctor $\operatorname{Pic}^{\Phi}_{X/S}$ of $\operatorname{Pic}_{X/S}$, which consists of all elements having Hilbert-polynomial Φ with respect to the given projective embedding of X.

For suitable Φ there exists a finite union $D(\Phi)$ of connected components of $\operatorname{Div}_{X/S}$ such that the functor $\operatorname{Pic}^{\Phi}_{X/S}$ is the quotient

$$D(\Phi) \;\longrightarrow\!\!\!\!\!\rightarrow\; \operatorname{Pic}^{\Phi}_{X/S}$$

by a proper and smooth equivalence relation. One now shows that therefore it is representable by a scheme.

For general Φ there exists an $n_\Phi \in \mathbb{Z}$ such that the translate $\mathrm{Pic}^{\Phi}_{X/S} + \mathcal{O}_X(n_\Phi)$ is of the special case above.

Since $\mathrm{Pic}^{\Phi}_{X/S}$ is an open and closed subfunctor of $\mathrm{Pic}_{X/S}$ one finds that

$$\mathrm{Pic}_{X/S} \;=\; \coprod_{\Phi \in \mathbb{Q}[t]} \mathrm{Pic}^{\Phi}_{X/S}$$

is representable by a scheme over S. □

A further theorem on the representability is the following.

Theorem 2. (Murre and Oort)
Let X be a proper scheme over a field k. Then $\mathrm{Pic}_{X/k}$ is representable by a scheme which is locally of finite type over k.

<u>Proof:</u> cf. [Mu], [Oo]
One reduces to the projective case which was done by Grothendieck [FGA, n° 232, Sect. 6].

If now X is proper over k, we define $\mathrm{Pic}^0_{X/k}$ as the connected component which contains the unit element. It is a group scheme of finite type over k.

If X is proper over S, we define the groupfunctor

$$\mathrm{Pic}^0_{X/S}(T) \quad := \quad \left\{ \xi \in \mathrm{Pic}_{X/S}(T) : \; \xi|_{X_t} \in \mathrm{Pic}^0_{X_s/k(s)}\big(k(t)\big) \quad \forall t \in T \right\}.$$

The case of curves

In the case of curves more can be said.

Theorem 3.
Let $f : X \longrightarrow S$ be a proper, flat curve, locally of finite presentation. Then $\mathrm{Pic}_{X/S}$ is a formally smooth functor over S.

<u>Proof:</u> cf. [BLR, Prop. 8.4.2]
That $\mathrm{Pic}_{X/S}$ is formally smooth over S means that for each affine S-scheme Z and for each closed subscheme $Z_0 \subseteq Z$ which is given by a sheaf of ideals $\mathcal{N}$ with $\mathcal{N}^2 = 0$ the canonical map is surjective:

$$\mathrm{Hom}(Z, \mathrm{Pic}_{X/S}) \;\longrightarrow\!\!\!\!\rightarrow\; \mathrm{Hom}(Z_0, \mathrm{Pic}_{X/S}).$$

This follows from considering the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_X & \longrightarrow & \mathcal{O}^*_{X \times_S Z} & \longrightarrow & \mathcal{O}^*_{X \times_S Z_0} & \longrightarrow & 0\,. \\ & & n & \longmapsto & 1 + n & & & & \end{array}$$

In fact applying $(f \times \mathrm{id}_Z)_*$ yields

$$\mathrm{R}^1(f \times \mathrm{id}_Z)_* \, \mathcal{O}^*_{X \times_S Z} \longrightarrow \mathrm{R}^1(f \times \mathrm{id}_{Z_0})_* \, \mathcal{O}^*_{X \times_S Z_0} \longrightarrow \mathrm{R}^2(f \times \mathrm{id}_Z)_* \, \mathcal{N} \otimes_{\mathcal{O}_S} \mathcal{O}_X \,.$$

Since X is a curve over S and $\mathcal{N}$ is quasi-coherent, the last term is 0. So applying $\mathrm{H}^0(Z, \bullet)$ and observing Z affine, one gets

$$\begin{array}{ccc} \mathrm{H}^0\big(Z, \mathrm{R}^1(f \times \mathrm{id}_Z)_* \, \mathcal{O}^*_{X \times_S Z}\big) & \twoheadrightarrow & \mathrm{H}^0\big(Z_0, \mathrm{R}^1(f \times \mathrm{id}_{Z_0})_* \, \mathcal{O}^*_{X \times_S Z_0}\big) \,. \\ = \mathrm{Pic}_{X/S}(Z) & & = \mathrm{Pic}_{X/S}(Z_0) \end{array}$$

and thus the above morphism is surjective. □

If X is a proper curve over a field k, then $\mathrm{Pic}_{X/k}$ is a scheme locally of finite type over k by Theorem 2. So $\mathrm{Pic}^0_{X/k}$ is a scheme of finite type over k. In this case smoothness and formal smoothness are the same. So we see that $\mathrm{Pic}^0_{X/k}$ is a smooth k-scheme.

Next we want to study $\mathrm{Pic}^0_{X/k}$ in terms of divisors. So let X be a proper curve over a field k and D an effective Cartier divisor on X. Locally at a point $x \in X$ the divisor D is given by a regular element l. We define the order of D in x as

$$\mathrm{ord}_x(D) \;:=\; \mathrm{length}_k\big(\mathcal{O}_{X,x}/(l)\big) \,.$$

If X is regular at x, then $\mathcal{O}_{X,x}$ is a discrete valuation ring and $\mathrm{ord}_x(D)$ is just the order of l in that ring. We now define the *degree* of D

$$\deg(D) \;:=\; \sum_{x \in X} \mathrm{ord}_x(D) \cdot [k(x) : k] \,.$$

It has the following properties:

1. $\deg$ is additive, so we can also define it for non-effective divisors.
2. $\deg$ is not altered by field extensions.
3. If X is reduced, the degree of a divisor on X is the same as the degree of its pullback to the normalization of X.
4. The degree of the divisor of a meromorphic function is zero.

So we can define the degree for line bundles. Let $\mathcal{L}$ be a line bundle on X, then there are two effective Cartier divisors D and D' on X with $\mathcal{L} \cong \mathcal{O}(D - D')$. We define the *degree* of $\mathcal{L}$

$$\deg(\mathcal{L}) \;:=\; \deg(D) - \deg(D') \,.$$

If X is a flat, proper curve of finite presentation over an arbitrary base S and $\mathcal{L}$ a line bundle on X, the function $s \mapsto \deg(\mathcal{L}|_{X_s})$ is locally constant on S.

Next if $X = X_1 \cup \ldots \cup X_r$ is the decomposition in irreducible components, we define the *partial degree* of $\mathcal{L}$ on X_i:

$$\deg_{X_i}(\mathcal{L}) \ := \ \deg(\mathcal{L}|_{X_i}) \,.$$

The partial degrees are related to the total degree by the formula

$$\deg(\mathcal{L}) \ = \ \sum_{i=1}^{n} d_i \cdot \deg_{X_i}(\mathcal{L}) \,,$$

where d_i is the multiplicity of X_i in X, i.e. $d_i := \mathrm{length}(\mathcal{O}_{X,\eta_i})$ for the generic point η_i of X_i.

Now let X be a smooth proper geometrically irreducible curve of genus g over a field k. Assume that X has a rational point x_0. Then there is a morphism

$$X^g \ \longrightarrow \ X^{(g)} \ = \ X^g/\mathfrak{S}_g \ \longrightarrow \ \mathrm{Pic}^0_{X/k} \,, \qquad x_1 + \ldots + x_g \ \mapsto \ \mathcal{O}_X\big(\Sigma(x_i - x_0)\big) \,,$$

where $\mathfrak{S}_g$ is the symmetric group on g letters and $X^{(g)}$ is the symmetric product of X. The latter morphism is an epimorphism and birational. The whole Picard variety decomposes

$$\mathrm{Pic}_{X/k} \ = \ \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d_{X/k} \,.$$

where $\mathrm{Pic}^d_{X/k}$ is the connected component of $\mathrm{Pic}_{X/k}$ representing the line bundles of degree d. It is a $\mathrm{Pic}^0_{X/k}$-torsor.

In the following we consider arbitrary proper curves over a field k.

Proposition 4.
Let X be a proper curve over a field k. Then $\mathrm{Pic}^0_{X/k}$ consists of all elements of $\mathrm{Pic}_{X/k}$ whose partial degree on each irreducible component of $X \otimes_k \overline{k}$ is zero.

Proof: cf. [BLR, Cor. 9.3.13]
Let k be algebraically closed, $X_{\mathrm{red}} = \bigcup X_i$ be the irreducible components with normalizations $\widetilde{X}_i$ and $g : \coprod \widetilde{X}_i \longrightarrow X$. Then $\mathrm{Pic}_{X/k}$ is an extension

$$1 \ \longrightarrow \ L \ \longrightarrow \ \mathrm{Pic}_{X/k} \ \xrightarrow{g^*} \ \prod_i \mathrm{Pic}_{\widetilde{X}_i/k} \ \longrightarrow \ 1$$

by a connected linear group L. Therefore we have

$$\mathrm{Pic}^0_{X/k} \ = \ (g^*)^{-1}\Big(\prod_i \mathrm{Pic}^0_{\widetilde{X}_i/k}\Big) \,.$$

The proposition now follows from the fact, that $\mathrm{Pic}^0_{\widetilde{X}_i/k}(k)$ are exactly the line bundles having degree zero on X_i. □

Description of $\mathrm{Pic}^0_{X/k}$

Let X be a proper curve over a perfect field k and $\widetilde{X}$ the normalization of X_{red}. We want to introduce an intermediate curve lying between X_{red} and $\widetilde{X}$.

$$X \hookleftarrow X_{\mathrm{red}} \longleftarrow X' \longleftarrow \widetilde{X}.$$

There are only finitely many non-smooth points of X_{red}. We define X' by identifying all points of $\widetilde{X}$ lying above such a non-smooth point of X_{red}. (This can be formalized with the amalgamated sum, cf. [BLR, p. 247].) So the singularities of X' are just ordinary multiple points.

The above maps induce morphisms on the Picard-schemes.

$$\mathrm{Pic}^0_{X/k} \longrightarrow \mathrm{Pic}^0_{X_{\mathrm{red}}/k} \longrightarrow \mathrm{Pic}^0_{X'/k} \longrightarrow \mathrm{Pic}^0_{\widetilde{X}/k}.$$

The next theorem tells more about the structure of these morphisms.

Theorem 5.

a) *the map* $\mathrm{Pic}^0_{X/k} \twoheadrightarrow \mathrm{Pic}^0_{X_{\mathrm{red}}/k}$ *is an epimorphism with a smooth connected unipotent group as kernel.*

b) *the map* $\mathrm{Pic}^0_{X_{\mathrm{red}}/k} \twoheadrightarrow \mathrm{Pic}^0_{X'/k}$ *is an epimorphism with a smooth connected unipotent group as kernel, which is trivial if and only if* $X' = X_{\mathrm{red}}$.

c) *the map* $\mathrm{Pic}^0_{X'/k} \twoheadrightarrow \mathrm{Pic}^0_{\widetilde{X}/k}$ *is an epimorphism with a torus as kernel, which is trivial if and only if each irreducible component is homeomorphic to its normalization and the configuration of the irreducible components of* $X \otimes_k \overline{k}$ *is tree-like, i.e. if and only if* $\mathrm{H}^1_{\text{ét}}(X \otimes_k \overline{k}, \mathbb{Z}) = 0$.

d) $\mathrm{Pic}^0_{\widetilde{X}/k}$ *is an abelian variety.*

Proof: cf. [BLR, Sect. 9.2]

Ad c) Let $X' = X = \bigcup_{i=1}^{n} X_i$ be the irreducible components and $g : \widetilde{X} = \coprod_{i=1}^{n} \widetilde{X}_i \longrightarrow X$ be the normalization. Let further $x_1, \ldots, x_N$ be the singular points of X and $\widetilde{x}_{\nu 1}, \ldots, \widetilde{x}_{\nu m_\nu}$ the points of $\widetilde{X}$ lying above x_ν. Then we have the exact sequence

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow g_* \mathcal{O}_{\widetilde{X}}^* \longrightarrow g_* \mathcal{O}_{\widetilde{X}}^* / \mathcal{O}_X^* \longrightarrow 1.$$

The sheaf $\mathcal{F} := g_* \mathcal{O}_{\widetilde{X}}^* / \mathcal{O}_X^*$ is concentrated at the points x_ν, so we get

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & f_* \mathcal{O}_X^* & \longrightarrow & f_* g_* \mathcal{O}_{\widetilde{X}}^* & \longrightarrow & f_* \mathcal{F} & \longrightarrow & \mathrm{R}^1 f_* \mathcal{O}_X^* & \longrightarrow & \big(\mathrm{R}^1 f_*\big) g_* \mathcal{O}_{\widetilde{X}}^* \longrightarrow 1. \\ & & \| & & \| & & \backslash\backslash & & & & \\ & & \mathrm{H}^0(X, \mathcal{O}_X)^* & & \prod_{i=1}^{n} \mathrm{H}^0(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}})^* & & \prod_{\nu=1}^{N} \Big(\prod_{\mu=1}^{m_\nu} k(\widetilde{x}_{\nu\mu})^* \Big) / k(x_\nu)^* & & & & \end{array}$$

So the kernel is a quotient of a torus, thus a torus. The remaining assertion follows by combinatorial arguments. $\square$

Remark: One can describe the extension c) explicitly (cf. [Zh]).

For the remaining part let us work with curves over a discrete valuation ring. Then in the following case the Picard functor is representable.

Theorem 6. (Raynaud)
Let S be the spectrum of a discrete valuation ring. Let $f : X \longrightarrow S$ be a proper, flat, normal curve with $f_*\mathcal{O}_X = \mathcal{O}_S$ and geometrically reduced special fiber. Then $\mathrm{Pic}^0_{X/S}$ is representable by a separated S-scheme.

For the proof see [Ra, Thm. 8.2.1] or [BLR, Thm. 9.4.2].

Let now R be a complete discrete valuation ring and X a semi-stable curve over R (i.e. a proper, flat scheme whose geometric fibers are reduced and connected curves with only ordinary double points as singularities). Let further the generic fiber X_K be smooth over K and the irreducible components of the special fiber X_k be smooth over k.

Then by Theorem 5 after a base ring extension $J_k := \mathrm{Pic}^0_{X_k/k}$ is an extension

$$1 \longrightarrow \mathbb{G}^r_{m,k} \xrightarrow{\alpha} \mathrm{Pic}^0_{X_k/k} \longrightarrow \mathrm{Pic}^0_{\widetilde{X}_k/k} \longrightarrow 1,$$

of an abelian variety by a torus. The rank of the torus is $r = \mathrm{rk}_{\mathbb{Z}}\, \mathrm{H}^1(X_k, \mathbb{Z})$ and the map α can be described as follows. Let t_i be the coordinates of the torus, then α is given by the line bundle $t_1^{a_1} \otimes \ldots \otimes t_r^{a_r}$ on $X_k \times_k \mathbb{G}^r_{m,k}$ for some basis $a_1, \ldots, a_r$ of $\mathrm{H}^1(X_k, \mathbb{Z})$.

We now want to investigate this situation with formal and rigid geometric methods and consider the formal completion $\overline{J} := \left(\mathrm{Pic}^0_{X/R}\right)^\wedge = \mathrm{Pic}^0_{\widehat{X}/R}$ of $\mathrm{Pic}^0_{X/R}$ along its special fiber. Then the torus lifts over R to a smooth formal torus

$$1 \longrightarrow \overline{\mathbb{G}}^r_{m,R} \longrightarrow \overline{J} \longrightarrow B \longrightarrow 1.$$

There $\overline{\mathbb{G}}_{m,R}$ is the formal completion of $\mathbb{G}_{m,R}$ along its special fiber. The quotient B is a formal abelian scheme. Actually it is the formal completion of an abelian scheme over R. On the rigid fibers we obtain

$$1 \longrightarrow \overline{\mathbb{G}}^r_{m,K} \longrightarrow \overline{J}_{\mathrm{rig}} \longrightarrow B_{\mathrm{rig}} \longrightarrow 1.$$

$\overline{J}_{\mathrm{rig}}$ parameterizes the formally smooth deformations of the trivial line bundle on X_K. Or phrased differently it is given by the divisors on X_K whose reductions are divisors with all partial degrees equal to zero.

As push-forward of $\overline{J}_{\mathrm{rig}}$ via the open immersion of the formal torus into the affine torus we get

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \overline{\mathbb{G}}^r_{m,K} & \longrightarrow & \overline{J}_{\mathrm{rig}} & \longrightarrow & B_{\mathrm{rig}} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{G}^r_{m,K} & \longrightarrow & \widetilde{J}_{\mathrm{rig}} & \longrightarrow & B_{\mathrm{rig}} & \longrightarrow & 1
\end{array}$$

On $X_K^{\mathrm{an}} \times_K \widetilde{J}_{\mathrm{rig}}$ there is a universal line bundle $(\widetilde{\mathcal{P}}, \widetilde{\rho})$, which induces a canonical morphism $\widetilde{J}_{\mathrm{rig}} \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Pic}^0_{X_K^{\mathrm{an}}/K} = \big(\mathrm{Pic}^0_{X_K/K}\big)^{\mathrm{an}}$. The kernel

$$M \;=\; \big\{\, p \in \widetilde{J}_{\mathrm{rig}} :\; (\widetilde{\mathcal{P}}, \widetilde{\rho})|_{X_K^{\mathrm{an}} \times \{p\}} \text{ trivial} \,\big\}$$

of this morphism is a lattice in $\widetilde{J}_{\mathrm{rig}}$ of full rank. The intersection $M \cap \overline{J}_{\mathrm{rig}}$ contains only the unit element of $\overline{J}_{\mathrm{rig}}$. The quotient

$$\widetilde{J}_{\mathrm{rig}}/M \;=\; \big(\mathrm{Pic}^0_{X_K/K}\big)^{\mathrm{an}}$$

exists and makes $\widetilde{J}_{\mathrm{rig}}$ into the universal covering of the rigid space $\big(\mathrm{Pic}^0_{X_K/K}\big)^{\mathrm{an}}$.

For more details see [BL].

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